Turkish Journal of Mathematics

Volume 23 | Number 3 Article 3

1-1-1999

Generalized Solutions of a Class of Linear and Quasi-Linear Degenerated Hyperbolic Equations

ROSSITZA SEMERDJIEVA

Turk J Math
23 (2000) , 375 – 388.

GENERALIZED SOLUTIONS OF A CLASS OF LINEAR AND QUASI-LINEAR DEGENERATED HYPERBOLIC EQUATIONS

R. Semerdjieva

Abstract

The equation $L(u) := k(y)u_{xx} - \partial_y(\ell(y)u_y) + r(x,y)u = f(x,y,u)$, where $k(y) > 0, \ell(y) > 0$ for $y > 0, k(0) = \ell(0) = 0$ and $lim_{y\to 0}k(y)/\ell(y)$ exists, is strictly hyperbolic for $y > 0$ and its order degenerates on the line $y = 0$. We consider the boundary value problem $Lu = f(x,y,u)$ in $G, u\mid_{AC}= 0$, where G is a simply connected domain in R^2 with piecewise smooth boundary $\partial G = AB \cup AC \cup BC; AB = \{(x,0) : 0 \le x \le 1\}, AC : x = F(y) = \int_0^y (k(t)/\ell(t))^{1/2}dt$ and $BC : x = 1 - F(y)$ are characteristic curves. The existence and uniqueness of a generalized solution to this problem are proved in the linear case (where $f = f(x,y)$); the nonlinear case is treated by using the Schauder Fixed Point Theorem.

1. Introduction

Investigations of degenerated hyperbolic equations in the plane are mainly restricted to the case where the type degenerates to parabolic one on the line of degeneration (see [4, 5, 11, 12] and their respective bibliographies). Bitsadze [4] mentioned that the case of order degeneration deserves special attention and requires a special treatment, since the results for equations of changable type are not applicable. Up to our knowledge, there are few papers examining boundary value problems for such equations (see [3, 6, 7, 9, 10]).

Our aim here is to study by functional analytic methods the generalized solvability of a boundary value problem for a class of quasilinear degenerated hyperbolic equations, whose principal part vanishes on the line of degeneration. In that case it is essential to work with weighted Sobolev spaces. We introduce appropriate Sobolev spaces and first investigate the corresponding linear problem, prove a priori estimates and derive from them existence and uniqueness of generalized solutions. In the non-linear case we obtain existence theorem by the Schauder fixed point principal and give a sufficient condition for uniqueness. A similar approach was used by Aziz, Lemert and Schneider [2] for studying the generalized solvability of the Tricomi problem for a class of mixed type equations.

Consider the equation

$$L(u) := k(y)u_{xx} - \partial_y(\ell(y)u_y) + r(x,y)u = f(x,y,u), \tag{1}$$

where $k(y) > 0, \ell(y) > 0$ for $y > 0, k(0) = \ell(0) = 0$ and $lim_{y\to 0}k(y)/\ell(y)$ exists. Equation (1) is strictly hyperbolic for $y > 0$ and its order degenerates on the line $y = 0$.

Let G be a simply connected domain on the x, y plane with piecewise smooth boundary $\partial G = AB \cup AC \cup BC$, where $AB = \{(x,0) : 0 \le x \le 1\}$, and $AC : x = F(y) = \int_0^y (k(t)/\ell(t))^{1/2}dt$ and $BC : x = 1 - F(y)$ are characteristics of (1) issued from the point $C(1/2, Y)$, where the constant $Y > 0$ is determined by $F(Y)$=1/2.

We consider the following boundary value problem.

Problem B. Find in the domain G a solution of (1) satisfying the boundary condition $u = 0$ on AC.

Since the operator L is selfadjoint, the congugate Problem B* is to find in the domain G a solution of the equation $Lv = f(x,y,v)$ satisfying the boundary condition $v = 0$ on BC. It turns out that the treatment of these boundary value problems is symmetric. (In fact, it is easy to obtain one of the problems from the other by appropriate change of variables). Therefore we consider mainly Problem B and only formulate results concerning Problem B*.

