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A CLASS OF BANACH ALGEBRAS WHOSE DUALS HAVE THE SCHUR PROPERTY

H. Mustafayev, A. Ülger

Abstract

Call a commutative Banach algebra A a γ -algebra if it contains a bounded group Γ such that $\overline{aco(\Gamma)}$ contains a multiple of the unit ball of A . In this paper, first by exhibiting several concrete examples, we show that the class of γ -algebras is quite rich. Then, for a γ -algebra A , we prove that A^* has the Schur property iff the Gelfand spectrum Σ of A is scattered iff $A^* = ap(A)$ iff $A^* = \overline{Span(\Sigma)}$.

Key words and phrases: Schur property, Segal algebras, almost periodic functionals.

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1. Introduction

Let A be a commutative Banach algebra. We call A , for want of a better name, a γ -algebra if A contains a bounded group Γ (for multiplication) such that $\overline{aco(\Gamma)} \supseteq cA_1$, for some constant $c > 0$. Here $\overline{aco(\Gamma)}$ is the absolute convex hull of Γ and A_1 is the closed unit ball of A . The example that motivated us to introduce this class of Banach algebras is the algebra $A = C(K)$ of the continuous complex valued functions on a compact (Hausdorff) space K . For this algebra, the set $\Gamma = \{e^{ig} : g \in C_R(K)\}$ is a bounded group for the multiplication and, by Russo-Day-Palmer Theorem [B – D; p.208], $\overline{aco(\Gamma)} = A_1$. In section 2 we give several examples of γ -algebras and study stability properties of these algebras under some Banach algebras construction procedures. In particular we show

that the quotient algebras of Segal algebras $[Re]$ by closed ideals with compact hulls are γ -algebras, and that the projective tensor product $A \hat{\otimes} B$ of two γ -algebras A and B is again a γ -algebra. In section 3 we study structural and geometric properties of γ -algebras. The main results are the following. We first prove that, in spite of the diversity of the examples, the only γ -algebras are the quotient algebras of the group algebra $\ell^1(\Gamma)$ for a discrete abelian group Γ . This result reduces the study of the γ -algebras to that of the quotient algebras of the discrete group algebra $\ell^1(\Gamma)$. The second main result establishes the following equivalences for a γ -algebra A with the spectrum Σ : The space A^* has the Schur property (i.e. weakly convergent sequences in A^* are norm convergent) iff $A^* = ap(A)$ iff $\overline{Span(\Sigma)} = A^*$ iff Σ is scattered (i.e. it has no nonempty perfect subset). These results generalize and unify several known results or parts of them e.g. [P-S; Main Theorem], [D-Ü; Proposition 3.5], [L-P1; Theorem 1]. The main ingredients of the proofs are a result of Loomis [Lo; Theorem 4.] stating that a continuous function on an locally compact abelian group with scattered spectrum is almost periodic, and Theorem 2.14 of Lust-Piquard [L-P 2].

Notation and Preliminaries.

Our notation and terminologies are quite standard. If X is a Banach space, we denote by X^* its dual and by X_1 its closed unit ball. For x in X and f in X^* , we denote by $\langle x, f \rangle$ the natural duality between X and X^* . We always consider X as naturally embedded into its second dual X^{**} . For any subset E of X^* , we denote by $\overline{Span(E)}$ and $\overline{Span(E)}^*$, respectively, the norm and weak* closures of $Span(E)$ in X^* .

