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CRITERIA OF LE PAGE TYPE IN INVOLUTIVE BANACH ALGEBRAS*

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Abstract

The Le Page inequality in Banach algebras is $\| xy \| \leq \alpha \| yx \|$. We consider inequalities where a linear involution appears and has consequences on the involution itself and commutativity results. [1]

Key words: and phrases. Banach algebra, Commutativity, $C^\star$-algebra, antilinear map, Spectral radius, Numerical radius.

1. Introduction

Le Page considered in a Banach algebra A ([2]) the following condition (C_1): $\| xy \| \leq \alpha \| yx \|$, for ever x and y in A. This inequality has been extended in many ways. In particular, Baker and Pym ([1]) considered a bilinear mapping in the left side of (C_1). But few results have been obtained for involutive Banach algebras. The basic ingredient in proofs is the Liouville theorem for homomorphic functions and involutions do not preserve holomorphy. Here we use the Liouville theorem for harmonic functions to examine the following conditions:

$$\| x^\star y \| \leq \alpha \| yx \| \qquad (C_2),$$
$$\| xy + y^\star x^\star \| \leq \alpha \| yx \| \quad (C_3),$$

* [1] 1991 Mathematics Subject Classification : Primary 46 J40; Secondary 46H20

$$\| x^\star y \| \leq \alpha \| yx^\star \| \qquad (C_4).$$

In some cases, a linear involution is forced to be an algebra involution. In others we obtain commutativity results. It is worth mentioning that in the unitary case condition (C_2) is strong enough to imply a $C^\star$-algebra structure (theorem **2.1**); in the non unitary case it always implies that the algebra is hermitian.

In the sequel A will always be a complex associative Banach algebra with a norm $\| \, . \, \|$, and $C(A)$ the center of A. The numerical radius and the spectral radius of an element x will be denoted by $v(x)$ and $\rho(x)$, respectively. A linear involution is any antilinear mapping $x \to x^\star$ with $x^{\star\star} = x$, for every x in A. If moreover $(xy)^\star = y^\star x^\star$ for every x, y in A, it is said to be an algebra involution. The algebra A is said to be involutive it is endowed with an algebra involution.

2. Contuditon (C_2)

We deal here with the condition

$$\| x^\star y \| \leq \alpha \| yx \|, \quad \forall x, y \in A \quad (C_2).$$

In view of Le Page type results, the question is to know wether or not (C_2) implies computativity. The basic argument in this type of problem is Liouville's theorem. In our setting we should apply it to a product of holomorphic and a harmonic function. As a matter of fact, it appears that it is not valid in this situation. Indeed consider $A = C(K)$, the algebra of complex continuous functions on a compact space K and the bounded function $f(\lambda) = (exp(\lambda x))y(exp(-\bar{\lambda} x^\star))$ with x, y in A and $xy \neq 0$. If Liouville's theorem were valid, then f should be constant. By a classical argument, we then obtain $xy = yx^\star$. Whence $xy = 0$; a contradiction.

However, we can proceed differently and condition (C_2) appears to be strong enough as the following results show.

Theorem 2.1. *Let A be a complex unital and involutive Banach algebra satisfying (C_2). Then A is a commutative $C^\star$-algebra.*

Proof. By (C_2), we have, in particular, $\| u^2 \| \leq \alpha \| uu^\star \|$ for every $u \in A$. Hence, for every hermitian element h in $A, \| exp(ih) \| \leq \alpha \| (exp(i\frac{h}{2}))^2 \| \leq \alpha \| (exp(i\frac{h}{2})exp(-i\frac{h}{2}) \| \leq \alpha$. So, by Cuntz's theorem, the algebra A is a $C^\star$-algebra for an equivalent norm $| \, . \, |$. To see that it is commutative, observe that $|u|^2 = |uu^\star| \leq \beta|u^2|$, for some $\beta > 0$ and every u in A. Then we have by induction $|u^{2^n}| \geq (\frac{1}{\beta})^{2^n - 1}|u|^{2^n}$. Whence $\rho(u) \geq \frac{1}{\beta}|u|$, which implies commutativity.

In the non unitary case, we can say the following.

