

1-1-1999

## On the Boundedness of Integral Operators

İSMAİL EKİNCİOĞLU

ISROIL A. IKROMOV

Follow this and additional works at: <https://journals.tubitak.gov.tr/math>



Part of the [Mathematics Commons](#)

---

### Recommended Citation

EKİNCİOĞLU, İSMAİL and IKROMOV, ISROIL A. (1999) "On the Boundedness of Integral Operators," *Turkish Journal of Mathematics*: Vol. 23: No. 2, Article 4. Available at: <https://journals.tubitak.gov.tr/math/vol23/iss2/4>

This Article is brought to you for free and open access by TÜBİTAK Academic Journals. It has been accepted for inclusion in Turkish Journal of Mathematics by an authorized editor of TÜBİTAK Academic Journals. For more information, please contact [academic.publications@tubitak.gov.tr](mailto:academic.publications@tubitak.gov.tr).

## ON THE BOUNDEDNESS OF INTEGRAL OPERATORS

*İ. Ekinçioğlu & İ. A. Ikromov*

### Abstract

In this paper, we consider integral operators defined by kernel functions. It is well known the boundedness of such kind of operators as Shur Lemma type statements. But, the norm of operators was estimated by two integrals of kernel function. We obtain estimation of operators norm by one integral of kernel function.

### 1. Introduction

Let  $L^p(\mathbb{R}^n)$  be the space of all integrable functions with degree  $p$  and denote by  $L^p_{loc}(\mathbb{R}^n)$  the space of all local integrable functions with degree  $p$  in particular denote by  $L^\infty_{loc}(\mathbb{R}^n)$  the space of local bounded functions. Let  $\mathcal{D}_1 \subset L^p(\mathbb{R}^n)$  be a subspace of  $L^p(\mathbb{R}^n)$ .

Consider a linear operator  $T : \mathcal{D}_1 \rightarrow \mathcal{D}_2$ , where  $\mathcal{D}_2 \subset L^q(\mathbb{R}^n)$ .

**Definition** *Operator  $T$  is called an operator of type  $(p, q)$  if there exists a real number  $c$  such that for any  $f \in \mathcal{D}_1$  it holds the following inequality  $\|Tf\|_q \leq c \|f\|_p$ , where  $\|\cdot\|_p$  is a natural norm of the space  $L_p(\mathbb{R}^n)$  (see [3]).*

Note that if  $\mathcal{D}_1$  is a dense set on  $L^p(\mathbb{R}^n)$  then the operator  $T$  has a bounded extension from  $L^p(\mathbb{R}^n)$  to  $L^q(\mathbb{R}^n)$ . Let  $T$  be an integral operator with the kernel function  $K(x, y)$  and operator  $T$  is defined from  $C_0^\infty(\mathbb{R}^n)$  to  $L^2_{loc}(\mathbb{R}^n)$ . It is well known the statements about boundedness of integral operators. In the statements of type Shur Lemma the  $L^2$ -norm of operators is estimated by both  $\sup_y \int |K(x, y)| dx$  and  $\sup_x \int |K(x, y)| dy$ . i.e. by two integrals of kernel functions (see [1]).

We consider the integral operator of type

$$(Tf)(y) = \int_{R^n} \frac{K(x, y)f(x)dx}{|y|^n}.$$

\*

Denote by  $\sigma_A$  the set of  $R^n \times R^n$ :

$$\sigma_A \equiv \{(x, y) : x \in R^n, y \in R^n : |y| \leq A|X|\},$$

where  $A$  is a fixed positive.

Our main result consists of the following Theorem.

**Theorem 1.1.** *If  $\text{supp } K \subset \sigma_A$ , and there exists a number  $p > 2$  such that*

$$\int_{R^n} |K(\frac{y|x|}{|y|^2}, x)|^p dy = \rho(x) \in L^\infty(R^n),$$

*then the integral operator (1.1) has a bounded extension on  $L^p(R^n)$ .*

Now, consider also one variant of the statement for one-dimensional case. Let  $K(\xi, x)$  be a measurable function. Assume that  $\text{supp}K(\cdot, x) \subset [a, b]$  and consider integral operator

$$(Tf)(x) = \int_R \frac{K(\frac{x}{y}, x)}{y} f(y) dy \tag{0.1}$$

where  $f \in C_0^\infty(R \setminus \{0\})$ .

It holds the following theorem.

