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A SURVEY ON DRINFELD MODULAR FORMS

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Dedicated to Professor Masatoshi Ikeda on the occasion of his 70^{th} birthday

In his famous paper [4] of 1974, V.G. Drinfeld introduced what he then called "elliptic modules". These are in several respects similar to elliptic curves or abelian varieties. Elliptic modules = Drinfeld modules may be described

- analytically through lattices in a complete algebraically closed field C of characteristic p by some sort of "Weierstrass uniformization";
- algebraically as a module structure on the additive group $\mathbb{G}_a$ over C.

The interplay between these viewpoints results in a rich theory of moduli schemes and modular forms. In the case of Drinfeld modules of rank two (for which the analogy with elliptic curves is most compelling), the moduli scheme is a curve, and modular forms are holomorphic functions on Drinfeld's upper half-plane Ω with a prescribed transformation behavior under arithmetic groups acting on Ω.

In the present paper, largely expository and without proofs, we restrict to the typical case where the base ring A that substitutes the integers $\mathbb{Z}$ is a polynomial ring $\mathbb{F}_q[T]$ and where the arithmetic group acting equals $\Gamma = \mathrm{GL}(2, \mathbb{F}_q[T])$. We survey the Weierstrass uniformization of Drinfeld modules and the related analytic functions (section 2) and report on the structure of the ring of modular forms in the classical (section 1) and the Drinfeld case (section 3), and on the role played by the respective Eisenstein series, discriminant, and j-invariant functions. In the fourth section, we give a brief account of the congruence properties, proved in [8], of normalized Eisenstein series g_k. In order to investigate zero distribution and growth properties of several classes of modular forms, we introduce and discuss the Bruhat-Tits tree $\mathcal{T}$ and the building map λ from Ω to $\mathcal{T}$. This is used in section 7 to present recent results of G. Cornelissen and the author about Drinfeld-Eisenstein series. In the final section 8, the only one that contains original results and detailed proofs, we determine the distribution of zeroes of the lattice functions $\alpha_k(z)$ in the fundamental domain $\mathcal{F}$ of Γ on Ω (Theorem 8.11, Corollary 8.12). Rigid analytic contour integration then enables us to give a neat description of the behavior of $|\alpha_k(z)|$ on $\mathcal{F}$ (Formulas 8.14 and 8.19, Theorem 8.20).

Even in our restricted framework, we had to omit several important topics indispensable for a deeper study: the theories of Hecke operators, of Goss polynomials, of conditionally convergent Eisenstein series, Fourier analysis on $\mathcal{T}$ ... are not even mentioned. The reader will find some hints in [8] and [9].

Our leading principle has been to present and discuss definitions, constructions, and ideas but to leave out full proofs as long as they refer to established results and are available in the literature. This applies to sections 1 to 6 of the present paper. In contrast, the results of sections seven (proofs of which will appear in [11]) and eight (for which full proofs are given) are new.

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1. The classical setting (e.g. [20], [22], [23]).

Let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be the $\mathbb{Z}$-module generated by two $\mathbb{R}$-linearly independent complex numbers ω_1, ω_2, i.e., a *lattice* in $\mathbb{C}$. With Λ we associate its Weierstrass function

(1.1) $$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda}{}' \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2},$$

where the $\sum'$ indicates summation over the non-zero elements ω of Λ. Then $\wp_\Lambda$ is meromorphic and Λ-invariant on $\mathbb{C}$, and satisfies the differential equation

(1.2) $$\wp_\Lambda'^{\,2} = 4\wp_\Lambda^3 - g_2\wp_\Lambda - g_3$$

with certain constants $g_i = g_i(\Lambda)$ $(i = 2, 3)$, where

$$\Delta := g_2^3 - 27g_3^2 \neq 0. \tag{1.3}$$

This means, $(\wp(z), \wp'(z)) \in \mathbb{C}^2$ is a point on the affine curve E_Λ^{aff} (smooth since $\Delta \neq 0$) with equation

$$Y^2 = 4X^3 - g_2X - g_3. \tag{1.4}$$

(1.5) Such curves (or rather their projective models E_Λ given in $\mathbb{P}^2(\mathbb{C})$ by the homogeneous equation

$$Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3)$$

are known as elliptic curves. The above yields a biholomorphic isomorphism of the complex torus $\mathbb{C}/\Lambda$ with $E_\Lambda(\mathbb{C})$, well-defined through its restriction to $(\mathbb{C} - \Lambda)/\Lambda$ by $z \longmapsto (\wp(z) : \wp'(z) : 1)$. Note that $E_\Lambda(\mathbb{C})$ inherits a group structure from $\mathbb{C}/\Lambda$, which may however be defined in purely algebraic terms on the algebraic curve E_Λ, and which turns E_Λ into an abelian variety. In fact, each elliptic curve $E/\mathbb{C}$ has the form $E = E_\Lambda$ for some lattice Λ as above, and two such, E_Λ and $E_{\Lambda'}$, are isomorphic as abelian varieties (i.e., as algebraic curves through some isomorphism preserving origins) if and only if Λ' and Λ are *homothetic* ($\Lambda' = c\Lambda$, some $c \in \mathbb{C}^*$).

(1.6) Each lattice is homothetic to some lattice $\Lambda_\omega = \mathbb{Z}\omega + \mathbb{Z}$, where $\omega \in \mathbb{C} - \mathbb{R}$ is uniquely determined up to the action of $\mathrm{GL}(2, \mathbb{Z})$ through fractional linear transformations:

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\omega = \frac{a\omega + b}{c\omega + d}.$$

We may even choose ω in the upper half-plane $H := \{\omega \in \mathbb{C} \mid \mathrm{im}(\omega) > 0\}$, which then is determined up to the action of $\Gamma := \mathrm{SL}(2, \mathbb{Z})$ = the *modular group*.

(1.7) In order to determine the constants g_i in (1.2), we consider the *Eisenstein series*

$$E_k(\Lambda) := \sum_{\lambda \in \Lambda}{}' \frac{1}{\lambda^k} \quad (k \geq 4 \text{ even}).$$

(The series converges for $k > 2$ but yields zero for k odd.) For $\Lambda = \Lambda_\omega$ we also put

$$E_k(\omega) = E_k(\Lambda_\omega) = \sum_{a,b \in \mathbb{Z}}{}' \frac{1}{(a\omega + b)^k},$$

which is holomorphic on H. Then the Laurent expansion of $\wp_\Lambda(z)$ around $z=0$ is

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{k\geq 2}(k+1)E_{k+2}(\Lambda)z^k. \tag{1.8}$$

Comparing the first coefficients yields

$$g_2(\Lambda) = 60E_4(\Lambda), \quad g_3(\Lambda) = 140E_6(\Lambda). \tag{1.9}$$

It is obvious from $E_k(c\Lambda) = c^{-k}E_k(\Lambda)$ that

$$E_k(\frac{a\omega+b}{c\omega+d}) = (c\omega+d)^k E_k(\omega), \quad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma. \tag{1.10}$$

Holomorphic functions on H satisfying this rule (plus a holomorphy condition at ∞: see below) are called *modular forms of weight* k for Γ; they form a $\mathbb{C}$-vector space M_k. Hence e.g. $E_k \in M_k$ and $\Delta \in M_{12}$, where $\Delta(z) = g_2^3(z) - 27g_3^2(z)$, $g_i(z) = g_i(\Lambda_z)$.