Theorem 2.2. *Let A be a non-unital complex Banach algebra with a continuous involution. If A satisfies (C_2), then*

(i) $A^2 \subset C(A)$. *Hence if A admits a left or a right approximate unit, then A is commutative;*

(ii) If (C_2) is verified for $u \in A^1 = A \bigoplus C$, and $v \in A$, then A is commutative;

(iii) A is hermitian.

□

Proof. Notice that (C_2) implies $\| u^\star v \| \leq \beta \| vu^\star \|$, for a $\beta > 0$, u and v in A. Hence $\| uv \| \leq \beta \| vu \|$, for every u and v in A. This is the Le Page condition. Whence **(i)** and **(ii)** ([3]). On the other hand (C_2) gives, for every normal element u in A, $\rho(u)^2 \leq \alpha^{\frac{1}{2^n}}(\| (uu^\star)^{2^{n-1}} \|)^{\frac{1}{2^{n-1}}}$. Hence $\rho(u)^2 \leq \rho(uu^\star)$. Whence **(iii)**. □

Remark 2.1 *1) The existence of a unit is essential in theorem 2.1. Just consider the trivial case of any vector space with the product zero and any involution. 2) To give a non trivial example, let B be any involutive Banach algebra (commutative or not, unital or not) which is not a $C^\star$-algebra. Consider B as a vector space endowed with the zero product. Then $C(K) \times B$, with the usual operations and the involution $(f, b) \mapsto (\bar{f}, b^\star)$*

satisfies (C_2). It is not a $C^\star$-algebra. 3) We have seen that (C_2) implies the Le Page condition (C_1). This implication does not hold if $x \mapsto x^\star$ is not an algebra involution. Indeed if B is any involutive unital and non commutative algebra, consider $C(K) \times B$ with the usual operations and the antilinear mapping $(f,b) \mapsto (\bar{f},0)$. Condition ($C_2$) is verified but not condition (C_1) for B^2 is not in the center of B.

3. CONDITION (C_3)

In this section, we look at the condition

$$\| xy + y^\star x^\star \| \leq \alpha \| yx \| \qquad (C_3).$$

If $x \mapsto x^\star$ is an algebra involution, (C_3) reads $\| Re(uv) \| \leq 2\alpha \| vu \|$, which yields $\| uv \| \leq 4\alpha \| vu \|$. This is the Le Page condition. If we have only a linear involution, we get the following results, under a more general condition than (C_3).

Theorem 3.1. *Let $(A, \| . \|)$ be a complex Banach algebra, p a linear seminorm on A, $q : A \to R_+$ any positive map and $\varphi : A \to A$ an antilinear mapping. If*

$$p[uv + \varphi(v)\varphi(u)] \leq q(vu), \forall u, v \in A \quad (C_5),$$

then

(i) $p(xyz - yzx) = 0, \forall x, y, z \in A$. In particular $A^2 \subset C(A)$ if p is a norm;

(ii) $p(\varphi(zx)\varphi(y) - \varphi(z)\varphi(xy)) = 0, \forall x, y, z \in A$;

(iii) If p is continuous and A has a left or a right approximate unit, then

(a) any commutator is in the kernel of p. Hence A is commutative if p is a norm.

(b) $p(\varphi(xy) - \varphi(x)\varphi(y)) = 0, forevery\, x, y \in A$, if φ is continuous and is such that $\varphi(e_i) = e_i$, where $(e_i)_i$ is the approximate unit;

(iv) If (C_5) is verified with $u \in A$ and $v \in A^1 = A \oplus C$, then we have **(a)** *and* **(b)** *of* **(iii)**.

Proof. If A is not unital, consider $A^1 = A \oplus C$. For x, y, z in A consider the function

$$f(\lambda) = (exp(\lambda x))yz(exp(-\lambda x)) + \varphi(z(exp(-\lambda x)))\varphi((exp(\lambda x))y).$$

For every l in the topological dual $(A,p)'$, of A, for ρ, the function lop is harmonic. It is also bounded by (C_5). It is then constant. Differentiating relative to the real part and the imaginary part of λ, we have

$$l[xyz - yzx - \varphi(zx)\varphi(y) + \varphi(z)\varphi(xy)] = 0$$

and

$$l[xyz - yzx + \varphi(zx)\varphi(y) - \varphi(z)\varphi(xy)] = 0$$

Whence **(i)** and **(ii)**, by the Hahn-Banach theorem.
(iii) is an immediate consequence of **(i)** and **(ii)**.
To get **(iv)**, consider the function.