**Theorem 1.2.** *Let there exists a number  $p > 2$  such that*

$$\int_{R^n} |K(\xi, x)|^p d\xi = c(x) \in L^\infty(R),$$

*then the integral operator (1.2) has a bounded extension on  $L^2(R)$ .*

---

\* 1991 mathematics Subject Classification: 44A15, 47B37

**2. Shur Lemma on the Boundedness of Integral Operators**

Let  $T_B$  be an integral operator with the kernel function  $B(x, y)$  and operator  $T_B$  is defined from  $C_0^\infty(R^n)$  to  $L_{loc}^2(R^n)$ :

$$(T_B f)(x) = \int B(x, y) f(y) dy.$$

It is well known the Shur Lemma and its generalizations on the boundedness of the operators [1,3]. The following Lemma was proved in (see [2], p.47).

**Lemma 2.1.** *Let there exists non-negative numbers  $\varepsilon, c$  and a measurable function  $M(x, y)$  such that the following inequalities*

$$\begin{aligned} |B(x, y)| &\leq M(x, y), \\ \int_{R^n} M(x, y) |x|^{-\varepsilon} dx &\leq c |y|^{-\varepsilon}, \\ \int_{R^n} M(x, y) |y|^{-\varepsilon} dy &\leq c |x|^{-\varepsilon} \end{aligned}$$

*hold. Then the integral operator  $T_B$  has a bounded extension on  $L^2(R^n)$  and its norm is estimated by  $c$ .*

Firstly we consider the application of Lemma 2.1 to the special integral operator.

Let  $\rho \in L^\infty(R^n \times R^n)$  be a fixed function and  $\mathcal{D} = C_0^\infty(R^n \setminus \{0\})$ . Consider integral operator  $T : \mathcal{D} \rightarrow L_{loc}^2(R^n)$  given by

$$Tf(y) = \int_{R^n} \frac{\rho(x, y) f(x) dx}{|x|^\alpha |y|^\beta}, \tag{0.2}$$

where  $\alpha, \beta$  are fixed positive numbers and  $\alpha + \beta = n$ .

The following lemma is needed for the sequel.

**Lemma 2.2.** *Let  $A$  be a fixed positive number. If  $\text{supp } \rho \subset \sigma_A$  and  $\alpha < \beta$  then the operator  $T$  defined by the formula (2.1) has a type (2.2).*

**Proof.** Without loss of generality we can assume that  $A = 1$ . Let  $\alpha < \varepsilon < \beta$  be a fixed positive number. Then it is easy to show that the following inequalities:

$$\int_{R^n} \frac{|\rho(x, y)dx|}{|x|^\varepsilon|x|^\alpha|x|^\beta} \leq \frac{H}{|y|^\varepsilon},$$

$$\int_{R^n} \frac{|\rho(x, y)dy|}{|x|^\alpha|y|^\beta|y|^\varepsilon} \leq \frac{H}{|x|^\varepsilon}$$

hold, where  $H$  is a some constant.

Consequently, by the Lemma 2.1 the integral operator  $T$  has a bounded extension on  $L^2(R^n)$ , therefore, it has a type (2,2) and the proof is completed  $\square$

### 3. Proof of Theorems

In this section we will prove the Theorem 1.1 and Theorem 1.2.

**Proof of Theorem 1.1** We show that the integral operator (1.1) has a type (2.2). Without loss of generality we can assume that  $A = 1$ . Let  $f \in C_0^\infty(R^n \setminus \{0\})$  be a fixed function. Then we have

$$|Tf(y)|^2 = \int_{R^n} \int_{R^n} \frac{K(y, x_1)\overline{K(y, x_2)}}{|y|^{2n}} f(x_1)\overline{f(x_2)} dx_1 dx_2.$$

We will represent the integral  $|Tf(y)|^2$  as a sum of two integrals.

$$|Tf(y)|^2 = \int \int_{|x_1| \geq |x_2|} \frac{K(y, x_1)\overline{K(y, x_2)}}{|y|^{2n}} f(x_1)\overline{f(x_2)} dx_1 dx_2$$

$$+ \int \int_{|x_1| \leq |x_2|} \frac{K(y, x_1)\overline{K(y, x_2)}}{|y|^{2n}} f(x_1)\overline{f(x_2)} dx_1 dx_2$$