(1.11) Suppose the holomorphic function f on H satisfies the functional equation (1.10) under Γ. Then in particular, f is $\mathbb{Z}$-periodic and has a Fourier expansion

$$f(z) = \tilde{f}(q) = \sum_{n\in\mathbb{Z}} a_n q^n$$

with a Laurent series $\tilde{f}$ in $q(z) = \exp(2\pi i z)$. The required holomorphy condition for f at ∞ is that $a_n = 0$ for $n < 0$. If even $a_n = 0$ for $n \leq 0$, f is called a *cusp form*. The Fourier expansion of E_k is given by

$$E_k(z) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n>0}\sigma_{k-1}(n)q^n, \tag{1.12}$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ and $\zeta(k) = \frac{(2\pi)^k}{2k!}|B_k|$ with the k-th Bernoulli number B_k, $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, ... Comparing coefficients yields that $\Delta(z)$ is a cusp form of weight 12.

(1.13) It is well-known that $j := g_2^3/\Delta$ is a complete invariant for elliptic curves, that is, two curves in Weierstrass form (1.4) are isomorphic if and only if their j-invariants agree.

(As customary, we briefly write the affine equation (1.4) but think of and work with the attached homogeneous equation). Together with (1.6) we get:

(1.14) The j-invariant mapping $z \longmapsto j(z) = g_2^3(z)/\Delta(z)$ gives rise to an isomorphism of Riemann surfaces between $\Gamma \setminus H$, the quotient space of H (mod Γ), and the affine line $\mathbb{C} = \mathbb{A}^1(\mathbb{C})$.

(1.15) Next, define $\mathcal{F} := \{z \in \mathbb{C} \mid -\frac{1}{2} \leq \mathrm{re}(z) \leq \frac{1}{2},\ |z| \geq 1\}$. It is a *fundamental domain* for the action of Γ on H, i.e., $\mathcal{F}$ represents $\Gamma \setminus H$ and up to the obvious identifications on the boundary of $\mathcal{F}$, its elements are inequivalent modulo Γ. There are two special points on $\Gamma \setminus H$ (the *elliptic points*), represented by the 4-th and 3-rd roots of unity i and ρ, and characterized by the fact that their stabilizers Γ_i and Γ_ρ are cyclic groups of orders 4 and 6, respectively. (The stabilizers of points inequivalent to i or ρ are simply the subgroups $\{\pm 1\}$ of Γ.) We have the following basic relation for $0 \neq f \in M_k$:

(1.16)
$$\sum_{z \in \Gamma \setminus H}{}^{*} \nu_z(f) + \frac{1}{2}\nu_i(f) + \frac{1}{3}\nu_\rho(f) + \nu_\infty(f) = \frac{k}{12}.$$

Here $\nu_z(f)$ is the vanishing order of f at z (for $z = \infty$, it is the vanishing order of the power series $\tilde{f}$ in q, see (1.11)), and the sum $\sum^*$ is over the non-elliptic points of $\Gamma \setminus H$. As an easy consequence, $M_4 = \mathbb{C}E_4$, $M_6 = \mathbb{C}E_6$, and more generally, the algebra $M := \bigoplus_{k \geq 0} M_k$ of modular forms is the polynomial ring

(1.17)
$$M = \mathbb{C}[E_4, E_6] = \mathbb{C}[g_2, g_3]$$

in the algebraically independent functions g_2 and g_3.

We will see analogues of the above (and of many more properties of classical modular forms) in the function field setting.

2. The Drinfeld setting ([4], [3], [16], [7], [8], [13]).

Following the general philosophy about similarities between number fields and function fields, we now transfer the contents of section 1 to characteristic p.

(2.1) We let $\mathbb{F}_q$ be the finite field with q elements, of characteristic p, and $A = \mathbb{F}_q[T]$ the polynomial ring in one indeterminate T over $\mathbb{F}_q$. Its quotient field $K = \mathbb{F}_q(T)$ is provided with the degree valuation $v : K \longrightarrow \mathbb{Z} \cup \{\infty\}$ defined by $v(a/b) = \deg\ b - \deg\ a$ $(a, b \in A)$ and the corresponding absolute value $|x| = q^{-v(x)}$. The completion of K with respect to v is the field $K_\infty = \mathbb{F}_q((\pi))$ of formal Laurent series in the uniformizer $\pi = T^{-1}$; its ring of integers, maximal ideal, residue class field are denoted by $O_\infty = \mathbb{F}_q[[\pi]]$, $\mathfrak{m}_\infty = \pi O_\infty$, and $k(\infty) \cong \mathbb{F}_q$, respectively.

The main difference with the classical case stems from the fact that the algebraic closure $\overline{K}_\infty$ of K_∞ has infinite degree over K_∞ and therefore fails to be complete w.r.t the unique extension (also denoted by "$|\ .\ |$") to $\overline{K}_\infty$. We put C for the completion of the valued field $\overline{K}_\infty$ and note that, due to Krasner's lemma, C is both complete and algebraically closed, having the algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$ as its residue field. For such fields there is a function theory like classical complex function theory with results of similar strength [5].

The reader is now invited to flick forward and have a brief look at the dictionary of section 5. We first introduce A-lattices Λ in C, Λ-periodic functions, Drinfeld A-modules, ..., which substitute $\mathbb{Z}$-lattices Λ in $\mathbb{C}$, the Weierstrass function $\wp_\Lambda$, elliptic curves, ..., respectively.

(2.2) An *A-lattice in C* is a finitely generated (hence free) discrete A-submodule Λ of C. Discreteness means that the intersection of Λ with each ball of finite radius is finite. Equivalently, $\Lambda = A\omega_1 + \cdots + A\omega_r$ with K_∞-linearly independent elements $\omega_1, \ldots, \omega_r$ of C. (We will mainly deal with the case where the rank r equals two.) With such a Λ, we associate the function

$$e_\Lambda(z) = z \prod_{\lambda \in \Lambda}{}' (1 - \frac{z}{\lambda}), \tag{2.3}$$

which converges locally uniformly for $z \in C$. The function e_Λ is entire, Λ-periodic, surjective, $\mathbb{F}_q$-linear, has its zeroes, all simple, at Λ, and is given by an everywhere convergent power series

$$e_\Lambda(z) = \sum_{k \geq 0} \alpha_k(\Lambda) z^{q^k}. \tag{2.4}$$

Fix some $a \in A$ and consider the commutative diagram

$$(2.5) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda & \longrightarrow & C & \xrightarrow{e_\Lambda} & C & \longrightarrow & 0 \\ & & \downarrow a & & \downarrow a & & \downarrow \Phi_a^\Lambda & & \\ 0 & \longrightarrow & \Lambda & \longrightarrow & C & \xrightarrow{e_\Lambda} & C & \longrightarrow & 0\,, \end{array}$$

where the two left vertical arrows are multiplication by a, while the map ϕ_a^Λ is defined through the diagram. A closer look shows that

$$(2.6) \qquad \phi_a^\Lambda(z) = az + l_1(a,\Lambda)z^q + \cdots l_N(a,\Lambda)z^{q^N},$$

where $N = \text{rank}(\Lambda) \cdot \deg\ a$ and $l_N(a,\Lambda) \neq 0$.

