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SOME THEOREMS INVOLVING INEQUALITIES ON P-VALENT FUNCTIONS

Hüseyin Irmak, Ömer Faruk Çetin

Abstract

In this paper, some theorems involving inequalities on p -valent functions (that is p -valently close-to-convex functions, p -valently starlike functions, and p -valently convex functions) are given. Moreover, some applications in the theorems which are important for geometric function theory are also included.

Keywords: Analytic, p -valent, p -valently close-to-convex functions, p -valently starlike functions, p -valently convex functions, open unit disk, and Jack's Lemma.

1. Introduction and Definitions

Let $T(p)$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad (p \in N = \{1, 2, 3, \dots\}), \quad (1)$$

which are analytic and p -valent in the open unit disk $U = \{z : z \in \mathcal{C} \text{ and } |z| < 1\}$. A function $f(z) \in T(p)$ is said to be in the subclass $TK(p)$ of p -valently close-to-convex functions with respect to the origin in U if it satisfies the inequality (cf.[1-3]):

$$\operatorname{Re} \left\{ \frac{f'(z)}{z^{p-1}} \right\} > 0, \quad (z \in U; p \in N). \quad (2)$$

On the other hand, a function $f(z) \in T(p)$ is said to be in the subclass $TS(p)$ of p -valently starlike functions with respect to the origin in U if it satisfies the inequality

(cf.[1-3]):

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0, \quad (z \in U; p \in N). \quad (3)$$

Furthermore, a function $f(z) \in T(p)$ is said to be in the subclass $TC(p)$ of p -valently convex functions with respect to the origin in U if it satisfies the inequality (cf.[1-3]):

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0, \quad (z \in U; p \in N). \quad (4)$$

A function $f(z) \in T(p)$ is said to be in the subclass $V_q(p)$ if it satisfies the inequality:

$$\left| \frac{(p-q)!}{p!} \cdot \frac{D_z^q f(z)}{z^{p-q}} - 1 \right| < 1, \quad (5)$$

$$(z \in U; p > q; p \in N; q \in N_0 = N \cup \{0\}),$$

and, a function $f(z) \in T(p)$ is said to be in the subclass $W_q(p)$ if it satisfies the inequality:

$$\left| \frac{zD_z^{q+1}f(z)}{D_z^q f(z)} - (p-q) \right| < p-q, \quad (6)$$

$$(z \in U; p > q; p \in N; q \in N_0).$$

Here and throughout this paper, D_z^q denotes the q th-order ordinary differential operator. For a function $f(z) \in T(p)$,

$$D_z^q f(z) = \frac{p!}{(p-q)!} z^{p-q} + \sum_{k=p+1}^{\infty} \frac{k!}{(k-q)!} a_k z^{k-q}, \quad (p > q; p \in N; q \in N_0). \quad (7)$$

To establish our results, we need the following Lemma given by Jack [4] (also, by Miller and Mocanu [5]).

Lemma Let $w(z)$ be analytic in U with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point z_0 , then

$$z_0 w'(z_0) = c w(z_0), \tag{8}$$

where c is a real number and $c \geq 1$.

2. Some Theorems Involving Inequalities on P-valent Functions

We first prove the following theorem.

Theorem 1. If $f(z) \in T(p)$ satisfies the inequality

$$\operatorname{Re} \left\{ q + \frac{z D_z^{q+1} f(z)}{D_z^q f(z)} - p \right\} < \frac{1}{2}, \quad (z \in U; p > q; p \in N; q \in N_0), \tag{9}$$

then $f(z) \in V_q(p)$.

Proof. Let the function $f(z) \in T(p)$. Then defining the function $w(z)$ by

$$\frac{(p-q)!}{p!} \cdot \frac{D_z^q f(z)}{z^{p-q}} = 1 + w(z), \quad (z \in U; p > q; p \in N; q \in N_0), \tag{10}$$

we have that $w(z)$ is analytic in U and $w(0) = 0$. It follows from (10) that

$$\frac{(p-q)!}{p!} \cdot \frac{z D_z^{q+1} f(z)}{z^{p-q}} = z w'(z) + (p-q)[1 + w(z)]. \tag{11}$$

Then, we have from (10) and (11) that

$$F(z) = q + \frac{z D_z^{q+1} f(z)}{D_z^q f(z)} - p = \frac{z w'(z)}{1 + w(z)}. \tag{12}$$

Now, suppose that there exists a point $z_0 \in U$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1. \tag{13}$$

Then, using the Lemma and letting $w(z_0) = e^{i\theta} (w(z_0) \neq -1)$ in the equation (12), we have

$$\begin{aligned} \operatorname{Re}\{F(z_0)\} &= \operatorname{Re}\left\{\frac{z_0 w'(z_0)}{1+w(z_0)}\right\} = \operatorname{Re}\left\{\frac{c w(z_0)}{1+w(z_0)}\right\} \\ &= c \operatorname{Re}\left\{\frac{e^{i\theta}}{1+e^{i\theta}}\right\} = \frac{c}{2} \geq \frac{1}{2}, \end{aligned} \tag{14}$$

which contradicts the hypothesis (9). Therefore, we conclude that $|w(z)| < 1$ for all $z \in U$, and the definition (10) immediately yields the inequality

$$\left| \frac{(p-q)!}{p!} \cdot \frac{D_z^q f(z)}{z^{p-q}} - 1 \right| < 1. \quad (z \in U; p > q; p \in N; q \in N_0), \tag{15}$$

that is, that $f(z) \in V_q(p)$. Thus, the proof is completed.