We denote by $L^2(k)$ the space of functions with integrable square with respect to the weighted measure $kdxdy$ and set

$$(u,v)_{L^2(k)} = \int_G k(y)u(x,y)v(x,y)dxdy, \quad \| u \|_{L^2(k)} = (u,u)^{1/2}_{L^2(k)}.$$

Set also

$$(u,v)_{k,\ell} = \int_G [k(y)u_x(x,y)v_x(x,y) + \ell(y)u_y(x,y)v_y(x,y) + u(x,y)v(x,y)]dxdy.$$

Let $C^\infty_{AC}(\bar{G})$ and $C^\infty_{BC}(\bar{G})$ be the sets of functions $u, v \in C^\infty(\bar{G})$ such that, respectively, $u\mid_{AC}= 0$ or $v\mid_{BC}= 0$. Denote, respectively, by $H^1(k,\ell), H^1_{AC}(k,\ell), H^1_{BC}(k,\ell)$ the corresponding weighted Sobolev spaces as completions of the spaces $C^\infty(\bar{G}), C^\infty_{AC}(\bar{G})$ and $C^\infty_{BC}(\bar{G})$ with respect to the norm

$$\| u \|_{k,\ell} = (u,u)^{1/2}_{k,\ell} = \left(\int_G (ku_x^2 + \ell u_y^2 + u^2)dxdy\right)^{1/2}.$$

Definition *A function $u \in H^1_{AC}(k,\ell)$ is called generalized solution of Problem B if the identity*

$$B[u,v] := -\int_G (ku_xv_x - \ell u_yv_y - ruv)dxdy = \int_G f(x,y,u)vdxdy \tag{2}$$

is satisfied for any function $v \in H^1_{BC}(k,\ell)$.

In an analogous way one can define a generalized solution of Problem B*.

2. A priori estimates

Let U and V denote, respectively, the subsets of $C^\infty_{AC}(\bar{G})$ and $C^\infty_{BC}(\bar{G})$ consisting of functions vanishing, respectively, in a neighborhood of AC and BC.

Lemma 1 *(i) The sets U and V are dense, respectively, in the spaces $H^1_{AC}(k,\ell)$ and $H^1_{BC}(k,\ell)$.*

(ii) If $k,\ell \in C[0,Y]$ and $v \in V$, then the boundary problem

$$h(u) := (1-x)u_x = v \ in\ G,\ u\mid_{AC}= 0,$$

has a unique solution $u \in C^1(\bar{G})$.

(iii) If $k,\ell \in C[0,Y]$ and $u \in U$, then the boundary problem

$$h^\star(v) := -xv_x = u \ in\ G,\ v\mid_{BC}= 0,$$

has a unique solution $v \in C^1(\bar{G})$.

Proof. We prove only that U is dense in $H^1_{AC}(k,\ell)$. Let U^1 be the subset of $C^1(\bar{G})$ consisting of functions that vanish in a neighborhood of AC; then U is dense in U^1. Indeed, the domain G is convex, thus star-like with respect to any internal point (x_0, y_0). Therefore any function $f \in U^1$ may by approximated together with its first derivatives uniformly on $\bar{G}$ by C^1-functions defined in a neighborhood of $\bar{G}$. Namely,

$$f_t(x,y) = f(x_0 + t(x - x_0), y_0 + t(y - y_0)) \to f(x,y)$$

uniformly on $\bar{G}$ (together with first derivatives) as $t \uparrow 1$. Moreover, functions $f_t, t \in (0,1)$, will vanish in a neighborhood of AC for t close enough to 1.