Furthermore, $a \longmapsto \phi_a^\Lambda$ is additive, multiplicative (i.e., $\phi_{ab}(z) = \phi_a(\phi_b(z))$) and $\mathbb{F}_q$-linear. In other words, defining the multiplication in the set

$$\text{End}_{C,\mathbb{F}_q}(\mathbb{G}_a) = \{\sum l_i X^{q^i} \mid l_i \in C\}$$

of $\mathbb{F}_q$-linear polynomials over C through insertion, $\phi^\Lambda : a \longmapsto \phi_a^\Lambda$ defines a homomorphism of $\mathbb{F}_q$-algebras from A to $\text{End}_{C,\mathbb{F}_q}(\mathbb{G}_a)$. By means of ϕ^Λ, $C = \mathbb{G}_a(C)$ is equipped with a new structure of A-module: $a * z = \phi_a^\Lambda(z)$. Such an A-module structure subject to (2.6) is known as a Drinfeld A-module of rank r over C. Note that the rank of the Drinfeld module ϕ^Λ equals the rank of the lattice Λ.

We regard Drinfeld modules (and notably those of rank two) as analogues of elliptic curves, where the functional equation

$$e_\Lambda(az) = \phi_a^\Lambda(e_\Lambda(z))$$

for e_Λ derived from (2.5) corresponds to (1.2) or rather to the multiplication equation derived from (1.2). The point is that (1.2) defines a $\mathbb{Z}$-module structure on the elliptic curve $\mathbb{C}/\Lambda \xrightarrow{\cong} E_\Lambda(\mathbb{C})$, while (2.5) and (2.6) define the above A-module structure on the additive group scheme $\mathbb{G}_a$.

(2.7) Each homomorphism $\phi : a \longmapsto \phi_a$ subject to (2.6) of $\mathbb{F}_q$-algebras from A to $\mathrm{End}_{C,\mathbb{F}_q}(\mathbb{G}_a)$ is uniquely determined through

$$\phi_T(X) = TX + l_1 X^q + \cdots + l_r X^{q^r},$$

where $l_1, \ldots, l_r \in C$, $l_r \in C^*$ may be freely chosen. Similar to (1.5), each rank-r Drinfeld A-module ϕ comes from a rank-r lattice Λ as above, and for two Drinfeld modules $\phi = \phi^\Lambda$, $\phi' = \phi^{\Lambda'}$ we have equivalence between

(i) ϕ and ϕ' are isomorphic as Drinfeld modules;
(ii) $\exists\, c \in C^*$ such that $\Lambda' = c\Lambda$;
(iii) $\exists\, c \in C^*$ such that $l_i' = c^{1-q^i} l_i$, where the l_i, l_i' are the coefficients of ϕ_T, ϕ_T', respectively.

The functional equation

$$e_\Lambda(Tz) = \phi_T(e_\Lambda(z)) = Te_\Lambda(z) + l_1 e_\Lambda(z)^q + \cdots + l_r e_\Lambda(z)^{q^r}$$

yields the recursion formula

$$[k]\alpha_k = l_1 \alpha_{k-1}^q + \cdots + l_r \alpha_{k-r}^{q^r} \tag{2.8}$$

for the coefficients $\alpha_k = \alpha_k(\Lambda)$ of $e_\Lambda(z)$, with $\alpha_k = 0$ for $k < 0$ and $\alpha_0 = 1$. Here we have abbreviated $[k] = T^{q^k} - T \in A$. Hence each α_k is a polynomial in $l_1, \cdots, l_r$.

(2.9) As in (1.9), the $l_i = l_i(T, \Lambda)$ may be expressed through lattice sums. For $k \in \mathbb{N}$, $k \equiv 0 \pmod{q-1}$, define the *k-th Eisenstein series* $E_k(\Lambda)$ through

$$E_k(\Lambda) = \sum_{\lambda \in \Lambda}{}' \frac{1}{\lambda^k}.$$

Due to our non-archimedean situation and the discreteness of Λ, the series converges (in arbitrary order) for any $k > 0$, but evaluates to zero if $k \not\equiv 0 \pmod{q-1}$. An elementary but somewhat complicated calculation (e.g. [7] II 2.11) yields

$$l_k = \sum_{1 \le j \le k-1} E_{q^{k-j}-1}, l_j^{q^{k-j}} + [k] E_{q^k-1}. \tag{2.10}$$

(2.11) Let us first inspect the rank-one situation. By (2.7), each rank-one Drinfeld module is isomorphic with the Carlitz module ρ defined by $\rho_T(X) = TX + X^q$. It corresponds to a rank-one lattice $L = \overline{\pi}A$ with some $\overline{\pi} \in C$ well-defined up to a $(q-1)$-th root of unity, i.e., up to an element of $\mathbb{F}_q^*$. Choose and fix such a $\overline{\pi}$. From (2.10) we see that

$$1 = [1]E_{q-1}(L) = [1]\overline{\pi}^{1-q}E_{q-1}(A), \quad \text{i.e.,}$$

(2.12) $$\overline{\pi}^{q-1} = [1]E_{q-1}(A) = (T^q - T)\sum_{a\in A}{}' \frac{1}{a^{q-1}}.$$

In particular, $|\overline{\pi}^{q-1}| = |[1]| = q^q$. Furthermore, putting

(2.13) $$D_k = [k][k-1]^q \cdots [1]^{q^{k-1}},$$

the coefficients of e_L are given by

(2.14) $$\alpha_k(L) = \frac{1}{D_k},$$

as follows from (2.8).

3. Drinfeld modular forms ([16], [8]).

(3.1) Next, we consider the case where $r = \mathrm{rank}(\Lambda) = 2$. Then $\Lambda = A\omega_1 + A\omega_2$, and the Drinfeld module $\phi = \phi^\Lambda$ is given by ϕ_T, which we write

$$\phi_T(X) = TX + gX^q + \Delta X^{q^2}.$$

Due to (2.10), $g = g(\Lambda) = [1]E_{q-1}(\Lambda)$ and $\Delta = \Delta(\Lambda) = [1]^q E_{q-1}^{q+1}(\Lambda) + [2]E_{q^2-1}(\Lambda) \neq 0$. As in (1.6), we may scale Λ such that $\Lambda = \Lambda_\omega = A\omega + A$, where $\omega \in \Omega := C - K_\infty$ is uniquely determined up to the action of $\Gamma := \mathrm{GL}(2, A)$. The set Ω is called the *Drinfeld upper half-plane*; it is provided with a natural structure of rigid analytic space in the sense of Tate (cf. [14], [5], [12], [13]). With respect to that structure, the functions $E_k(\omega) := E_k(\Lambda_\omega)$ are holomorphic on Ω, and the obvious rule $E_k(c\Lambda) = c^{-k}E_k(\Lambda)$ for $c \in C^*$ translates to

$$E_k\left(\frac{a\omega + b}{c\omega + d}\right) = (c\omega + d)^k E_k(\omega) \quad \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma\right).$$

