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## ON THE ASYMPTOTIC BEHAVIOUR OF HEEGNER POINTS

*Jan Nekovář & Norbert Schappacher*

Dedicated to Professor Masatoshi Ikeda on the occasion of his 70<sup>th</sup> birthday

### Abstract

We prove that all but finitely many Heegner points on a given modular elliptic curve (or, more generally, on a given quotient of the modular Jacobian variety  $J_0(N)$ ) are of infinite order in the Mordell-Weil group where they naturally live, i.e., over the corresponding ring class field.

### 1. Notations

**1.1** Let  $N > 1$ . The quasi-projective curve  $Y_0(N)$  defined over  $\mathbf{Q}$  classifies isogenies  $[E \xrightarrow{\lambda} E']$  of elliptic curves with cyclic kernel  $\ker \lambda \cong \mathbf{Z}/N\mathbf{Z}$ . Over  $\mathbf{C}$ , the isogeny  $[\mathbf{C}/\mathbf{Z} + \mathbf{Z}\tau \xrightarrow{[\times N]} \mathbf{C}/\mathbf{Z} + \mathbf{Z}N\tau]$  corresponds to the point  $\Gamma_0(N) \cdot \tau$  of the quotient  $\Gamma_0(N) \backslash \mathcal{H} = Y_0(N)(\mathbf{C})$  of the complex upper half-plane  $\mathcal{H}$ . The dual isogeny  $[\mathbf{C}/\mathbf{Z} + \mathbf{Z}N\tau \rightarrow \mathbf{C}/\mathbf{Z} + \mathbf{Z}\tau]$  induced by the identity on  $\mathbf{C}$  corresponds to the point  $\Gamma_0(N) \cdot w_N(\tau)$ , where  $w_N(\tau) = \frac{-1}{N\tau}$  denotes the Fricke involution.

We write as usual  $X_0(N)$  for the smooth projective curve defined over  $\mathbf{Q}$  which is the compactification of  $Y_0(N)$  and classifies cyclic  $N$ -isogenies between generalized elliptic curves. And we denote by  $J_0(N)$  the Jacobian of  $X_0(N)$ . We embed  $X_0(N)$  in  $J_0(N)$  by sending  $\infty$  to 0, where  $\infty$  is the cusp corresponding to the Néron polygon with a single side.

Finally, we fix a nonzero quotient defined over  $\mathbf{Q}$ ,  $J_0(N) \rightarrow A$  of the abelian variety  $J_0(N)$ , and we let  $X_0(N) \xrightarrow{\pi_A} A$  be the nonconstant morphism defined over  $\mathbf{Q}$  which arises from composing the fixed embedding of  $X_0(N)$  into  $J_0(N)$  with the projection of  $J_0(N)$  onto  $A$ . The following results will therefore apply in particular to the case of a (modular) elliptic curve  $A$  over  $\mathbf{Q}$ .

**1.2** Let  $K$  be an imaginary quadratic field such that all prime numbers dividing  $N$  split in  $K$ . Note right away that, for any given  $N$ , there are infinitely many  $K$  satisfying this so-called “Heegner-condition” (which was introduced by Birch). It implies that there exists an ideal  $\mathfrak{n}$  of the ring of integers  $\mathfrak{o}_K$  such that one has  $\mathfrak{o}_K/\mathfrak{n} \cong \mathbf{Z}/N\mathbf{Z}$ .

More explicitly, let  $D$  be the discriminant of  $K$  and let  $\sqrt{D}$  be the square root of  $D$  which belongs to  $\mathcal{H}$ . Then  $\mathfrak{o}_K = \mathbf{Z} + \mathbf{Z}\alpha$  and  $\mathfrak{n} = N\mathbf{Z} + \mathbf{Z}\alpha$ , with  $\alpha = \frac{-B+\sqrt{D}}{2}$  such that  $\alpha^2 + B\alpha + AN = 0$  and  $B^2 - 4AN = D$ . Or again,  $\mathfrak{o}_K = \mathbf{Z} + \mathbf{Z}\frac{NA}{\alpha}$ ,  $\mathfrak{n}^{-1} = \mathbf{Z} + \mathbf{Z}\frac{A}{\alpha}$ .