On the other hand, every C^1-function g defined on a neighborhood of $\bar{G}$ may be approximated by C^∞-functions uniformly on $\bar{G}$ together with its first derivatives. Indeed, one can use the standard smoothing operators $I_\epsilon(g) = g \star \varphi_\epsilon$ as $\epsilon \to 0$, where $\varphi_\epsilon(x,y) = \epsilon^{-2}\varphi(x/\varepsilon, y/\varepsilon), \varepsilon > 0, \varphi$ is a positive C^∞-function in $\mathbf{R}^2$ vanishing outside the unit disc, $\int \varphi dxdy = 1$, and $\star$ denotes the convolution operation. In addition, if g vanishes in a neighborhood of AC, then for small values of ε the function $I_\varepsilon(g)$ will vanish also in a neighborhood of AC. Hence the set U is dense in U^1.

Next we show that U^1 is dense in $C^\infty_{AC}(\bar{G})$. Let $\omega : \mathbf{R} \to \mathbf{R}$ be an increasing C^∞-function such that $w(t) = 0, t \leq 1$ and $w(t) = 1$ if $t \geq 2$, and let $C_w = sup\{w'(t) : t \in \mathbf{R}\}$. Fix a function $f \in C^\infty_{AC}(\bar{G})$ and set

$$C_f = sup_G \mid f \mid + sup_G \mid \partial_x f \mid + sup_G \mid \partial_y f \mid .$$

Consider the functions

$$u_n(x,y) = \omega[n(x - F(y))] f(x,y), \quad n = 1, 2, \ldots$$

Then $u_n = 0$ if $x - F(y) \leq 1/n$, so u_n vanishes in a neighborhood of AC, and moreover,

$$\parallel f - u_n \parallel_{k,\ell} \to 0 \text{ as } n \to \infty.$$

Indeed, since

$$1-\omega[n(x-F(y))] \equiv 0 \text{ if } x-F(y) \geq 2/n$$

we have $f-u_n=0$ in $G\setminus G_n$, where

$$G_n=\{(x,y)\in G : x-F(y)\leq 2/n\}.$$

Taking into account that f vanishes on AC we obtain for $(x,y)\in G_n$ and some $\widetilde{x}\in (F(y),x)$

$$\mid f(x,y) \mid = \mid f(x,y)-f(F(y),y) \mid = (x-F(y)) \mid \partial_x f(\widetilde{x},y) \mid \leq \frac{2}{n}C_f.$$

Thus the following estimates hold:

$$\| f-u_n \|_{L^2(G)} = \| f(1-\omega[n(x-F(y))])||_{L^2(G)} \leq \| f \|_{L^2(G_n)} \leq \frac{2}{n}C_f\sqrt{\mu(G_n)};$$

$$\| \partial_x(f-u_n) \|_{L^2(k)} \leq \| \partial_x f(1-\omega[n(x-F(y))])||_{L^2(k)} + \| nfw'[n(x-F(y))] \|_{L^2(k)} \leq$$

$$(C_f+\frac{2}{n}C_f.n.C_w)\left(\int_{Gn} k(y)dxdy\right)^{1/2};$$

$$\| \partial_y(f-u_n)||_{L^2(\ell)} \leq \| \partial_y f(1-w[n(x-F(y))]) \|_{L^2(\ell)} + \| nfw'[n(x-F(y))] \|_{L^2(k)} \leq$$

$$C_f\left(\int_{Gn} \ell(y)dxdy\right)^{1/2} + 2C_f.C_w\left(\int_{Gn} k(y)dxdy\right)^{1/2},$$

where $\mu(G_n)$ is Lebesgue measure of G_n. Since $\mu(G_n)\to 0$ as $n\to\infty$ the claim follows.

It is easy to see by the formula

$$u(x,y)=\int_{F(y)}^{x}\frac{v(t,y)}{1-t}dt,$$

that (ii) holds, and (iii) follows from the analogous formula

$$v(x,y) = -\int_{1-F(y)}^{x} \frac{u(t,y)}{t} dt.$$

□

The following lemma is a variation of the well-known Fridrich's inequality. Weights and specific properties of our domain are used, therefore we provide a proof.

Lemma 2 *If* $w \in H^1_{AC}(k,\ell)$ *or* $w \in H^1_{BC}(k,\ell)$, *then* $\| w \|_{L^2(k)} \leq 1/2 \| w_x \|_{L^2(k)}$.