(3.2) In order to find the right substitute for (1.11), we must describe the structure of Ω "at infinity". For $z \in C$, define $|z|_i := \inf_{x \in K_\infty} |z-x| = \min_{x \in K_\infty} |z-x|$, which plays the role of the complex imaginary part. It is an exercise to show that

$$\left|\frac{az+b}{cz+d}\right|_i = \frac{|\det\ \gamma|}{|cz+d|^2}|z|_i \quad \text{for } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \gamma \in \mathrm{GL}(2, K_\infty).$$

For c in the value set $q^{\mathbb{Q}}$ of C, the subsets $\Omega_c := \{z \in \Omega \mid |z|_i \geq c\}$ are admissible and stable under $\Gamma_\infty := \{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \mid a, d \in \mathbb{F}_q^*,\ b \in A\}$, and satisfy

$$\Omega_c \cap \gamma(\Omega_c) \neq \emptyset \Rightarrow \gamma \in \Gamma_\infty,$$

provided that $c > 1$. Hence for such c, the canonical map from $\Gamma_\infty \setminus \Omega_c$ to $\Gamma \setminus \Omega$ is injective and even an open immersion of rigid spaces. To determine the quotient $\Gamma_\infty \setminus \Omega_c$, we first divide out the group $\Gamma_\infty^u := \{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \mid b \in A\}$, whose associated quotient map is the restriction of e_A to Ω_c, and then factor modulo the action of $\Gamma_\infty/\Gamma_\infty^u \cong \{\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \mid a, d \in \mathbb{F}_q^*\}$. For certain reasons, we take

$$t(z) := e_L(\overline{\pi} z)^{-1} = \overline{\pi}^{-1} e_A(z)^{-1} \tag{3.3}$$

as the coordinate on $\Gamma_\infty^u \setminus \Omega_c = A \setminus \Omega_c$. (The change $e_A \longrightarrow e_A^{-1}$ doesn't matter since e_A has neither zeroes nor poles on Ω_c, and the factor $\overline{\pi}$ serves for normalizing purposes, as does $2\pi i$ classically.) Since $t(\frac{a}{d}z) = \frac{d}{a}t(z)$ for $a, d \in \mathbb{F}_q^*$, the group $\Gamma_\infty/\Gamma_\infty^u$ acts on $t(\Omega_c) \subset C$ like the group of $(q-1)$-th roots of unity. Hence $t^{q-1} : \Gamma_\infty \setminus \Omega_c \hookrightarrow C$ defines an injection (in fact: as open immersion of analytic spaces). The following is crucial.

3.4 Proposition ([8] 5.5 + 5.6). *Let $c \in q^{\mathbb{Q}}$, $c > 1$. For $z \in \Omega_c$, $\log_q |t(z)|$ depends only on $|z|_i$. There exists a real constant $c_0 > 1$ such that $|z|_i \leq -\log_q |t(z)| \leq c_0 |z|_i$. The map $z \longmapsto t^{q-1}(z)$ identifies $\Gamma_\infty \setminus \Omega_c$ with the pointed ball $B_r - \{0\} = \{z \in C \mid 0 < |z| \leq r\}$, where $r = r(c)$ tends to zero with $c \longrightarrow \infty$.*

(We don't need the precise formula for $r(c)$, given *loc. cit.*) We are now ready to make the following basic definition.

3.5 Definition. *Let k be a non-negative integer and m a residue class $(\mathrm{mod}\ q-1)$. A modular form of weight k and type m for Γ is a function $f : \Omega \longrightarrow C$*

that satisfies

(i) $f(\gamma z) = \frac{(cz+d)^k}{(\det\ \gamma)^m} f(z) \quad ((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) = \gamma \in \Gamma,\ \gamma z = \frac{az+b}{cz+d})$;

(ii) f *is holomorphic on* Ω*;*

(iii) f *is holomorphic at infinity.*

Here the last condition signifies that the Laurent series expansion of f restricted to Ω_c with respect to t (which exists, since f by (i) is invariant under Γ_∞^u) has no polar terms.

We further define

$$
\begin{aligned}
M_{k,m} &= C\text{-vector space of modular forms of weight } k \text{ and type } m \\
M &= \bigoplus_{k,m} M_{k,m},\ M_0 = \bigoplus_k M_{k,0}, \text{ and for } 0 \neq f \in M_{k,m} \\
\nu_z(f) &= \text{vanishing order of } f \text{ at } z \in \Omega \\
\nu_\infty(f) &= \text{vanishing order of } f \text{ at } \infty = \text{order of the power} \\
&\ \text{series } \tilde{f} \text{ such that } f(z) = \tilde{f}(t(z)) \text{ for } |z|_i \text{ large.}
\end{aligned}
$$

Remark. Note that we are dealing here with $\Gamma = \mathrm{GL}(2, A)$, which in contrast to $\mathrm{SL}(2, A)$ admits the non-trivial characters $\gamma \longmapsto (\det\ \gamma)^m$. This explains the slightly more general transformation rule (i) compared to e.g. (1.10). The weight and type are not quite independent: If $0 \neq f \in M_{k,m}$ then $k \equiv 2m \pmod{q-1}$ (apply (i) to the scalar matrix $(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix})$). Further, if f has a t-expansion $\sum a_n t^n(z)$ then $a_n \neq 0$ implies that $n \equiv m \pmod{q-1}$.

We next give a list of natural examples of modular forms. In each case it is easy to verify conditions (i) and (ii) of (3.5); the holomorphy at infinity comes out by calculating the t-expansion (see [8]). Recall that Λ_z is the lattice $Az + A$.

3.7 Examples. (i) For $k \in \mathbb{N}$, $k \equiv 0 \pmod{q-1}$, the function $E_k :\ z \longmapsto E_k(\Lambda_z) = \sideset{}{'}\sum_{a,b \in A} \frac{1}{(az+b)^k}$ is non-zero and lies in $M_{k,0}$.

(ii) Fix $0 \neq a \in A$, and consider the polynomial

$$\phi_a^{\Lambda_z}(X) = \sum_{0 \leq i \leq 2 \deg a} l_i(a, \Lambda_z) X^{q^i}$$

of (2.6). Letting z vary on Ω, $l_i(a, z) := l_i(a, \Lambda_z)$ becomes a function in z, actually

$l_i(a,\cdot) \in M_{k,0}$ with $k = q^i - 1$. For $a = T$ and $i = 1, 2$, we get functions $g \in M_{q-1,0}$ and $\Delta \in M_{q^2-1,0}$ (compare (3.1)). Note that Δ nowhere vanishes on Ω.
(iii) Similarly, write

$$e_{\Lambda_z}(w) = \sum_{i \geq 0} \alpha_i(z) w^{q^i}.$$

Then α_i is a modular form of weight $q^i - 1$ and type 0.
(iv) Let $H \subset \Gamma$ be the subgroup of matrices $\{\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \mid a \in \mathbb{F}_q^*,\ b \in A\}$. For $k > 0$, $0 \leq m \leq \frac{k}{q+1}$, $k \equiv 2m \pmod{q-1}$, the *Poincaré series*

$$P_{k,m}(z) = \sum_{\gamma \in H \setminus \Gamma} \frac{(\det\ \gamma)^m}{(cz+d)^k} t^m(\gamma z)$$

is well-defined, converges, and defines an element $0 \neq P_{k,m} \in M_{k,m}$. Here, as usual, an element γ of Γ representing a class in $H \setminus \Gamma$ is written as $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

Comparing with section 1, our present Eisenstein series E_k (but in view of (1.8) also the α_i) correspond to the elliptic Eisenstein series E_k of (1.7), whereas the $l_i(a,z)$ are similar to certain functions derived from coordinates on elliptic curves. In particular, g and Δ play the role of the classical functions g_2, g_3 and Δ, respectively.