For every  $f \geq 1$  relatively prime to  $N$ , we write  $\mathfrak{o}_f = \mathbf{Z} + f\mathfrak{o}_K$  for the order of conductor  $f$  in  $K$ . Its discriminant is  $D_f = Df^2$ . And we put  $\mathfrak{n}_f = \mathfrak{o}_f \cap \mathfrak{n}$ . Since  $(f, N) = 1$ ,  $\mathfrak{n}_f$  is a proper  $\mathfrak{o}_f$ -ideal, that is to say,  $\mathfrak{o}_f = \{x \in K \mid x\mathfrak{n}_f \subseteq \mathfrak{n}_f\}$ .

More explicitly, we have that  $\mathfrak{o}_f = \mathbf{Z} + \mathbf{Z}\alpha_f$  and  $\mathfrak{n}_f = N\mathbf{Z} + \mathbf{Z}\alpha_f$  for  $\alpha_f = f\alpha$ . Thus  $\alpha_f = \frac{-B_f+\sqrt{D_f}}{2}$ ,  $\alpha_f^2 + B_f\alpha_f + A_fN = 0$  with  $B_f = fB$ ,  $A_f = f^2A$ ,  $B_f^2 - 4A_fN = D_f = f^2D$ . Or again,  $\mathfrak{o}_f = \mathbf{Z} + \mathbf{Z}\frac{NA_f}{\alpha_f} = \mathbf{Z} + \mathbf{Z}\frac{NfA}{\alpha}$ ,  $\mathfrak{n}_f^{-1} = \mathbf{Z} + \mathbf{Z}\frac{A_f}{\alpha_f} = \mathbf{Z} + \mathbf{Z}\frac{fA}{\alpha}$ .

Given our choice of  $\mathfrak{n}$ , the *Heegner point*  $y_f$  of conductor  $f$  on  $Y_0(N)$  is defined to be the point represented by the isogeny  $[\mathbf{C}/\mathfrak{o}_f \rightarrow \mathbf{C}/\mathfrak{n}^{-1}]$  which is induced by the identity on  $\mathbf{C}$ . Its image  $\pi_A(y_f) \in A(K_f)$  is called the *Heegner point of conductor  $f$  on  $A$* .

The point  $w_N(y_f)$  is represented by the dual isogeny

$$[\mathbf{C}/\mathfrak{n}^{-1} \rightarrow \mathbf{C}/\mathfrak{o}_f] = \left[ \mathbf{C}/\mathbf{Z} + \mathbf{Z}\frac{fA}{\alpha} \rightarrow \mathbf{C}/\mathbf{Z} + \mathbf{Z}\frac{NfA}{\alpha} \right],$$

which is induced by multiplication by  $N$ . The point  $w_N(y_f)$  therefore corresponds to the point  $\Gamma_0(N) \cdot \tau_f$  of  $\Gamma_0(N) \backslash \mathcal{H}$ , where  $\tau_f = -\frac{fA}{\alpha} = \frac{f(B+\sqrt{D})}{2N}$ .

**1.3** The field of definition  $K_f$  of  $y_f$  — and therefore also that of  $\pi_A(y_f)$  — is the field generated over  $K$  by the  $j$ -invariants of elliptic curves with complex multiplication by the order  $\mathfrak{o}_f$ . It is the *ring class field* of conductor  $f$  of  $K$ , i.e., the abelian extension of  $K$  which is unramified outside of  $f$  and in which a prime ideal of  $K$  not dividing  $f$  is totally split if and only if it is not only principal, but can be generated by an element which, modulo  $f$ , is congruent to a rational number.

If  $f$  and  $f'$  are relatively prime, then  $K_f$  and  $K_{f'}$  are linearly disjoint over the Hilbert class field  $K_1$  of  $K$ . Also,  $K_{ff'}$  is the compositum of  $K_f$  and  $K_{f'}$ . The same holds true for the rings of integers.

**1.4** We can now state our main finiteness result, in which  $N > 1$  is a fixed positive integer,  $D$  varies over the discriminants of imaginary quadratic fields  $K$  satisfying the Heegner condition of 1.2, and  $f$  varies over positive integers prime to  $N$ .