Now let A be a Banach algebra. For f in A^* and a in A , we denote by $f.a$ the functional defined on A by $\langle f.a, x \rangle = \langle f, ax \rangle$. The functional f is said to be almost periodic on A if the set $H(f) = \{f.a : a \in A_1\}$ is relatively compact in A^* . By $ap(A)$ we denote the space of the almost periodic functionals on A . This is a closed subspace of A^* . Now let Γ be a discrete abelian group, G be its dual group, and $\ell^1(\Gamma)$ be its group algebra. For a in $\ell^1(\Gamma)$, we denote by \hat{a} the Fourier transform of a . We consider each element f of G as a multiplicative linear functional on $\ell^1(\Gamma)$ acting through the formula $\langle a, f \rangle = \hat{a}(f)$. For a subset E of G , we denote by $\overline{Span(E)}$ the norm closed subspace of $\ell^\infty(\Gamma)$ generated by E . By $AP(\Gamma)$ we denote the space of the almost periodic functions

on Γ . As is well known, $AP(\Gamma) = \overline{Span(G)} = ap(\ell^1(\Gamma))$. For φ in $\ell^1(\Gamma)$, the spectrum of φ is denoted by $\sigma(\varphi)$. This is the set

$$\sigma(\varphi) = G \cap \overline{\{\varphi.a : a \in \ell^1(\Gamma)\}^*}. \quad [K; p.170]$$

For a closed subset E of G , by $k(E)$ we denote the kernel of E . This is largest closed ideal of $\ell^1(\Gamma)$ whose hull is E . Finally, by $k(E)^\perp$, we denote the annihilator of $k(E)$ in $\ell^\infty(\Gamma)$.

Examples of γ -Algebras.

As introduced in the preceding section, by a γ -algebra we mean a commutative Banach algebra A that contains a bounded group Γ such that $\overline{aco(\Gamma)} \supseteq cA_1$, for some constant $c > 0$. Such an algebra necessarily has a unit element. Beside the commutative C^* -algebras, several well known Banach algebras are γ algebras. In this section we give several examples of γ -algebras and study some stability properties of them.

a) Group Algebra. Let G be a locally compact abelian group and $L^1(G)$ be its group algebra.

Proposition 2.1 *The algebra $L^1(G)$ is a γ -algebra iff the group G is discrete.*

Proof. Since a γ -algebra is unital, if $L^1(G)$ is a γ -algebra then G is discrete. Conversely, if G is discrete then the Dirac measures $\delta_g (g \in G)$ are in $L^1(G)$ and $\Gamma = \{\delta_g : g \in G\}$ is a bounded group in $L^1(G)$ ($\delta_f \star \delta_g = \delta_{f+g}$), and the closed absolute convex hull of Γ is the closed unit ball of $\ell^1(G) = L^1(G)$. □

As an immediate corollary of this proposition we have following result.

Corollary 2.2. *The Fourier algebra $A(G)$ is a γ -algebra iff the group G is compact.*

b) Segal Algebra. Let G be a locally compact abelian group. We recall that a Segal algebra $S(G)$ on G is a Banach algebra such that

i) $S(G)$ is a translation invariant dense subalgebra of $L^1(G)$,

ii) For $a \in S(G)$ and $g \in G$, $\|a_g\| = \|a\|$, where a_g is defined by $a_g(f) = a(f+g)$, and $\|\cdot\|$ is the norm of $S(G)$,

iii) For each $a \in S(G)$, the mapping $g \mapsto a_g$ is continuous from G into $S(G)$ at the unit element of G .

About Segal algebras, ample information can be found in Reiter's book [Re]. We now give some concrete examples of Segal algebras.

1) The algebra $S(G) = L^1(G) \cap L^p(G)$ ($1 \leq p < \infty$), equipped with the norm $\|a\| = \|a\|_1 + \|a\|_p$, is a Segal algebra.

2) The algebra $S(G) = L^1(G) \cap C_o(G)$, equipped with the norm $\|a\| = \|a\|_1 + \|a\|_\infty$, is a Segal algebra.

A Segal algebra $S(G)$ is a commutative semisimple regular Banach algebra whose Gelfand spectrum is \hat{G} , the dual group of G . For $a \in S(G)$, the Gelfand transform of a is just the Fourier transform of $a \in L^1(G)$. From these facts it follows that, for a closed subset K of \hat{G} , $I_K = \{a \in S(G) : \hat{a}(K) = 0\}$ is the largest, and the set $J_K = \overline{\{a \in S(G) : \text{Supp}(\hat{a}) \cap K = \emptyset\}}$ is the smallest, closed ideals of $S(G)$ whose hull is K . We shall also need the following two facts about Segal algebras [Re; p.128]:

(\star) There is a constant $c > 0$ such that, for $a \in S(G)$, $\|a\|_1 \leq c \|a\|$.