Next we prove the following theorem. □

Theorem 2. *If $f(z) \in T(p)$ satisfies the inequality*

$$\operatorname{Re}\left\{1 + z \left(\frac{D_z^{q+2} f(z)}{D_z^{q+1} f(z)} - \frac{D_z^{q+1} f(z)}{D_z^q f(z)} \right)\right\} < \frac{1}{2}, \quad (z \in U; p > q; p \in N; q \in N_0), \tag{16}$$

then $f(z) \in W_q(p)$.

Proof. Let the function $f(z) \in T(p)$. Now consider the function $w(z)$ defined by

$$\frac{z D_z^{q+1} f(z)}{D_z^q f(z)} = (p-q)[1+w(z)], \quad (z \in U; p > q; p \in N; q \in N_0). \tag{17}$$

Clearly the function $w(z)$ is analytic in U and $w(0) = 0$. It follows from the definition of $w(z)$ that

$$\begin{aligned} 1 + \frac{z D_z^{q+2} f(z)}{D_z^{q+1} f(z)} &= (p-q) \left(1 + w(z) + z w'(z) \frac{D_z^q f(z)}{D_z^{q+1} f(z)} \right) \\ &= (p-q)[1+w(z)] + \frac{z w'(z)}{1+w(z)}. \end{aligned} \tag{18}$$

Now suppose that there exists a point $z_0 \in U$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1. \tag{19}$$

Then the Lemma gives $w(z_0) = e^{i\theta}$ and $z_0 w'(z_0) = cw(z_0) (c \geq 1)$. Therefore, we obtain from (17)-(20) that

$$\begin{aligned} \operatorname{Re} \left\{ 1 + z \left(\frac{D_z^{q+2} f(z)}{D_z^{q+1} f(z)} - \frac{D_z^{q+1} f(z)}{D_z^q f(z)} \right) \Big|_{z=z_0} \right\} &= \operatorname{Re} \left\{ \frac{z_0 w'(z_0)}{1 + w(z_0)} \right\} \\ &= \operatorname{Re} \left\{ \frac{cw(z_0)}{1 + w(z_0)} \right\} = c \operatorname{Re} \left\{ \frac{e^{i\theta}}{1 + e^{i\theta}} \right\} = \frac{c}{2} \geq \frac{1}{2}, \end{aligned} \tag{20}$$

which contradicts the condition (16). Hence, we conclude that $|w(z)| < 1$ for all $z \in U$, that is, that $f(z) \in W_q(p)$. This completes the proof of Theorem 2.

By taking $q = 0$ in Theorems 1 and 2, we have the following corollaries. □

Corollary 1. *If $f(z) \in T(p)$ satisfies the inequality*

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} - p \right\} < \frac{1}{2}, \quad (z \in U; p \in N), \tag{21}$$

then

$$\left| \frac{f(z)}{z^p} - 1 \right| < 1, \quad (z \in U; p \in N). \tag{22}$$

Corollary 2. *If $f(z) \in T(p)$ satisfies the inequality*

$$\operatorname{Re} \left\{ 1 + z \left(\frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)} \right) \right\} < \frac{1}{2}, \quad (z \in U; p \in N), \tag{23}$$

then $f(z) \in TS(p)$ and

$$\left| \frac{zf'(z)}{f(z)} - p \right| < p, \quad (z \in U; p \in N). \tag{24}$$

By taking $q = 1$ in Theorems 1 and 2, we have the following.

Corollary 3. If $f(z) \in T(p)$ satisfies the inequality

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} - p \right\} < \frac{1}{2}, \quad (z \in U; p \in N), \quad (25)$$

then $f(z) \in TK(p)$ and

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| < p, \quad (z \in U; p \in N). \quad (26)$$

Corollary 4. If $f(z) \in T(p)$ satisfies the inequality

$$\operatorname{Re} \left\{ 1 + z \left(\frac{f'''(z)}{f''(z)} - \frac{f''(z)}{f'(z)} \right) \right\} < \frac{1}{2}, \quad (z \in U; p \in N), \quad (27)$$

then $f(z) \in TC(p)$ and

$$\left| 1 + \frac{zf''(z)}{f'(z)} - p \right| < p - 1, \quad (z \in U; p \in N - \{1\}). \quad (28)$$

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