**1.5 Theorem.** *There are only a finite number of pairs  $(D, f)$  as above such that the point  $\pi_A(y_f) \in A(K_f)$  is a torsion point.*

**2. The first finiteness result**

Theorem 1.5 will result from a quantitative version of the following proposition—see 3.3 below. Proposition 2.1 has already been used in the literature—see [3].

**2.1 Proposition.** *There exists  $f_0 > 0$ , depending on the level  $N$  and the discriminant  $D$ , such that for every  $f > f_0$  relatively prime to  $N$ , the point  $\pi_A(y_f) \in A(K_f)$  is a point of infinite order on the abelian variety  $A$ .*

The proof proceeds in three steps, 2.2 – 2.4.

**2.2** Let  $K_\infty = \bigcup_{f \geq 1} K_f$ . We show that the subgroup of torsion points  $A(K_\infty)_{\text{tors}}$  is finite.

Let  $\ell$  be a prime number which is inert in  $K$ . Every prime ideal  $\lambda$  of  $K_f$  above  $\ell$  has norm  $N\lambda = \ell^2$ . — In fact, writing  $f = \ell^a f'$  with  $f'$  not divisible by  $\ell$  one sees that the prime divisors of  $\ell\mathfrak{o}_{K_1}$  are totally ramified in  $K_{\ell^a}/K_1$  and  $\ell\mathfrak{o}_K$  splits completely in  $K_{f'}$ .

Since  $\ell$  does not divide  $N$  (the primes dividing  $N$  split in  $K$ ), the abelian variety  $A$ , as any quotient of  $J_0(N)$ , has good reduction at  $\lambda$ , and for every  $f \geq 1$  prime to  $N$  the torsion subgroup of order prime to  $\ell$  of  $A(K_f)$  reduces injectively modulo  $\lambda$ . This gives

$$\text{card}(A(K_f)_{\text{tors}}^{\text{non-}\ell}) \mid \text{card}(\tilde{A}_\lambda(\mathbf{F}_{\ell^2})).$$

Taking a second prime  $\ell'$ , distinct from  $\ell$ , which remains in  $K$  we see that

$$\text{card}(A(K_f)_{\text{tors}}) \mid \text{card}(\tilde{A}_\lambda(\mathbf{F}_{\ell^2})) \cdot \text{card}(\tilde{A}_{\lambda'}(\mathbf{F}_{\ell'^2})[\ell^\infty]),$$

where the suffix  $[\ell^\infty]$  signifies taking the  $\ell$ -primary part. This proves 2.2.

**2.3** Over  $\mathbf{C}$ ,  $\pi_A$  lifts to a holomorphic mapping  $F$  on the completed upper half-plane.

$$\begin{array}{ccc} \mathcal{H} \cup \mathbf{P}^1(\mathbf{Q}) & \xrightarrow{F} & \mathbf{C}^{\dim A} \\ \downarrow & & \downarrow \\ \Gamma_0(N) \backslash \mathcal{H} \cup \mathbf{P}^1(\mathbf{Q}) & \xrightarrow{\pi_A} & A(\mathbf{C}) \cong \mathbf{C}^{\dim A} / \Lambda \end{array}$$

We fix  $F$  by requiring that  $F(\infty) = 0$ . Then, for every  $m \geq 1$ , there exists  $M_m \in \mathbf{R}$  such that for all  $\tau \in \mathcal{H}$  with  $\text{Im}(\tau) > M_m$  we have:

$$0 < \|F(\tau)\| < \frac{1}{m} \inf_{0 \neq \gamma \in \Lambda} \|\gamma\|.$$

**2.4** We now conclude the proof of proposition 2.1 first under the additional hypothesis that the Fricke involution  $w_N$  induces an automorphism of  $A$ : According to 2.2, put  $m = \text{card}(A(K_\infty)_{\text{tors}})$ , and pick  $M_m$  for this choice of  $m$  as in 2.3. Let  $f_0 = \frac{2N}{|D|^{1/2}} M_m$ . Then we find, in the notation introduced at the end of 1.2 above, that for every integer  $f$  greater than  $f_0$  and prime to  $N$  one has  $\text{Im}(\tau_f) = \frac{f|D|^{1/2}}{2N} > M_m$ . 2.3 now ensures that the point  $\pi_A(w_N(y_f)) \in A(K_f)$ , which corresponds to  $F(\tau_f)$ , is not an  $m$ -torsion point of  $A$ . In view of our choice of  $m$  it has to be of infinite order. Since the involution  $w_N$  induces an automorphism of  $A$ , the same holds for the Heegner point  $\pi_A(y_f)$  itself.