($\star\star$) $S(G)$ is an ideal in $L^1(G)$ and, for $a \in S(G)$ and $b \in L^1(G)$, $\|a \star b\| \leq \|a\| \cdot \|b\|_1$.

Proposition 2.3. *Let I be a closed ideal of a Segal algebra $S(G)$ with compact hull. Then the quotient algebra $S(G)/I$ is a γ -algebra.*

Proof. Let K be the hull of I . Choose a function $u \in S(G)$ such that $\hat{u} = 1$ on a neighborhood of K . Then, the element $u + I$ is the unit element of the algebra $A = S(G)/I$. Put $\Gamma = \{u_g + I : g \in G\}$. As $u_f \star u_g = u_{f+g}$ and $\|u_g + I\| = \|u + I\| \leq \|u\|$, Γ is a bounded group in A . To prove that $\overline{\text{aco}(\Gamma)}$ contains a multiple of A_1 , it is enough to show that, for $\varphi \in I^\perp$, $\|\varphi\| \leq c \text{Sup}_{g \in G} |\langle \varphi, u_g \rangle|$, for some positive constant c . To this end, observe that, since for each $a \in S(G)$, $a \star u - a \in J_K \subseteq I$, we have $\langle \varphi, a \rangle = \langle \varphi, u \star a \rangle$. From this equality, by ($\star\star$), we get that $\varphi = \varphi \star u$ and that

$$|\langle \varphi, a \rangle| \leq \|\varphi\| \cdot \|u \star a\| \leq \|\varphi\| \cdot \|u\| \cdot \|a\|_1.$$

This inequality shows that φ is bounded on $S(G)$ for the norm of $L^1(G)$. Hence, $S(G)$ being dense in $L^1(G)$, φ can be extended in a unique way to $L^1(G)$. We denote by $\bar{\varphi}$

this unique extension of φ . Since, by (\star) , $l^\infty(G)$ embeds continuously into $S(G)^\star$, there is a constant $c > 0$ such that $\|\varphi\| \leq c \|\bar{\varphi}\|_\infty$. Now, since $\varphi = \varphi \star u$, φ is a continuous function on G so that

$$\|\bar{\varphi}\|_\infty = \|\varphi\|_\infty = \text{Sup}_{g \in G} |(\varphi \star u)(-g)| = \text{Sup}_{g \in G} |\langle \varphi, u_g \rangle|$$

From this we conclude that $\|\varphi\| \leq c \text{Sup}_{g \in G} |\langle \varphi, u_g \rangle|$, and so $\overline{aco(\Gamma)} \supseteq cA_1$. □

This proposition has several immediate corollaries.

Corollary 2.4. *Let I be a closed ideal of $L^1(G)$ with compact hull. Then the algebra $L^1(G)/I$ is a γ -algebra.* □

Corollary 2.5. *Let $A(G)$ be the Fourier algebra of G , K be a compact subset of G , and I_K be the largest closed ideal of $A(G)$ whose hull is K . Then the algebra $A(K) = A(G)/I_K$ is a γ -algebra.* □

c) Projective tensor products and quotients of γ -algebras.

If A and B are two commutative Banach algebras then their projective tensor product $A \hat{\otimes} B$ is a commutative Banach algebra for the multiplication which is the linear extension of the multiplication $a \otimes b \cdot c \otimes d = ac \otimes bd$ on the simple tensors [B-D; Chapter VI].

Proposition 2.6 The projective tensor product $A \hat{\otimes} B$ of two γ -algebras A and B is also a γ -algebra.