Proof. We consider the case where $w \in C^\infty_{AC}$. Since

$$w(x,y) = \int_{F(y)}^{x} w_x(t,y) dt,$$

we have by Cauchy inequality

$$| w(x,y) |^2 \leq (x - F(y)) \int_{F(y)}^{1-F(y)} | w_x(t,y) |^2 \, dt.$$

Thus

$$\int_{F(y)}^{1-F(y)} | w(x,y) |^2 \, dx \leq \frac{(1-2F(y))^2}{2} \int_{F(y))}^{1-F(y)} | w_x(t,y) |^2 \, dt \leq \frac{1}{2} \int_{F(y)}^{1-F(y)} | w_x(t,y) |^2 \, dt.$$

Hence

$$\| w \|^2_{L^2(k)} = \int_0^Y \int_{F(y)}^{1-F(y)} k(y) \, | w(x,y) |^2 \, dx dy \leq$$

$$\frac{1}{2} \int_0^Y \int_{F(y)}^{1-F(y)} k(y) \, | w_x(t,y) |^2 \, dt dy = \frac{1}{2} \| w_x \|^2_{L^2(k)} .$$

□

Lemma 3 *Suppose* $k(y), \ell(y) \in C[0,Y], k(y) > 0, \ell(y) > 0$ *for* $y > 0$ $k(0) = \ell(0) = 0$ *and* $lim_{y\to 0}k(y)/\ell(y)$ *exists;* $r, r_x \in C(\bar{G})$.

(a) If $c_o := inf_{\bar{G}}(k + r - (1-x)r_x) > 0$ *and* $r \geq 0$ *on* BC*, then for* $v \in V$ *and* u *related to* v *as in (ii) of Lemma 1, it holds that*

$$B[u,v] = B[u,h(u)] \geq m_0 \parallel u \parallel^2_{k,\ell}, \;\; m_0 = min(1/4, c_0/2); \tag{3}$$

(b) If $c_0^\star := inf_{\bar{G}}(k + r + xr_x) > 0$ *and* $r \geq 0$ *on* AC*, then for* $u \in U$ *and* v *related to* u *as in (iii) of Lemma 1, it holds that*

$$B[u,v] = B[h^\star(v),v] \geq m_0^\star \parallel v \parallel^2_{k,\ell}, \;\; m_0^\star = min(1/4, c_0^\star/2). \tag{4}$$

Proof. We shall prove only (3) since the proof of (4) is similar. By Green's formula we obtain

$$B[u,h(u)] = -\int_G [ku_x\partial_x h(u) - \ell u_y \partial_y h(u) - ruh(u)]dxdy =$$

$$-\int_G [k(1-x)u_x u_{xx} - ku_x^2 - \ell(1-x)u_y u_{xy} - r(1-x)uu_x]dxdy =$$

$$\int_G ku_x^2 dxdy - \frac{1}{2}\int_G [k\partial_x u_x^2 - \ell\partial_x u_y^2 - r\partial_x(u^2)](1-x)dxdy =$$

$$\frac{1}{2}\int_G [ku_x^2 + \ell u_y^2 + (r - (1-x)r_x)u^2]dxdy - \frac{1}{2}\int_{\partial G}(ku_x^2 - \ell u_y^2 - ru^2)(1-x)dy.$$

It is easy to see that $-1/2\int_{\partial G}(ku_x^2 - \ell u_y^2 - ru^2)(1-x)dy \geq 0$. Indeed, $\partial G = AB \cup BC \cup CA$, where we denote the corresponding curves with beginning and end points. Obviously, the line integral on AB equals 0; on $BC : x = 1 - F(y)$ we have $h(u) = (1-x)u_x = v(x,y)$, thus $u_x = 0$ and the line integral on BC equals $1/2\int_0^Y(\ell u_y^2 + ru^2)F(y)dy \geq 0$; finally, on CA we have $u = 0$, which implies $ku_x^2 - \ell u_y^2 = 0$ on CA, therefore the line integral on CA equals 0.