Finally, in order to prove the proposition for an arbitrary quotient  $A$  of  $J_0(N)$ , not necessarily invariant under  $w_N$ , one only has to modify the preceding argument by applying 2.3 to  $w_N(A)$  rather than  $A$ .

**2.5 Remarks.** (1) For  $f$  as above, put  $G_f = \text{Gal}(K_f/K)$ . Assume that  $A$  is of dimension 1, i.e., a (modular) elliptic curve. For a ring class character  $\chi \in \widehat{G}_f$  of conductor dividing  $f$ , we define the  $L$ -function of  $A$  twisted by  $\chi$  by the following Euler product, which converges for  $\text{Re}(s) > 3/2$ .

$$L(A/K, \chi, s) = \prod_{\mathfrak{p} \subset \mathfrak{o}_K} \det(1 - \text{Frob}_{\mathfrak{p}} \cdot \chi(\text{Frob}_{\mathfrak{p}}) \mathbf{N}\mathfrak{p}^{-s} \mid V_l(A)_{I_{\mathfrak{p}}})^{-1}$$

Here,  $\text{Frob}_{\mathfrak{p}}$  denotes the arithmetic Frobenius, and we put  $\chi(\text{Frob}_{\mathfrak{p}}) = 0$  for  $\mathfrak{p} \mid f$ . It follows from the Heegner condition of 1.2 that the order of  $L(A/K, \chi, s)$  at  $s = 1$  is odd and therefore that  $L(A/K, \chi, 1) = 0$ . Write  $\langle \cdot, \cdot \rangle_f$  for the sesquilinear extension to  $A(K_f) \otimes_{\mathbf{Z}} \mathbf{C}$  of the canonical Néron-Tate height pairing on  $A(K_f)$ . Finally, put  $e_\chi = \frac{1}{\#G_f} \sum_{\sigma \in G_f} \chi(\sigma)^{-1} \sigma$ . Then the following formula is conjectured to hold, with the real period  $\omega_A$  and some nonzero rational number  $r = r(D, f)$ .

$$(2.6) \quad L'(A/K, \chi, 1) = r \frac{\omega_A}{\sqrt{|D|}} \langle e_\chi y_f, e_\chi y_f \rangle_f$$

In the particular case where  $f = 1$ , this is the well-known theorem of B.H. Gross and D. Zagier [4]. The generalization 2.6 is not completely proved yet. Assuming it, our

theorem shows that, for every sufficiently big  $f$ , there is at least one ring class character  $\chi \in \widehat{G}_f$  such that  $L'(A/K, \chi, 1) \neq 0$ . On the other hand, guided by results of Rohrlich's [11], one may wonder whether, given  $K$ , there are only a finite number of pairs  $(f, \chi)$ , with  $f \geq 1$  prime to  $N$  and  $\chi \in \widehat{G}_f$  a character of conductor  $f$ , such that  $L'(E/K, \chi, 1) = 0$ .

(2) The second step in the above proof transfers to our situation (and simplifies) an argument given in a more general context by S. Bloch and C. Schoen — see [12].

(3) The above proposition generalizes an analogous result proved in a particular case by P.F. Kurčánov [6]. The proof given by Kurčánov is certainly different from ours, but does rely on similar principles.