Proof. Let $\Gamma_1 \subseteq A$ ($\Gamma_2 \subseteq B$) be a bounded group such that $\overline{aco(\Gamma_1)} \supseteq c_1 A_1$ ($\overline{aco(\Gamma_2)} \supseteq c_2 B_1$) for some constant $c_1 > 0$ ($c_2 > 0$). Then $\Gamma = \{a \otimes b : a \in \Gamma_1, b \in \Gamma_2\}$ is a bounded group in $A \hat{\otimes} B$, and by the very definition of the projective tensor norm, $\overline{aco(\Gamma)}$ contains $c(A \hat{\otimes} B)_1$, where $c = \min\{c_1, c_2\}$. □

Proposition 2.7. Let A be a γ -algebra and B be an arbitrary Banach algebra. If there is a continuous onto homomorphism $h : A \rightarrow B$, then the algebra B is also a γ -algebra.

Proof. Let Γ_1 be a bounded group in A such that $\overline{aco(\Gamma_1)} \supseteq c_1 A_1$, for some constant $c_1 > 0$. As h is onto, by the Open Mapping Theorem, there is a constant $c_2 > 0$ such that $h(A_1) \supseteq c_2 B_1$. Let $\Gamma_2 = h(\Gamma_1)$. Then Γ_2 is a bounded group in B and $\overline{aco(\Gamma_2)} \supseteq h(\overline{aco(\Gamma_1)}) \supseteq c_1 h(A_1) \supseteq c_1 c_2 B_1$. This proves that B is also a γ -algebra. \square

3. Structure of γ -Algebras.

The above results show that the class of γ -algebras is quite rich and the algebras constituting it are quite various. The first main result of this chapter shows that, in spite of the diversity of γ -algebras, γ -algebras are exactly the quotient algebras of the discrete group algebra $\ell^1(\Gamma)$ for some discrete group Γ .

Theorem 3.1. *A commutative Banach algebra A is a γ -algebra iff it is isomorphic to a quotient algebra of the group algebra $\ell^1(\Gamma)$ for some discrete group Γ .*

Proof. Let Γ be a bounded group in A such that $\overline{aco(\Gamma)} \supseteq c A_1$, for some constant $c > 0$. Consider the mapping $\omega : \ell^1(\Gamma) \rightarrow A$, defined by $\omega(a) = \sum_{\gamma \in \Gamma} a(\gamma) \gamma^{-1}$. This is not really the Fourier transform but exactly as in the case of the Fourier transform one can easily see that ω is an algebra homomorphism. The group Γ being bounded, ω is bounded. Moreover, for $a = \delta_{\gamma^{-1}}$ (Dirac measure at γ^{-1}), $\omega(a) = \gamma$ so that $\omega(\ell^1(\Gamma)_1) \supseteq aco(\Gamma)$. It follows that $\overline{\omega(\ell^1(\Gamma)_1)} \supseteq c A_1$. From this inclusion, as in the most standard proof of the open mapping theorem (see e.g. [H-S; p.215]), we deduce that ω is onto. It follows that A is isomorphic to a quotient algebra of the algebra $\ell^1(\Gamma)$. For the reverse implication it is enough to apply Propositions 2.1 and 2.7. \square

For the proof of the second main result of this section we need the following three lemmas

Lemma 3.2. *Let Γ be a discrete abelian group and E be a closed subset of the dual group G of Γ . Then the following assertions are equivalent.*

- a) *The set E is scattered.*
- b) $k(E)^\perp \subseteq AP(\Gamma)$.