We will prove that the following inequality

$$\int_{R^n} \int \int_{|x_1| \geq |x_2|} \frac{|K(y, x_1)\overline{K(y, x_2)}|}{|y|^{2n}} |f(x_1)\overline{f(x_2)}| dx_1 dx_2 dy \leq L \|f\|^2 \quad (0.3)$$

holds, where  $L$  is a positive number. Let  $q$  be an adjoint number to  $p$ , i.e.,  $\frac{1}{p} + \frac{1}{q} = 1$ . Then by using the Hölder's inequality we obtain

$$\begin{aligned}
 & \int_{R^n} dy \int \int_{|x_1| \geq |x_2|} \frac{|K(y, x_1) \overline{K(y, x_2)}|}{|y|^{2n}} |f(x_1)f(x_2)| dx_1 dx_2 \\
 = & \int \int_{|x_1| \geq |x_2|} |f(x_1)f(x_2)| dx_1 dx_2 \int_{R^n} \frac{|K(y, x_1) \overline{K(y, x_2)}|}{|y|^{2n}} dy \leq \\
 \leq & \int \int_{|x_1| \geq |x_2|} |f(x_1)f(x_2)| dx_1 dx_2 \\
 \times & \left( \int_{R^n} \frac{|K(y, x_2)|^p dy}{|y|^{2n}} \right)^{\frac{1}{p}} \left( \int_{R^n} \frac{|K(y, x_1)|^q dy}{|y|^{2n}} \right)^{\frac{1}{q}}
 \end{aligned}$$

Changing of variables by  $y = \frac{z|x_2|}{|z|^2}$  we get for the jacobian the relation

$$J(x_2, z) = \frac{|x_2|^n}{|z|^{2n}} a(x_2, z),$$

where  $a(x_2, z)$  is some uniformly bounded function on  $R^n \times R^n$ .

It is clear that if  $(y, x_2) \in \sigma$ , then  $|z| = \frac{|x_2|}{|y|} \leq 1$ . Consequently, we obtain

$$\begin{aligned}
 \int_{R^n} \frac{|K(y, x_2)|^p dy}{|y|^{2n}} &= \int_{|z| \leq 1} \frac{|K(\frac{z|x_2|}{|z|^2}, x_2)|^p}{|x_2|^n} |a(x_2, z)| dz \\
 &\leq \frac{\|a\|_\infty}{|x_2|^n} \int_{|z| \leq 1} \left| K\left(\frac{z|x_2|}{|z|^2}, x_2\right) \right|^p dz.
 \end{aligned}$$

Hence

$$\int_{R^n} \frac{|K(y, x_2)|^p dy}{|y|^{2n}} \leq \frac{\rho_1(x_2)}{|x_2|^n}$$

where  $\rho_1 \in L^\infty(R^n)$ .

At the same time by using Hölder's inequality we get

$$\begin{aligned}
 \int_{R^n} \frac{|K(y, x_1)|^q dy}{|y|^{2n}} &= \int_{|z| \leq 1} \frac{|K(\frac{z|x_1|}{|z|^2}, x_1)|^q}{|x_1|^n} |a(x_1, z)| dz \\
 &\leq \frac{b_n \|a\|_\infty}{|x_1|^n} \left( \int_{|z| \leq 1} \left| K\left(\frac{z|x_1|}{|z|^2}, x_1\right) \right|^p dz \right)^{\frac{q}{p}},
 \end{aligned}$$

where  $b_n$  is some constant. Thus we have

$$\int_{R^n} \frac{|K(y, x_1)|^q}{|y|^{2n}} dy \leq \frac{\rho_2(x_1)}{|x_1|^n}$$

where  $\rho_2 \in L^\infty(R^n)$ .

Let  $\chi(x_1, x_2)$  be an indicator function of the set

$$\sigma_1 = \{(x_1, x_2) \in R^n \times R^n : |x_1| \geq |x_2|\}.$$

Consider an integral operator

$$T_1 f(x_1) = \int_{R^n} \frac{\chi(x_1, x_2) \rho_1(x_2) \rho_2(x_1)}{|x_1|^{\frac{n}{q}} |x_2|^{\frac{n}{p}}} f(x_2) dx_2.$$

By the Lemma 2.2 the integral operator  $T_1$  has a type (2,2) and consequently has a bounded extension on  $L^2(R^n)$ . This implies that for any  $f \in L^2(R^n)$  it holds the following inequality

$$\| T_1 f \| \leq M \| f \|, \tag{0.4}$$

where  $M$  is a constant and norm  $\| \cdot \|$  is  $L^2(R^n)$ .

It is not hard to prove that the following inequality

$$\int_{R^n} dy \int_{|x_1| \geq |x_2|} \frac{|K(y, x_1) \overline{K(y, x_2)}|}{|y|^{2n}} |f(x_1) f(x_2)| dx_1 dx_2 \leq \int_{R^n} |f(x_1)| |T_1 f(x_1)| dx_1$$

holds.