Hence

$$B[u,h(u)] \geq \frac{1}{2}\int_G [ku_x^2 + \ell u_y^2 + (r-(1-x)r_x)u^2]dxdy.$$

Finally, applying Lemma 2 we obtain that the last integral is not less than

$$\frac{1}{4}\int_G [ku_x^2 + 2\ell u_y^2 + 2(k+r-(1-x)r_x)u^2]dxdy \geq m_0 \parallel u \parallel_{k,\ell}^2,$$

which proves the statement. □

Theorem 1 *(Main a priori estimates). Under the assumptions of Lemma 3, the following estimates hold:*

$$m_0 \parallel v \parallel_{L^2(k)} \leq sup_{u\in H^1_{AC}(k,\ell)} \frac{\mid B[u,v] \mid}{\parallel u \parallel_{k,\ell}} \leq C \parallel v \parallel_{k,\ell} \quad \forall v \in H^1_{BC}(k,\ell); \tag{5}$$

$$m_0^\star \parallel u \parallel_{L^2(k)} \leq sup_{v\in H^1_{BC}(k,\ell)} \frac{\mid B[u,v] \mid}{\parallel v \parallel_{k,\ell}} \leq C \parallel u \parallel_{k,\ell} \quad \forall u \in H^1_{AC}(k,\ell), \tag{6}$$

where $m_0, m_0^\star$ are the constants from Lemma 3 and C=const>0.

Proof. The right-hand side of (5) follows immediately from Cauchy inequality. Indeed, we have

$$\mid B[u,v] \mid = \mid \int_G (ku_xv_x - \ell u_yv_y + ruv)dxdy \mid \leq C \parallel u \parallel_{k,\ell} \parallel v \parallel_{k,\ell}.$$

Since the set V is dense in the space $H^1_{BC}(k,\ell)$ it is enough to prove the left-hand side of (5) for $v \in V$. Fix $v \in V$. Then by (ii) of Lemma 1, there exists $u \in C^1_{AC}(\bar{G})$ such that $h(u) = (1-x)u_x = v$, so from formula (3) in Lemma 3 it follows

$$m_0 \parallel v \parallel_{L^2(k)} = m_0 \parallel (1-x)u_x \parallel_{L^2(k)} \leq m_0 \parallel u \parallel_{k,\ell} \leq \frac{B|u,v]}{\parallel u \parallel_{k,\ell}},$$

which proves (5).

The proof of (6) is similar, but instead of (3) one should use (4). □

3. Linear Case

We consider here the linear case, where the right-hand side f of (1) does not depend on u.

Theorem 2 *If $k(y), \ell(y) \in C[0,Y], k(y) > 0, \ell(y) > 0$ for $y > 0, k(0) = \ell(0) = 0$ and $lim_{y\to 0} k(y)/\ell(y)$ exists; $r, r_x \in C(\bar{G}), r \geq 0$ on $AC \cup BC$, and*

$$c_0 := inf_{\bar{G}}(k + r - (1-x)r_x) > 0, \quad c_0^\star := inf_{\bar{G}}(k + r + xr_x) > 0,$$

then for any $f(x,y) \in L^2(1/k)$ there exists a unique generalized solution w of Problem B such that

$$\| w \|_{k,\ell} \leq \frac{1}{m_0} \| f \|_{L^2(1/k)}, \tag{7}$$

where $m_0 = min(1/4, c_0/2)$.