(3.8) From (2.7) it is obvious that $j(\phi) := \frac{g^{q+1}}{\Delta}$ is a complete invariant for rank-two Drinfeld modules ϕ defined by

$$\phi_T(X) = TX + gX^q + \Delta X^{q^2}.$$

That is, two such, given by coordinates (g, Δ) and (g', Δ'), are isomorphic if and only if $\frac{g^{q+1}}{\Delta} = \frac{g'^{q+1}}{\Delta'}$. Therefore, the holomorphic Γ-invariant function $j : \Omega \longrightarrow C$ given by $z \longmapsto g(z)^{q+1}/\Delta(z)$ identifies the quotient $\Gamma \setminus \Omega$ with the affine line $C = \mathbb{A}^1(C)$ both set-theoretically and analytically.

(3.9) As in (1.15), there are *elliptic points* z on Ω, namely those whose stabilizer groups Γ_z in Γ are strictly larger than $Z(\mathbb{F}_q) = \{\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) \mid a \in \mathbb{F}_q^*\}$, which is the stabilizer of generic points. We have equivalence between

(i) z is elliptic;

(ii) z is Γ-conjugate to an element of $\mathbb{F}_{q^2} - \mathbb{F}_q$;
(iii) $\Gamma_z \cong \mathbb{F}_{q^2}^*$;
(iv) $j(z) = 0$.

Hence there is only one equivalence class of elliptic points, represented by any element e of $\mathbb{F}_{q^2} - \mathbb{F}_q$. Correspondingly, the map $j : \Omega \longrightarrow C$ is unramified off elliptic points, and ramified with index $(q^2-1)/(q-1) = q+1$ in elliptic points. As an easy consequence of (3.8) we get the relation (proof: [7] V sect. 5)

$$\sum_{z \in \Gamma \backslash \Omega}{}^{*} \nu_z(f) + \frac{\nu_e(f)}{q+1} + \frac{\nu_\infty(f)}{q-1} = \frac{k}{q^2-1}, \tag{3.10}$$

valid for any $f \in M_{k,m}$ which doesn't vanish identically. As in (1.16), the sum $\sum^*$ is over the non-elliptic classes in $\Gamma \setminus \Omega$. As an example

$$\begin{array}{lcll} \nu_e(g) & = & 1, & \nu_\infty(g) = \nu_z(g) = 0 \quad (z \text{ non-elliptic}), \\ \nu_\infty(\Delta) & = & q-1, & \nu_z(\Delta) = 0 \qquad\qquad\quad (z \in \Omega). \end{array} \tag{3.11}$$

Letting now $h := P_{q+1,1}$, we get

3.12 Theorem ([16], [8]). (i) $M_0 = \bigoplus_{k \geq 0} M_{k,0} = C[g, \Delta]$

(ii) $M = \bigoplus_{\substack{k \geq 0 \\ m (\mathrm{mod}\ q-1)}} M_{k,m} = C[g,h]$ *with algebraically independent forms* (g, Δ), (g, h), *respectively.*

About Δ we have the following result, which is analogous to Jacobi's formula

$$\Delta(z) = (2\pi i)^{12} q \prod_{n \geq 1} (1-q^n)^{24} \quad (q = e^{2\pi i z})$$

for the classical discriminant.

3.13 Theorem [6]. *For* $0 \neq a \in A$ *define the polynomial* $f_a(X) \in A[X]$ *derived from the Carlitz module* ρ *as* $f_a(X) = X^{q^{\deg\ a}} \rho_a(X^{-1})$. *Then* $\Delta(z)$ *has the product expansion in*

$t(z)$ convergent for $|z|_i$ sufficiently large:

$$\Delta(z) = -\overline{\pi}^{q^2-1} t^{q-1} \prod_{a \text{ monic}} f_a(t)^{(q^2-1)(q-1)}.$$

As a consequence, $\overline{\pi}^{1-q^2}\Delta$ has a t-expansion with coefficients in A. More generally, there is the next integrality result.

3.14 Theorem ([16], [8]).

(i) *The subset $M_{k,m}(A)$ of forms having their t-coefficients in A defines an A-structure on the C-vector space $M_{k,m}$.*

(ii) *Defining $g_{\text{new}} := \overline{\pi}^{1-q} g$ and $\Delta_{\text{new}} := \overline{\pi}^{1-q^2}\Delta$, we have*

$$M_0(A) := \bigoplus_k M_{k,0}(A) = A[g_{\text{new}}, \Delta_{\text{new}}].$$

(iii) *Similarly, $M(A) := \bigoplus_{k,m} M_{k,m}(A) = A[g_{\text{new}}, h]$.*

As we see from (3.10), h never vanishes on Ω, therefore $\nu_\infty(h) = 1$ and h^{q-1} must be proportional to Δ. Comparing the leading terms yields

3.15 Theorem ([8] 9.1). $\Delta_{\text{new}} = -h^{q-1}$.

(3.16) We finally mention how new modular forms may be constructed from differentiating old ones. First note that $\frac{dt}{dz} = -\overline{\pi} t^2$ since $\frac{de_L(z)}{dz} = 1$. Therefore,

$$\theta := \overline{\pi}^{-1} = \frac{d}{dz} = -t^2 \frac{d}{dt}$$

acts on the power series ring $A[[t]]$. Applying θ to (3.5) (i) gives the transformation rule

$$(\theta f)(\gamma z) = \frac{(cz+d)^{k+2}}{(\det\ \gamma)^{m+1}} (\theta f)(z) + \frac{k \cdot c}{\overline{\pi}} \frac{(cz+d)^{k+1}}{(\det\ \gamma)^{m+1}} f(z)$$

for θf. Letting $f = \Delta$ and dividing by Δ yields

$$E(\gamma z) = \frac{(cz+d)^2}{\det\ \gamma} E(z) - \frac{c(cz+d)}{\det\ \gamma}$$

for $E(z) := \frac{\theta\Delta(z)}{\Delta(z)}$. With a small calculation, we get the next assertion.

3.17 Proposition. *For $f \in M_{k,m}$ define $\partial_k f := \theta f + kEf$. Then*

(i) $\partial_k f \in M_{k+2,m+1}$

(ii) $\partial = (\partial_k)_{k\in\mathbb{N}_0}$ *is a graded differential operator of weight two on M. That is, for $f_i \in M_{k_i,m_i}$ $(i = 1, 2)$, the rule*

$$\partial_{k_1+k_2}(f_1 \cdot f_2) = \partial_{k_1}(f_1)f_2 + f_1\partial_{k_2}(f_2)$$

holds.