$$c) k(E)^\perp = \overline{Span(E)}.$$

Proof. a)→ b). Assume that the set E is scattered. Let φ be an element of $k(E)^\perp$. We have to show that the function φ is almost periodic on Γ . Let us first see that the spectrum of φ is contained in E . To see this, let f be an element in $\sigma(\varphi)$. Then, by definition of $\sigma(\varphi)$, $f = weak^* - lim \varphi.a_\alpha$ for some net $(a_\alpha)_{\alpha \in J}$ in A . If $f \notin E$, then by regularity of the algebra $\ell^1(\Gamma)$, there exists an element a in $\ell^1(\Gamma)$ such that $\langle f, a \rangle \neq 0$ and $\hat{a}(E) = 0$. Hence a is in the ideal $k(E)$ and, since $\langle f, a \rangle \neq 0$, $f.a = \langle f, a \rangle f \neq 0$. As $k(E)$ is an ideal, for any $b \in \ell^1(\Gamma)$, $a \star b \in k(E)$ so that $\langle \varphi, a \star b \rangle = \langle \varphi.a, b \rangle = 0$. This implies that $\varphi.a = 0$. However, as $f.a = weak^* - lim \varphi.a_\alpha.a = 0$, this contradicts the fact that $f.a \neq 0$. Hence $\sigma(\varphi) \subseteq E$, and the set E being scattered, by Loomis' Theorem [Lo; Theorem 4], φ is almost periodic on Γ . This proves the inclusion $k(E)^\perp \subseteq AP(\Gamma)$.

b)→ c). Assume b) holds. The inclusion $\overline{Span(E)} \subseteq k(E)^\perp$ being clear, we prove the reverse inclusion only. To prove this, let φ be an element of $k(E)^\perp$. As $\sigma(\varphi) \subseteq E$, and φ is almost periodic on Γ , $\varphi \in \overline{Span(E)}$, see e.g. [B; p. 110, Theorem 2.2.3]. This proves the inclusion $k(E)^\perp \subseteq \overline{Span(E)}$.

c)→ a). To prove this implication, the implication c)→ b) being clear, we shall prove that b)→ a). So assume that b) holds. To get a contradiction assume that the set E is not scattered. Then E contains a nonempty perfect set K . The set K being perfect, by Theorem 10 in [La; p.52], there exists a nontrivial regular continuous Borel measure μ on G supported by K . Let $\varphi : \Gamma \rightarrow C$, $\varphi(\lambda) = \int_G \overline{\langle f, \lambda \rangle} d\mu(f)$, be the Fourier-Stieltjes transform of μ . It is clear that the function φ is bounded on Γ so that it is in $\ell^\infty(\Gamma)$. For $a \in K(E)$,

$$\langle \varphi, a \rangle = \sum_{\lambda \in \Gamma} \varphi(\lambda) a(\lambda) = \int_G \sum_{\lambda \in \Gamma} a(\lambda) \overline{\langle f, \lambda \rangle} d\mu(f) =$$

$$\int_G \hat{a}(f) d\mu(f) = 0$$

since the support of μ is contained in E . This shows that $\varphi \in k(E)^\perp$. Hence by b), $\varphi \in AP(\Gamma)$. Let m be the (unique) invariant mean on $AP(\Gamma)$ [Bu; p.15, Corollory 1.26]. Then $m(\varphi) = \mu(e)$, where e is the unit element of the group G . Define now a new Borel

measure by $\tilde{\mu}(B) = \overline{\mu(B^{-1})}$ on the Borel σ -algebra of G . Then, as in [Bu; p.73, Corollary 4.13], we have

$$m(|\varphi|^2) = (\mu \star \tilde{\mu})(e) = \sum_{f \in G} |\mu(f)|^2 = 0$$

since μ is continuous. However this is not possible unless φ is identically zero [Bu; p.15, Corollary 1.23], which is not the case since μ is not trivial. This contradiction completes the proof. \square

Lemma 3.3. *Let A be an arbitrary Banach algebra. If the space A^* has the Schur property, then $ap(A) = A^*$.*

Proof. Suppose that A^* has the Schur property. Let f be an element of A^* . We have to show that the set $H(f) = \{f.a : a \in A_1\}$ is relatively compact in A^* . This is clearly equivalent to showing that the operator $T : A \rightarrow A^*$, defined by $T(a) = f.a$, is compact. To prove this, let $(a_n)_{n \in \mathbb{N}}$ be a sequence in A_1 . Since A^* has the Schur property, A does not contain an isomorphic copy of ℓ^1 [D; Theorem 3]. Hence, by Rosenthal's ℓ^1 -Theorem [R], the sequence $(a_n)_{n \in \mathbb{N}}$ has a subsequence, denoted again $(a_n)_{n \in \mathbb{N}}$, which is weakly Cauchy. It follows that the sequence $(T(a_n))_{n \in \mathbb{N}}$ is weakly Cauchy in A^* . As A^* has the Schur property, it is weakly sequentially complete. Consequently the sequence $(T(a_n))_{n \in \mathbb{N}}$ converges weakly, so in norm in A^* . This proves that each f in A^* is almost periodic on A so that $A^* = ap(A)$. \square