Proof. From (5) it follows immediately that for any fixed $v \in H^1_{BC}(k,\ell)$ we obtain from the bilinear form $B[u,v]$ a continuous linear functional $F_v(u) = B[u,v]$ in the Hilbert space $H^1_{AC}(k,\ell)$, with norm satisfying $m_0 \| v \|_{L^2(k)} \leq \| F_v \| \leq C \| v \|_{k,\ell}$. Thus by the Riesz Representation Theorem, there exists a unique function $Tv \in H^1_{AC}(k,\ell)$ such that

$$F_v(u) = B[u,v] = (u, Tv)_{k,\ell} \quad \forall u \in H^1_{AC}(k,\ell).$$

Obviously Tv depends linearly on v, so we obtain a linear continuous operator $T : H^1_{BC}(k,\ell) \to H^1_{AC}(k,\ell)$ such that

$$m_0 \| v \|_{L^2(k)} \leq \| Tv \|_{k,\ell} \leq C \| v \|_{k,\ell} \quad \forall v \in H^1_{BC}(k,\ell). \tag{8}$$

In an analogous way it follows from (6) that there exists a linear continuous operator $S : H^1_{AC}(k,\ell) \to H^1_{BC}(k,\ell)$ such that $B[u,v] = (Su, v)_{k,\ell}$ and

$$m_0^\star \| u \|_{L^2(k)} \leq \| Su \|_{k,\ell} \leq C \| u \|_{k,\ell} \quad \forall u \in H^1_{AC}(k,\ell).$$

Obviously both T and S are injective operators due to the left-hand sides of (8) and (9). It is easy to see that the range of T is dense in the space $H^1_{AC}(k,\ell)$. Indeed, suppose that $u \in H^1_{AC}(k,\ell)$ is orthogonal to the range of T, i.e. $(u, Tv)_{k,\ell} = 0, \forall v \in H^1_{BC}(k,\ell)$. Since $(u, Tv)_{k,\ell} = B[u,v] = (Su, v)_{k,\ell}$ then we have $(Su, v)_{k,\ell} = 0, \forall v \in H^1_{BC}(k,\ell)$, thus $Su = 0$, whence it follows $u = 0$. But this means that the closure of $R(T)$ coincides with the whole space $H^1_{AC}(k,\ell)$. In the same way one can prove that the range $R(S)$ of S is dense in the space $H^1_{BC}(k,\ell)$.

Fix $f \in L^2(1/k)$ and consider on the range $R(T)$ of T the linear functional

$$G_f(Tv) = \int_G fv dxdy.$$

The inequality

$$\mid \int_G fv dxdy \mid \leq \parallel f \parallel_{L^2(1/k)} \parallel v \parallel_{L^2(k)} \leq \frac{1}{m_0} \parallel f \parallel_{L^2(1/k)} \parallel Tv \parallel_{k,\ell}$$

shows that the linear functional G_f is continuous and its norm is less then $(1/m_0) \parallel f \parallel_{L^2(1/k)}$. Since $R(T)$ is dense in the space $H^1_{AC}(k,\ell)$, there is a unique linear continuous extension of G_f on $H^1_{AC}(k,\ell)$ with the same norm and, by Riesz Representation Theorem, there exists a unique function $w \in H^1_{AC}(k,\ell)$ such that $G_f(u) = (w,u)_{k,\ell}$ for any $u \in H^1_{AC}(k,\ell)$ and $\parallel w \parallel_{k,\ell} = \parallel G_f \parallel \leq (1/m_0) \parallel f \parallel_{L^2(1/k)}$. In particular, for $u = Tv, v \in H^1_{BC}(k,\ell)$ we have

$$B[w,v] = (w, Tv)_{k,\ell} = G_f(Tv) = \int_G fv dxdy,$$

hence w is a unique generalized solution of Problem B in the linear case. □

Remark *Under the assumptions of Theorem 2 one can prove in an analogous way existence of a unique generalized solution of Problem $B^\star$.*

4. Nonlinear Case

Next we consider the nonlinear case, where the right-hand side of (1) depends on u. Recall that a function $f(x,y,u) : G \times R \to R$ satisfies the *Carathéodory condition* (e.g. [8]), if for any fixed $u \in R$ it is a measurable function on G, and for almost all $(x,y) \in G$

the function f is continuous with respect to $u \in R$.