(iii) $M(A)$ *is stable under ∂.*

Again from comparing leading coefficients, we get

3.18 Corollary. $\partial g_{\mathrm{new}} = h$.

4. Some congruence properties of modular forms [8].

(4.1) Recall that $[k] = T^{q^k} - T$, $D_k = [k][k-1]^q \cdots [1]^{q^{k-1}}$. We further define $L_k = [k][k-1]\cdots[1]$. Their arithmetic meaning is:

$$\begin{array}{lcll} [k] & = & \prod f & f \text{ monic, prime, of degree a divisor of } k \\ D_k & = & \prod f & f \text{ monic of degree } k \\ L_k & = & \text{l.c.m.}\{f \mid f \text{ monic of degree } k\}. & \end{array}$$

The *special Eisenstein series* of weight of shape $q^k - 1$ are particularly important, see (2.10). We normalize them as follows:

(4.2) $$g_k := (-1)^{k+1}\overline{\pi}^{1-q^k} L_k E_{q^k-1}.$$

4.3 Proposition ([8] 6.9). *The g_k satisfy*

(i) $g_k = 1 + o(t)$

(ii) $g_k \in M_{q^k-1,0}(A)$;

(iii) $g_0 = 1$, $g_1 = g_{\mathrm{new}}$, *and for* $k \geq 2$,

$$g_k = -[k-1]g_{k-2}\Delta_{\mathrm{new}}^{q^{k-2}} + g_{k-1}g_{\mathrm{new}}^{q^{k-1}}.$$

Therefore there exists a unique polynomial $A_k(X,Y) \in A[X,Y]$, isobaric of weight $q^k - 1$ (where the weights of X and Y are $q-1$ and q^2-1, respectively), such that $g_k = A_k(g_{\mathrm{new}}, \Delta_{\mathrm{new}})$. With a view towards (3.15), we further put $B_k(X,Z) = A_k(X, -Z^{q-1})$. Then $B_k(g_{\mathrm{new}}, h) = g_k$.

We next consider congruence properties. For an ideal $\mathfrak{a}$ of A and $f, f' \in M(A)$, $f \equiv f' \pmod{\mathfrak{a}}$ means congruence $(\mathrm{mod}\ \mathfrak{a})$ of all the t-coefficients.

4.4 Proposition (*loc. cit.* 6.11): *Let* $\mathfrak{p}$ *be a prime ideal of* A *of degree* d. *Considering* g_k *as a power series in* t, *the congruence*

$$g_{k+d}(t) \equiv g_k(t^{q^d}) \pmod{\mathfrak{p}}$$

holds for $k \geq 0$. *In particular,* $g_d \equiv 1 \pmod{[d]}$.

As we will see below, this is "the only congruence mod $\mathfrak{p}$" for elements of $M(A)$. To make this precise, we define the homomorphism $\epsilon_{\mathfrak{p}} : M(A) \longrightarrow \mathbb{F}_{\mathfrak{p}}[[t]]$ composed of the canonical injection

$$\begin{array}{ccc} M(A) & \hookrightarrow & A[[t]] \\ f & \longmapsto & t\text{-expansion of } f \end{array}$$

and the reduction map $\sim : A \longrightarrow \mathbb{F}_{\mathfrak{p}} = A/\mathfrak{p}$ (everything derived from reduction $(\mathrm{mod}\ \mathfrak{p})$ will equally be denoted by a "$\sim$"). We further let $\epsilon_{\mathfrak{p},0}$ be the restriction of $\epsilon_{\mathfrak{p}}$ to $M_0(A)$.

4.5 Theorem (*loc. cit.* 12.1 + 12.3). *The kernel of* $\epsilon_{\mathfrak{p}}$ (*of* $\epsilon_{\mathfrak{p},0}$) *is the ideal of* $M(A) = A[g_{\mathrm{new}}, h]$ (*of* $M_0(A) = A[g_{\mathrm{new}}, \Delta_{\mathrm{new}}]$) *generated by* $g_d - 1$.

4.6 Remark. By the theorem, $\tilde{A}_d(\tilde{g}_{\mathrm{new}}, \tilde{\Delta}_{\mathrm{new}}) = 1 = \tilde{B}_d(\tilde{g}_{\mathrm{new}}, \tilde{h})$, where ($\sim$) means reduction $(\mathrm{mod}\ \mathfrak{p})$. Actually $\tilde{A}_d$ equals the reduction $\tilde{F}_d$ of the polynomial $F_d(X,Y)$ (*loc. cit.* 11.6), i.e., the *supersingular polynomial* $(\mathrm{mod}\ \mathfrak{p})$, which describes the Hasse invariant of Drinfeld A-modules in characteristic $\mathfrak{p}$. This has rather strong consequences. For example, $\tilde{F}_d$ is square-free, which gives the simplicity of zeroes of g_d as a function on Ω (see (7.13)).

5. Dictionary classical versus Drinfeld modular forms.

We summarize here some of the analogies and similarities between data and results in the classical and in the Drinfeld setting of modular forms.

Number field side	Function field side
$\mathbb{Z}$	A
$\mathbb{Q}$, absolute value "$\| \, . \, \|$"	K, absolute value "$\| \, . \, \|$"
$\mathbb{R}$	K_∞
$\mathbb{C}$	C
imaginary part $\mathrm{im}(z)$	$\|z\|_i$
H (or rather $H^\pm = \mathbb{C} - \mathbb{R}$)	$\Omega = C - K_\infty$
$\Gamma = \mathrm{SL}(2, \mathbb{Z})$ (or rather $\mathrm{GL}(2, \mathbb{Z})$)	$\Gamma = \mathrm{GL}(2, A)$
elliptic points on H (equivalent with 4-th or 3-rd roots of unity)	elliptic points on Ω (equivalent with (q^2-1)-th roots of unity)
$\mathbb{G}_m$ = multiplicative group	ρ = Carlitz module
$2\pi i$, $2\pi i\mathbb{Z}$	$\overline{\pi}$, $\overline{\pi}A = L$
$\wp_\Lambda$ Weierstrass function	e_Λ
elliptic curve	Drinfeld module of rank two
$q(z) = e^{2\pi i z}$	$t(z) = e_L(\overline{\pi}z)^{-1}$
elliptic modular forms	Drinfeld modular forms
Eisenstein series	Eisenstein series
g_2, g_3, Δ, j	g, Δ, j
(1.16)	(3.10)
(1.17)	(3.12)
Jacobi's formula	(3.13)
special Eisenstein series E_{p-1}, $p \geq 5$ prime	special Eisenstein series E_k, $k = q^d - 1$, $\mathfrak{p}$ a prime of degree d

6. The Bruhat-Tits tree.

The Bruhat-Tits tree $\mathcal{T}$ of $\mathrm{PGL}(2, K_\infty)$ is extensively discussed in [21]. Most of the explicit calculations below are taken from [9] or [10]. It is a $(q+1)$-regular tree (i.e., each vertex connects to $q+1$ non-oriented edges) upon which $\mathrm{GL}(2, K_\infty)$ acts (vertex- and edge-) transitively.