Therefore we get (3.1) by using (3.2) and Schwartz inequality. The case  $|x_2| \geq |x_1|$  may be considered by the analogy. This completes the proof of Theorem 1.2.

**Corollary 3.1** *Let  $K(y, x)$  be a homogeneous function of degree zero with respect to  $y$  and  $\text{supp}K(y, x) \subset \sigma_a$ . If the following relation*

$$\int_{|y| \leq 1} |K(y, x)|^p ds(y) \in L^\infty(R^n)$$

holds for some  $p > 2$ , where  $ds(y)$  is the Lebesgue measure on  $S^{n-1}$ , then the integral operator (1.1) has a bounded extension on  $L^2(\mathbf{R}^n)$ .

The Corollary 3.1 is proved by the analogy of Theorem 1.1.

**Proof of Theorem 1.2** The noting by  $T^*$  adwoint operator of  $T$  we see

$$(T^*f)(y) = \int_R \frac{K(\frac{x}{y}, x)}{y} f(x) dx.$$

We show that  $T^*$  has a type (2,2). Let  $f \in C_0^\infty(\mathbf{R} \setminus \{0\})$  be a fixed function.

Then we have

$$\begin{aligned} |T^*f(y)|^2 &= \left| \int \int_{|x_1| \geq |x_2|} \frac{K(\frac{x_1}{y}, x_1), \overline{K(\frac{x_2}{y}, x_2)} f(x_1) \overline{f(x_2)}}{y^2} dx_1 dx_2 + \right. \\ &\quad \left. + \int \int_{|x_1| \leq |x_2|} \frac{K(\frac{x_1}{y}, x_1), \overline{K(\frac{x_2}{y}, x_2)} f(x_1) \overline{f(x_2)}}{y^2} dx_1 dx_2 \right|. \end{aligned}$$

Now we prove the analogy of inequality (3.1). By using the Hölder's inequality we have

$$\begin{aligned} &\int dy \left| \int \int_{|x_1| \geq |x_2|} \frac{K(\frac{x_1}{y}, x_1), \overline{K(\frac{x_2}{y}, x_2)} f(x_1) \overline{f(x_2)}}{y^2} dx_1 dx_2 \right| \leq \\ &\leq \int \int_{|x_1| \geq |x_2|} |f(x_1) f(x_2)| \left( \int_R \frac{|K(\frac{x_1}{y_1}, x_1)|^p}{y_1^2} dy_1 \right)^{\frac{1}{p}} \\ &\quad \left( \int_R \frac{|K(\frac{x_2}{y_2}, x_2)|^q}{y_2^2} dy_2 \right)^{\frac{1}{q}} dx_1 dx_2. \end{aligned}$$

Let us use a change of variables  $\frac{x_1}{y_1} = \zeta$  we have

$$\int_R \frac{|K(\frac{x_1}{y_1}, x_1)|^p}{y_1^2} dy_1 = \int_a^b \frac{|K(\zeta, x_1)|^p}{|x_1|} d\zeta \leq \frac{c_1(x_1)}{|x_1|}$$

where  $\rho_1 \in L^\infty(\mathbf{R})$ .

By the analogy we get



EKİNCİOĞLU, IKROMOV

$$\int_{\mathbf{R}} \frac{|K(\frac{x_2}{y_2}, x_2)|^q}{y_2^2} dy_2 \leq \frac{\rho_2(x_2)}{|x_2|},$$

where  $\rho_2 \in L^\infty(\mathbf{R})$  by the condition of the theorem, since  $q < 2$ . Now by using Lemma 2.1 as above we arrive to proof of Theorem 1.2.

### References

- [1] L. Hörmander, The Analysis of Linear Partial Differential Operators III, Springer-Verlag, (1985).
- [2] D. H. Phong, E. M. Stein, Oscillatory integrals with polynomial phases *Inventiones mathematicae*, **110**, 39-62, (1992).
- [3] E. M. Stein, Harmonic Analysis Real-Variable Methods, Orthogonality and Oscillatory Integrals, Princeton Uni.Press, Princeton, New Jersey, 1993.

İ. EKİNCİOĞLU

Received 02.03.1998

Department of Mathematics

Kocatepe University 03200 Afyon-TURKEY

İ. A. IKROMOV

Samarkand Division of Uzbek,

Academy of Sciences

Temur Malik Street No.3

703000, Samarkand, UZBEKISTAN