### 3. Effectivity questions

**3.1** It follows from the Pólya-Vinogradov theorem that we may always find distinct prime numbers  $\ell, \ell'$  as in 2.2 such that  $\ell, \ell' < |D|^c$  for an absolute constant  $c$ . This gives the following bound for  $m$ .

$$\text{card}(A(K_\infty)_{\text{tors}}) \leq \text{card}(\widetilde{A}_\lambda(\mathbf{F}_{\ell^2})) \cdot \text{card}(\widetilde{A}_{\lambda'}(\mathbf{F}_{\ell'^2})) \leq ((\ell+1)(\ell'+1))^{2 \dim A} \leq |D|^{4c \dim A}$$

**3.2** In 2.3, we may always take

$$M_m = c_1 + \frac{\log(m)}{2\pi},$$

for some constant  $c_1$  depending on the map  $\pi_A$ . Indeed, a local parameter at  $\infty$  is given by  $e^{2i\pi\tau}$ , and  $|e^{2i\pi\tau}| = e^{-2\pi\text{Im}(\tau)}$ .

**3.3 Proof of theorem 1.5.** Let us put together 2.4, 3.1 and 3.2. We see that there exists an absolute constant  $c_0$  and a constant  $c_1$  depending on  $A$  such that for all positive integers  $f$  prime to  $N$  and satisfying

$$f > \frac{2N}{|D|^{1/2}} \{c_1 + c_0 \dim A \log |D|\},$$

the Heegner point  $\pi_A(y_f)$  has infinite order in  $A$ . For  $|D|$  sufficiently big, this inequality holds for any  $f \geq 1$ . This concludes the proof of theorem 1.5.

### 4. The anticyclotomic $\mathbf{Z}_p$ -extension and Mazur's module of Heegner points

We will now restrict to the case where  $A/\mathbf{Q}$  is of dimension 1, i.e.,  $A$  is a (modular) elliptic curve, assumed to be of conductor  $N$ . Let  $p$  be a prime number which stays prime

in  $K$ , and such that  $a_p = p + 1 - \#(\tilde{A}_p(\mathbf{F}_p))$ , the eigenvalue of the Hecke-operator  $T_p$  on  $A$ , is not divisible by  $p$ . In other words, assume that  $p$  is ordinary for  $A$ .

Let  $H_\infty = \bigcup H_n$  be the anticyclotomic  $\mathbf{Z}_p$ -extension of our fixed imaginary quadratic field  $K$ , i.e.,  $H_\infty$  is the unique  $\mathbf{Z}_p$ -extension of  $K$  contained in  $K_{p^\infty} = \bigcup K_{p^{n+1}}$ . We consider the Heegner points  $z_n = \text{tr}_{K_{p^{n+1}}/H_n}(\pi_A(y_{p^{n+1}})) \in A(H_n)$  and following Mazur [9], no. 19, we write  $\mathcal{E}_\infty$  for the projective limit (with respect to the trace maps) of the submodules  $\mathcal{E}_n$  of  $(E(H_n) \otimes \mathbf{Z}_p)/(\text{torsion})$  which are generated by all the conjugates of  $z_n$ . For  $n \geq 2$ , the points on different levels are linked by the following distribution relations, which are immediate consequences of [10], p. 430.

$$\text{tr}_{H_{n+1}/H_n}(z_{n+1}) = a_p z_n - z_{n-1}$$

$\mathcal{E}_\infty$  is an Iwasawa module, i.e., a finitely generated module over  $\Lambda = \mathbf{Z}_p[[\Gamma]] = \mathbf{Z}_p[[T]]$ , where  $\Gamma = \text{Gal}(H_\infty/K)$ . Moreover, as Mazur observed [9], no. 19,  $\mathcal{E}_\infty$  is a  $\Lambda$ -module (in fact, free) of rank 1 if and only if  $z_n$  is a point of infinite order for (any, and thus for all)  $n \gg 0$ . Mazur conjectured that this is always the case. Note that this conjecture is a special instance of the question formulated at the end of remark 2.5(1).