(6.1) For abbreviation, let G denote the group scheme GL(2) with center Z = scalar matrices in G, and put

$$\mathcal{K} := G(O_\infty), \quad \mathcal{J} := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{K} \mid c \equiv 0 \ (\mathrm{mod}\ \pi)\}.$$

Then the sets $X(\mathcal{T})$ of vertices and $Y(\mathcal{T})$ of oriented edges of $\mathcal{T}$ are given by

$$\begin{array}{rcl} X(\mathcal{T}) & = & G(K_\infty)/Z(K_\infty) \cdot \mathcal{K} \\ Y(\mathcal{T}) & = & G(K_\infty/Z(K_\infty) \cdot \mathcal{J}, \end{array}$$

and the action of $G(K_\infty)$ is left multiplication. The canonical map from $Y(\mathcal{T})$ to $X(\mathcal{T})$ associates with each oriented edge e its origin $o(e)$. The edge $\overline{e}$ is e with orientation reversed: $o(\overline{e}) = t(e)$ = terminus of e, $t(\overline{e}) = o(e)$.

For $k \in \mathbb{Z}$ and u in a system of representatives of $K_\infty/\pi^k O_\infty$ let $m(k,u)$ be the matrix $m(k,u) = \left(\begin{smallmatrix} \pi^k & , & u \\ 0 & , & 1 \end{smallmatrix}\right)$. Then

$$S_X := \{m(k,u) \mid k \in \mathbb{Z},\ u \in K_\infty/\pi^k O_\infty\} \text{ and } S_Y := S_X \,\dot{\cup}\, S_X \left(\begin{smallmatrix} 0 & 1 \\ \pi & 0 \end{smallmatrix}\right)$$

are systems of representatives for $X(\mathcal{T})$ and $Y(\mathcal{T})$, respectively. We let $v(k,u)$ $(e(k,u))$ be the vertex (oriented edge) represented by $m(k,u)$. Moreover, we put $v_k = v(-k,0)$ and $e_k = e(-k,0)$. An *end* of $\mathcal{T}$ is an equivalence class of infinite half lines $\bullet - - - \bullet - - - \bullet - - - \bullet - \cdots$, two of which are identified if they differ in a finite graph. The set of ends of $\mathcal{T}$ is canonically identified with $\mathbb{P}^1(K_\infty)$. E.g., "∞" is given by $(v_0, v_1, v_2, \ldots)$ and "0" by $(v_0, v_{-1}, v_{-2}, \ldots)$. The straight line $(\ldots, v_{k-1}, v_k, v_{k+1}, \ldots)$ joining 0 to ∞ is called the *principal axis* $A(0,\infty)$ of $\mathcal{T}$.

A first important result on $\mathcal{T}$ is due to Goldman-Iwahori ([15]; they actually prove a more general assertion valid for arbitrary dimensions).

6.2 Theorem. *The set $\mathcal{T}(\mathbb{R})$ of real points of $\mathcal{T}$ is in canonical bijection with the set of* *similarity classes of non-archimedean norms on the two-dimensional vector space* K_∞^2.

We briefly describe the map on vertices of $\mathcal{T}$ and refer e.g. to [12] for a thorough discussion. To each $x \in \mathcal{T}(\mathbb{Z}) = X(\mathcal{T})$, given by a matrix $g_x \in G(K_\infty)$, there corresponds an O_∞-lattice $L_x := O_\infty^2 g_x^{-1} \subset K_\infty^2$, which is well-defined up to scalars. Hence the norm "$|\ .\ |_x$" on K_∞^2 with unit ball L_x is well-defined up to scaling, i.e., up to similarity. Points of $\mathcal{T}(\mathbb{R})$ situated in the interior of an edge correspond to classes of norms whose unit balls are not O_∞-lattices in K_∞.

(6.3) The above allows to define a map λ from Ω to the set $\mathcal{T}(\mathbb{Q})$ of elements of $\mathcal{T}(\mathbb{R})$ with rational barycentric coordinates. Namely, for $z \in \Omega$ let $\nu_z:\ K_\infty^2 \longrightarrow \mathbb{R}$ be the norm given by $\nu_z(u,v) := |uz+v|$. By construction of the bijection in (6.2), the class of ν_z corresponds to an element in $\mathcal{T}(\mathbb{Q})$. The reason is that the value group $|C^*|$ equals $q^{\mathbb{Q}} \hookrightarrow \mathbb{R}_+^*$. Hence

$$\begin{array}{rcl} \lambda:\ \Omega & \longrightarrow & \mathcal{T}(\mathbb{Q}) \\ z & \longmapsto & \text{class of } \nu_z \end{array}$$

is well-defined. Moreover, it is onto $\mathcal{T}(\mathbb{Q})$ and $G(K_\infty)$-equivariant. Furthermore, the values $|z|$ and $|z|_i$ of some $z \in \Omega$ depend only on $\lambda(z) \in \mathcal{T}(\mathbb{Q})$.

We next consider group actions.

6.4 Theorem. *The half-line $h_\infty := (v_0, v_1, v_2, \ldots)$ of $\mathcal{T}$ is a fundamental domain for the action of Γ on $\mathcal{T}$. That is, each vertex is Γ-equivalent to precisely one of the v_k ($k \geq 0$). Similarly, each oriented edge is equivalent to precisely one of the e_k or $\overline{e}_k$ ($k \geq 0$).*

(This has been proved on and on in the literature. Perhaps the first proof is due to A. Weil [25].) Since v_k corresponds to the matrix $m(-k,0)$, its stabilizer Γ_{v_k} in Γ is

$$\text{(6.5)} \qquad \begin{array}{rcll} \Gamma_k & := & m(k,0)\Gamma m(-k,0) \cap Z(K_\infty)\cdot\mathcal{K} & \\ & = & G(\mathbb{F}_q) = \mathrm{GL}(2,\mathbb{F}_q) & k=0 \\ & = & \{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \mid a,d \in \mathbb{F}_q^*,\ b \in A,\ \deg\ b \leq k\} & k>0. \end{array}$$

The stabilizer of e_k or $\overline{e}_k$ is $\Gamma_{v_k} \cap \Gamma_{v_{k+1}} = \Gamma_k \cap \Gamma_{k+1}$. Directly from the definition of λ we get:

6.6 Proposition. *The inverse image* $\lambda^{-1}(v_k)$ *of the vertex* $v_k \in X(\mathcal{T})$ *is* $\mathcal{F}_k := \{z \in C \mid |z| = |z|_i = q^k\}$. *The invese image of the half-line* h_∞ *of* (6.4) *is* $\mathcal{F} := \{z \in C \mid |z| = |z|_i \geq 1\}$.

Together with (6.4) there results the next corollary, which states that $\mathcal{F}$ is as close to a fundamental domain for Γ on Ω as is possible in our situation.

6.7 Corollary. (i) *Each* $z \in \Omega$ *is* Γ*-equivalent to an element of* $\mathcal{F}$.
(ii) *Let* z, z' *be elements of* $\mathcal{F}$ *such that* $z' = \gamma z$ *with* $\gamma \in \Gamma$. *Then* $|z| = |z'|$ *and* $\gamma \in \Gamma_k$ *if* $|z| = q^k$, $\gamma \in \Gamma_k \cap \Gamma_{k+1}$ *if* $q^k < |z| < q^{k+1}$.

Of course, (6.7) may be proved directly without reference to $\mathcal{T}$.