Theorem 3 *If the assumptions of Theorem 2 are satisfied and*

(i) $k \in C^1[0, Y], k'/k^{1/2} \in L^q(G)$ *for some* $q > 2$*;*

(ii) the function $f(x, y, u)$ *satisfies Carathéodory condition and*

$$| f(x, y, u) | \leq A(x, y)\sqrt{k} + mk | u |,$$

where $A(x, y) \in L^2(G)$ *and m=const,* $0 \leq m < m_0/2$,

then there exists a generalized solution of Problem B.

If, in addition, the condition

(iii) $| f(x, y, u_1) - f(x, y, u_2) | \leq \rho k | u_1 - u_2 |, (x, y) \in G, u_1, u_2 \in \mathbf{R}$ *holds with constant* $\rho < 2m_0$*, then the generalized solution is unique.*

Proof. It is well known that the Carathéodory condition implies that for any measurable function $u(x, y)$ the function $f(x, y, u(x, y)), (x, y) \in G$, is also measurable; moreover, condition (ii) implies that the nonlinear operator

$$\Phi(u) = f(x, y, u(x, y))$$

acts continuously from the space $L^2(k)$ into the space $L^2(1/k)$ (see [8]). In addition, from (ii) it follows

$$\| \Phi(u) \|_{L^2(1/k)} \leq \| A(x, y) + m\sqrt{k} | u | \|_{L^2(G)} \leq A_0 + m \| u \|_{L^2(k)},$$

where $A_0 = \| A(x, y) \|_{L^2(G)}$.

By Theorem 2 there exists a linear continuous operator

$$W : L^2(1/k) \to H^1_{AC}(k, \ell),$$

corresponding to any right-hand side $f(x, y) \in L^2(1/k)$ of (1) the unique generalized solution $w = W(f)$ of (linear) Problem B, and moreover, $\| W(f) \|_{k,\ell} \leq (1/m_0) \| f \|_{L^2(1/k)}$.

Lemma 2 shows that any function $w \in H^1_{AC}(k, \ell)$ may be regarded as a function belonging to the space $L^2(k)$. Consider the embedding operator

$$i : H^1_{AC}(k,\ell) \to L^2(k),$$

which corresponds to any function $w \in H^1_{AC}(k,\ell)$ the same function regarded as an element of the space $L^2(k)$. By Lemma 2 we have that $\| i(w) \|_{L^2(k)} \leq (1/2) \| w \|_{k,l}$.

Moreover, the orepator i is compact. Indeed, suppose $D \subset H^1_{AC}(k,\ell)$ is a bounded subset. We shall prove that the set of functions $\widetilde{D} = \{k^{1/2}w : w \in D\}$ is precompact in the space $L^2(G)$. By the Relich-Kondraschov Theorem (e.g., see [1]) for any $p \in$(1,2) the natural embedding of the Sobolev space $W^{1,p}(G)$ into $L^2(G)$ is a compact operator. So, it is enough to show that there exists $p \in$ (1,2) such that the set $\widetilde{D}$ is bounded in $W^{1,p}(G)$.

Fix $p \in$ (1,2) such that $q = 2p/(2-p)$, where q is the number from condition (i). Then for any $w \in D$ we obtain

$$\| k^{1/2}w \|^p_{W^{1,p}(G)} = \int_G [| \partial_x(k^{1/2}w) |^p + | \partial_y(k^{1/2}w) |^p + | k^{1/2}w |^p] dxdy \leq$$

$$\int_G \{k^{p/2} | w_x |^p + 2^{p-1}(k/\ell)^{p/2}\ell^{p/2} | w_y |^p + [(1/2)(k'/\sqrt{k})^p + k^{p/2}] | w |^p\} dxdy \leq$$

$$C \left(\int_G (kw_x^2 + \ell w_y^2 + w^2) dxdy \right)^{p/2} = C \| w \|^p_{k,\ell},$$

where, due to condition (i),

$$C = \left(\int_G \left\{ 1 + 2^{\frac{2p-2}{2-p}} \left(\frac{k}{\ell} \right)^{\frac{p}{2-p}} + \left[\frac{1}{2} \left(\frac{k'}{\sqrt{k}} \right)^p + k^{p/2} \right]^{\frac{2}{2-p}} \right\} dxdy \right)^{\frac{2-p}{2}} < \infty.$$

Thus the set $\widetilde{D}$ is bounded in $W^{1,p}(G)$.