For the proof of the next lemma we refer the reader to [D-Ü; Corollary 4.3].

Lemma 3.4. *Let A be any Banach algebra and I be a closed ideal of it. Then $ap(A/I) = ap(A) \cap I^\perp$.*

The second main result of this section is the following result.

Theorem 3.5. *Let A be a γ -algebra, and let Σ be its Gelfand spectrum. Then the following assertions are equivalent.*

- a) The space Σ is scattered.
- b) The space A^* has the Schur property.
- c) $ap(A) = A^*$.
- d) $\overline{Span(\Sigma)} = A^*$.

Proof. Since A is a γ -algebra, by Theorem 3.1, A is isomorphic to the quotient algebra $\ell^1(\Gamma)/I$ for some discrete abelian group Γ and a closed ideal I of $\ell^1(\Gamma)$. So we can assume that $A = \ell^1(\Gamma)/I$.

a) \rightarrow b). Assume that Σ is scattered. This means that the hull of I is scattered. Then by [L-P1;Theorem 1] or [L-P2;Theorem 2.14], A^* has the Schur property.

b) \rightarrow c). This implication follows from Lemma 3.3 above.

c) \rightarrow d). Assume that $ap(A) = A^*$. Since $A = \ell^1(\Gamma)/I$ and, by Lemma 3.4, $ap(\ell^1(\Gamma)/I) = ap(\ell^1(\Gamma)) \cap I^\perp = AP(\Gamma) \cap I^\perp = A^* = I^\perp$, we conclude that $I^\perp \subseteq AP(\Gamma)$. Let E be the hull of I . As $I \subseteq k(E), k(E)^\perp \subseteq I^\perp \subseteq AP(\Gamma)$. Hence, by Lemma 3.2, E is scattered, so a set of synthesis, and $I = k(E)$. Hence, by Lemma 3.2 again, $I^\perp = k(E)^\perp = \overline{Span(E)}$. This proves that $\overline{Span(\Sigma)} = A^*$.

d) \rightarrow a). Assume that the equality $\overline{Span(\Sigma)} = A^*$ holds. As always one has the inclusion $\overline{Span(\Sigma)} \subseteq ap(A)$, we conclude that $ap(A) = A^*$. This implies that $I^\perp \subseteq AP(\Gamma)$. From this, as above, we conclude that Σ is scattered. □

Corollary 3.6. *Let X be a weak*-closed subspace of $\ell^\infty(\Gamma)$ which is also an $\ell^1(\Gamma)$ -module. Then X has the Schur property iff $X \subseteq AP(\Gamma)$.*

Proof. Since X is weak*-closed and an $\ell^1(\Gamma)$ -module, $X = I^\perp$ for some closed ideal I of $\ell^1(\Gamma)$. Let $E = hull(I)$. Then $I \subseteq k(E)$ so that $k(E)^\perp \subseteq I^\perp = X$. Now suppose that $X \subseteq AP(\Gamma)$. Then, by Lemma 3.2, the set E is scattered, so a set of synthesis and $I = k(E)$. Hence, by the preceding theorem, X has the Schur property. Conversely, assume that X has the Schur property. Then $k(E)^\perp$ has the Schur property. Hence, by Lemma 3.2, E is scattered, $X = k(E)^\perp$ and $X \subseteq AP(\Gamma)$. □

From examples in section 2 and the preceding theorem, the following corollaries are immediate. For related results we refer the reader to the papers of Lust-Piquard [L-P1

and L-P2], and Granirer [G].