$$f(\lambda) = (exp(\lambda x))y(exp(-\lambda x)) + \varphi(yexp(-\lambda x))\varphi(exp(\lambda x))$$

and proceed as above. □

Remark 3.2: *Let A be an involutive Banach algebra and p a seminorm on A such that*

$$p(\frac{x+x^\star}{2}) \leq \rho(x), \forall x \in A.$$

We, of course, have

$$p\left[\frac{uv+v^\star u^\star}{2}\right] \leq \rho(vu),\ \forall u \in A^1, \forall v \in A.$$

Then, by **(iv)**, *$p(xy-yx)$=0, for every x and y in A.*

As an application, notice that, if

$$\rho(\frac{x+x^\star}{2}) \leq \rho(x), \forall x \in A,$$

then

$$p(\frac{x+x^{\star}}{2}) \leq \rho(x), \forall x \in A,$$

where p is Palmer's seminorm.

In case of a norm p, we have the following results.

Corollary 3.3: *Each of the following conditions implies the commutativity of* A.

(1) $\nu(\frac{x+x^{\star}}{2}) \leq \alpha\rho(x), \forall x \in A$.

(2) $\| \frac{x+x^{\star}}{2} \| \leq \alpha\rho(x), \forall x \in A$.

4. CONDITION (C_4)

Our last condition is:

$$\| u^{\star}v \| \leq \alpha \| vu^{\star} \|, \ \forall u, v \in A \qquad (C_4)$$

If $x \mapsto x^{\star}$ is a generalized algebra involution, then (C_4) is equivalent to the Le Page condition. In case it is only a linear involution, we obtain the following results.

Theorem 4.1: *Let* A *be a complex unital Banach algebra with the unit* e *and* $\varphi : A \to A$ *a continuous antilinear mapping such that* $\varphi(exp(x))$ *is invertible, for every* x *in* A. *If* A *satisfies* (C_4), *then*

(i) $[\varphi(e)]^{-1}\varphi(x)y = y[\varphi(e)]^{-1}\varphi(x), \forall x, y \in A$.

(ii) If φ *is unital i.e.,* $\varphi(e) = e$, *then* $\varphi(x)y = y\varphi(x), \forall x, y \in A$.

(iii) If φ *is an antimorphism, then* $\varphi(xy - yx) = 0, \forall x, y \in A$. *In particular,* A *is commutative, when* φ *is one to one.*

Proof **(i)** *Consider the function* $f(\lambda) = [\varphi(exp(\lambda x))]y[\varphi(exp(\lambda x))]^{-1}$. *It is bounded. Straightforward calculations show that it is harmonic. It is then constant. Differentiating relative to the real part of* λ *(or the imaginary part) yields the result.*

(ii) *An immediate consequence of* **(i)**.

(iii) *Follows from* **(ii)**.

Remark 4. 2: *1) The hypothesis that $\varphi(exp(x))$ is invertible is meaningful in that it is not always fulfilled. Indeed take a Banach algebra endowed with two different algebra involutions $\star$ and $\equiv$, and consider $\varphi : A \to A$, given by $\varphi(x) = x^\star - x^\equiv$.*
2) If φ is linear, then the function f is in fact holomorphic;
3) One may state a more general theorem, considering $\varphi : A \to B$, where B is another Banach algebra;
4) One may also state theorem **4.1** *with p being a seminorm.*

Corollary 4.3: *Let $\varphi : A \to A$ be a linear involution $x \mapsto x^\star$ such that $e^\star = e$. Suppose that one of the following conditions holds:*
(a) $(xy)^\star = y^\star x^\star$, for every x, y in A.
(b) $(xy)^\star = x^\star y^\star$, for every x, y in A.
If A satisfies (C_4), then $x^\star y = y x^\star$, for every x, y in A.

Proof *Observe that, by* **(a)** *or* **(b)**, *$\varphi(exp(x))$ is invertible.*

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