Recall the definitions of the relevant Selmer groups. For any number field  $F$  and any  $m \geq 2$ , the  $m$ -Selmer group of  $A$  over  $F$  is defined to be the torsion group

$$\text{Sel}_m(A/F) = \ker\left(H^1(F, A_m) \longrightarrow \prod_v H^1(F_v, A)_m\right).$$

Via direct limits, we obtain  $\mathbf{Q}_p/\mathbf{Z}_p$ -modules

$$\text{Sel}_{p^\infty}(A/F) = \varinjlim \text{Sel}_{p^n}(A/F), \quad \text{Sel}_{p^\infty}(A/H_\infty) = \varinjlim \text{Sel}_{p^\infty}(A/H_n).$$

**4.1 Theorem.** *If  $z_n$  is a point of infinite order for  $n \gg 0$ , then the Selmer group  $\text{Sel}_{p^\infty}(A/K)$  contains a subgroup isomorphic to  $\mathbf{Q}_p/\mathbf{Z}_p$ .*

The following immediate consequence of this is particularly interesting when the order of vanishing of  $L(A/\mathbf{Q}, s)$  at  $s = 1$  is at least 2.

**4.2 Corollary.** *Assume that the  $p$ -part  $\text{III}(A/K)(p^\infty)$  of the Tate-Šafarevič group of  $A$  over  $K$  is finite and that  $z_n$  is a point of infinite order for  $n \gg 0$ . Then  $\dim A(K) \otimes \mathbf{Q} \geq 1$ .*

**4.3 Proof of theorem 4.1 (argument suggested by K. Rubin).** Put  $\mathcal{H}_\infty = \varinjlim \mathcal{E}_n \otimes \mathbf{Q}_p/\mathbf{Z}_p$ .

By the assumption of the theorem,  $\mathcal{E}_\infty$  is a rank one  $\Lambda$ -module, so its coinvariants  $(\mathcal{E}_\infty)_\Gamma$  admit a quotient isomorphic to  $\mathbf{Z}_p$ . Therefore the invariants  $\mathcal{H}_\infty^\Gamma \subset \text{Sel}_{p^\infty}(A/H_\infty)^\Gamma \subset H^1(H_\infty, A_{p^\infty})^\Gamma$  contain a copy of  $\mathbf{Q}_p/\mathbf{Z}_p$ . However, in our case the  $p^\infty$ -Selmer group along the anticyclotomic extension is “controlled” in the sense that the canonical map  $\text{Sel}_{p^\infty}(A/K) \rightarrow \text{Sel}_{p^\infty}(A/H_\infty)^\Gamma$  has finite kernel and cokernel. This control can be wielded locally, the only interesting place being at  $p$ . More precisely, write the local descent sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & A((H_n)_p) \otimes \mathbf{Q}_p/\mathbf{Z}_p & \longrightarrow & H^1((H_n)_p, A_{p^\infty}) & \longrightarrow & \widehat{A((H_n)_p) \otimes \mathbf{Z}_p} \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & A((K)_p) \otimes \mathbf{Q}_p/\mathbf{Z}_p & \longrightarrow & H^1((K)_p, A_{p^\infty}) & \longrightarrow & \widehat{A((K)_p) \otimes \mathbf{Z}_p} \longrightarrow 0 \end{array}$$

where the last vertical arrow is given by the dual of the trace map on local points. Its kernel is bounded independently of  $n$  since  $p$  is ordinary for  $A$  and thus the universal local traces have finite index in the local points  $A((K)_p)$ . This control theorem is due to Mazur [8]; for our situation see Manin [7], Thm. 4.5 together with Cor. 4.11(a).

**4.4 Remarks.** (1) Bertolini [1] (see also [2]) has established an Iwasawa theoretic analogue of Kolyvagin’s method to prove in particular (under additional hypotheses on the prime  $p$ ) that, if  $\mathcal{E}_\infty$  is indeed of rank 1, then it agrees with the dual of the Selmer group up to a torsion module for which he can exhibit an annihilating power series.

(2) 4.1 provides an example of how the behaviour of higher Heegner points (granting the non-triviality assumption) govern the arithmetic of  $E$  over  $K$ , and therefore over  $\mathbf{Q}$ . Another striking instance of such a relationship was given by Kolyvagin in [5]. It also depends on an initial non-triviality conjecture, and is more like an  $\ell$ -adic descent, for some fixed prime  $\ell$  different from the primes entering into the conductors of the Heegner points. It would be interesting to be able to combine these two theories.

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