Corollary 3.7. *Let G be a locally compact abelian group, $S(G)$ be a Segal algebra on it and I be a closed ideal of $S(G)$ with compact hull. Then the space $(S(G)/I)^*$ has the Schur property iff the hull of I is scattered. \square*

Corollary 3.8. *Let A and B be two γ -algebras. Then the space $(A\hat{\otimes}B)^*$ has the Schur property iff the spectrums \sum_A and \sum_B of A and B are both scattered. Moreover, in this case, $(A\hat{\otimes}B)^*$ = the closure in $K(A, B^*)$ of the space $\text{Span}\{f \otimes g : f \in \sum_A, g \in \sum_B\}$. Here $K(A, B^*)$ is the space of the compact linear operators $T : A \rightarrow B^*$, and $f \otimes g : A \rightarrow B^*$ is defined by $\langle f \otimes g, a \rangle = f(a)g$. \square*

Corollary 3.9. *Let G be a locally compact abelian group, E be a compact subset of it, and $A(E) = A(G)/k(E)$. Then the space $A(E)^*$ has the Schur property iff the set E is scattered. \square*

In general it is not easy to prove that the spectrum of a given Banach algebra A is scattered. In the case of γ -algebras we have the following result.

Theorem 3.10. *Let A and B be two γ -algebras, and assume that we have a continuous one-to-one homomorphism $h : A \rightarrow B$ whose range is dense in B . Then the space A^* has the Schur property iff the space B^* has the Schur property.*

Proof. We first observe that, by Theorem 3.1, every γ -algebra is (Silov) regular. Now assume that the spectrum \sum_A of A is scattered. Then the set $h^*(\sum_B)$ being a compact subset of \sum_A , is also scattered. As h^* is a homeomorphism from \sum_B onto $h^*(\sum_B)$, we conclude that \sum_B is also scattered. Now assume that \sum_B has the Schur property. Let us see that $h^*(\sum_B) = \sum_A$. If $f \in \sum_A \setminus h^*(\sum_B)$, by regularity of A , there exists an element $a \in A$ such that $\langle a, f \rangle = 1$ and $\langle h(a), g \rangle = 0$ for each $g \in \sum_B$. As the space \sum_B is scattered, the algebra B is semisimple by Theorem 3.1. Hence $h(a) = 0$. As h is one-to-one, $a = 0$. This contradiction proves that $h^*(\sum_B) = \sum_A$, and \sum_A is scattered. \square

From this theorem we easily deduce the following result.

Corollary 3.11. *Let A be a γ -algebra, which is algebraically isomorphic to a dense subalgebra of $C(K)$ for some compact space K . Then A^* has the Schur property iff K is scattered. \square*

We end the paper with a few questions.

Questions 1) Let A be a commutative semisimple regular Banach algebra. If A^* has the Schur property, is it then true that the spectrum of A is scattered?

2) Let Γ be a discrete abelian group. In view of Lemma 3.3, the following question arises naturally. Let I be a closed ideal of $\ell^1(\Gamma)$. What should the hull of I be for the inclusion $I^\perp \subseteq WAP(\Gamma)$ to hold? Here $WAP(\Gamma)$ is the space of the weakly almost periodic functions on Γ .

3) Let G be a locally compact (abelian or not) group, $A_p(G)$ be its Figa-Talamanca-Herz algebra [H] and I be a closed ideal of it with compact hull. Is the algebra $A_p(G)/I$ a γ -algebra?

4) Let X be a closed subspace of $\ell^\infty(\Gamma)$ which is also an $\ell^\infty(\Gamma)$ -module. Suppose that X has the Schur property. Is then true that

- i) X is weak*-closed in $\ell^\infty(\Gamma)$?
- ii) $X \subseteq AP(\Gamma)$?
- iii) $X = \overline{Span(E)}$ for some closed scattered subset E of $\hat{\Gamma}$?
- 5) How to prove Theorem 3.10 without using Theorem 3.5?

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