Next we apply Schauder Fixed Point Theorem. Consider the operator

$$K = i \circ W \circ \Phi : L^2(K) \to L^2(k).$$

It is continuous, as composition of continuous operators, and compact, since i is a compact operator. In addition, if $\| u \|_{L^2(k)} \leq R := A_0/(2m_0 - m)$, then

$$\| K(u) \|_{L^2(k)} \leq (1/2) \| W(\Phi(u)) \|_{k,\ell} \leq$$

$$(1/2m_0) \| \Phi(u) \|_{L^2(1/k)} \leq (1/2m_0)(A_0 + mR) \leq R.$$

Hence the operator K maps the ball with radius R in the space $L^2(k)$ into itself, so K has a fixed point u_0. But then u_0 is a generalized solution of Problem B.

Suppose now that condition (iii) holds. If u_1 and u_2 are two generalized solitions of problem B, then their difference $u_1 - u_2$ is a generalized solution of linear Problem B with right-hand side $f(x, y, u_1) - f(x, y, u_2)$. So by Lemma 2, Theorem 2 and condition (iii) we obtain

$$\| u_1 - u_2 \|_{L^2(k)} \leq \frac{1}{2} \| u_1 - u_2 \|_{k,\ell} \leq$$

$$\frac{1}{2m_0} \| f(x, y, u_1) - f(x, y, u_2) \|_{L^2(1/k)} \leq \frac{\rho}{2m_0} \| u_1 - u_2 \|_{L^2(k)} .$$

Since $\rho/2m_0 < 1$, these inequalities imply $u_1 = u_2$, which proves the theorem. □

Acknowledgement

The author would like to thank the Department of Mathematics of Middle East Technical University for its hospitality.

References

[1] Adams, R. A.: Sobolev spaces, Academic press, New York 1975.

[2] Aziz, A. K., Lemert, R., Schneider, M.: The existence of generalized solutions for a class of linear and nonlinear equations of mixed type. Note di Math. **10**, 47-64 (1990).

[3] Bitsadze, A. V.: Theory of equations of mixed type, whose order degenerates along the line of change of type. Continuous medium mechanics and related problems of analysis, Moscow, 47-52 (1972).

[4] Bitsadze, A. V.: Some classes of partial differential equations, Gordon and Breach Science Publishers, New York 1988.

[5] Carroll, R. W., Showalter, R. E.: Singular and degenerate Cauchy problems, Academic press, New-York 1976.

[6] Kapustin, N. Yu.: Solvability in the class L^2 of a characteristic problem for a hyperbolic equation with order degeneration. Dokl. Acad. Nauk USSR **311**, 784-788 (1990).

[7] Kapustin, N. Yu.: Solvability in the class $L2$ of a problem with a boundary condition on the characteristic for a hyperbolic equation with order degeneration. Differents. Uravn. **26**, 66-75 (1990).

[8] Krasnosel'skii, M. A.: Topological methods in the theory of nonlinear integral equations, Pergamon Press, Oxford 1964.

[9] Semerdjieva, R. I.: A hyperbolic equation with degeneration of order. Serdica **8**, 57-63 (1982).

[10] Semerdjieva, R. I.: A difference method for solution of a class of hyperbolic equations with degeneration of order. Differens. Uravn. **25**, 1755-1766 (1989).

[11] Smirnov, M. M.: Degenerated hyperbolic equations, Vyshaya Skola, Minsk 1977.

[12] Tersenov, S. A.: Introduction to the theory of equations, degenerating on the boundary, Novosibirsk State University, Novosibirsk 1973.

Rossitza SEMERDJIEVA
Department of Mathematics,
Middle East Technical University
06531, Ankara-TURKEY
email: rsem@rorqual.cc.metu.edu.tr

Received 23.10.1998