(6.8) We need to introduce certain $\mathbb{Z}$-valued functions on $Y(\mathcal{T})$. Such a function φ is called *alternating* if $\varphi(e) + \varphi(\overline{e}) = 0$ for $e \in Y(\mathcal{T})$, and *harmonic in* $v \in X(\mathcal{T})$ if the sum $\sum \varphi(e)$ over the edges e with $o(e) = v$ vanishes. We put $\underline{H}(\mathcal{T}, \mathbb{Z})$ for the group (actually the $G(K_\infty)$-module) of alternating and everywhere harmonic $\mathbb{Z}$-valued functions on $Y(\mathcal{T})$. Typical examples of such functions arise from holomorphic functions f on Ω as follows.

A holomorphic function f on Ω is bounded on each $\lambda^{-1}(v)$, $v \in X(\mathcal{T})$. We let $\|f\|_v := \sup\{|f(z)| \mid z \in \lambda^{-1}(v)\}$ be its *spectral norm.* For f not identically vanishing, we define the function $r(f)$ on $Y(\mathcal{T})$ through

$$r(f)(e) := \log_q \frac{\|f\|_{t(e)}}{\|f\|_{o(e)}}. \tag{6.9}$$

Then $r(f)$ is alternating and satisfies $r(f_1 \cdot f_2) = r(f_1) + r(f_2)$.

The next result is due to Marius van der Put ([18], [5]).

6.10 Theorem. (i) *Let* f *be invertible on* Ω. *Then* $|f(z)|$ *is constant on each* $\lambda^{-1}(v)$ *and* $r(f) \in \underline{H}(\mathcal{T}, \mathbb{Z})$.
(ii) *The following sequence is exact, where the middle term denotes the multiplicative group of invertible holomorphic functions on* Ω*:*

$$\begin{array}{ccccc} 1 \longrightarrow C^* \longrightarrow \mathcal{O}_\Omega(\Omega)^* & \longrightarrow & \underline{H}(\mathcal{T}, \mathbb{Z}) \longrightarrow 0 \\ f & \longmapsto & r(f) \end{array}.$$

By means of the "contour integration" formalism of [14] p. 93-95, the first part of the theorem may be generalized.

6.11 Proposition. *Let $f \in \mathcal{O}_\Omega(\Omega)$ be a holomorphic function on Ω and $v \in X(\mathcal{T})$ a vertex. Suppose that $f(z) \neq 0$ whenever $\lambda(z)$ lies in the interior of some edge e with $o(e) = v$. Then*

$$\sum_{o(e)=v} r(f)(e) \quad = \quad \textit{number of zeroes of } f \textit{ on } \lambda^{-1}(v), \quad \textit{counted with multiplicity},$$

where all the $r(f)(e)$ are integers.

(6.12) The above will allow us to relate the growth of e.g. modular forms on Ω with the distribution of their zeroes. As a first example, consider the function j on Ω, which has its only zeroes at the elliptic points, all of multiplicity $q+1$, and no poles on Ω (see (3.11)). It follows from (6.7) that the elliptic points in $\mathcal{F}$ are precisely the elements of $\mathbb{F}_{q^2} - \mathbb{F}_q \subset \mathcal{F}_0$. Since j is Γ-invariant, $r(j)$ is a function on the edges of the graph $\Gamma \setminus \mathcal{T}$, which is isomorphic under the quotient map with the half-line h_∞ of (6.4). Further, $r(j)$ is harmonic at vertices inequivalent with v_0. Now all the $q+1$ edges e of $\mathcal{T}$ with $o(e) = v_0$ are equivalent to e_0, thus

$$\begin{aligned}(q+1)r(j)(e_0) &= \#\{\text{zeroes of } j \text{ in } \mathcal{F}_0, \text{ with multiplicities}\} \\ &= (q+1)\#(\mathbb{F}_{q^2} - \mathbb{F}_q) \\ &= (q+1)q(q-1).\end{aligned}$$

Hence finally

$$r(j)(e_0) = q(q-1) \text{ and } r(j)(e_k) = q^{k+1}(q-1) \text{ for } k > 0. \tag{6.13}$$

The last equality stems from the fact:

(6.14) The q edges $e \neq e_k$ with $o(e) = v_k$ are identified under the quotient map $(\bmod\ \Gamma)$.

7. Zeroes and growth of Eisenstein series.

In view of the functional equation (3.5) (i), the divisor (set of zeroes with multiplicities) of a modular form f is invariant under Γ. For the investigation of the zeroes of f, we may therefore restrict to the "fundamental domain" $\mathcal{F}$ for Γ on Ω.

7.1 Proposition. *Let* $0 < k \equiv 0 \pmod{q-1}$. *The Eisenstein series* E_k *satisfies* $|E_k(z)| \leq 1$ *for* $z \in \mathcal{F}$, *with equality if* $z \notin \mathcal{F}_0$.

Proof. We have $E_k(z) = \sum_{a,b \in A}{}' \dfrac{1}{(az+b)^k}$, where $|az+b| = \max\{|az|, |b|\} \geq 1$ since $|z| = |z|_i \geq 1$. Further, if $z \notin \mathcal{F}_0$ then $|z| > 1$ and $|az+b)^{-k}| < 1$ whenever $(a,b) \neq (0,c)$ with $c \in \mathbb{F}_q$. But $\sum_{c \in \mathbb{F}_q}{}' \dfrac{1}{c^k} = -1$ has absolute value 1, thus $|E_k(z)| = 1$ in this case. □

7.2 Corollary. *Let* $z \in \mathcal{F}$ *be a zero of* E_k. *Then already* $z \in \mathcal{F}_0$.

7.3 Proposition. *The number of zeroes of* E_k *on* $\mathcal{F}_0$ *(counted with multiplicities) is* $q \cdot k$.

Proof. E_k doesn't vanish at ∞ since the constant term of its t-expansion is

$$\lim_{|z|_i \to \infty} \sum_{a,b}{}' \frac{1}{(az+b)^k} = \sum_{b}{}' \frac{1}{b^k} \equiv -1 \pmod{\pi}.$$

Hence formula (3.10) multiplied by $q(q^2-1)$ yields

$$(*) \qquad q(q^2-1) \sum_{z \in \Gamma \setminus \Omega}{}^{*} \nu_z(E_k) + q(q-1)\nu_e(E_k) = qk.$$

By (7.2), the sum $\sum_{z \in \Gamma \setminus \Omega}{}^{*}$ may be written as a sum over $z \in \Gamma_0 \setminus \mathcal{F}_0$. Now the quotient map from $\mathcal{F}_0$ to $\Gamma_0 \setminus \mathcal{F}_0$ is (q^3-q)-to-one off elliptic points and identifies the q^2-q elliptic points. We thus see that the left hand side of $(*)$ is just the number in question. □

The functional equation of E_k under $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ together with the multiplicativity of the map r implies

$$(7.4) \qquad r(E_k)(\gamma e) = k \cdot S(\gamma, e) + r(E_k)(e),$$

where $S(\gamma, e) := r(cz+d)(e)$, $e \in Y(\mathcal{T})$. The quantity $S(\gamma, e)$ is discussed and calculated in [10] 2.1 + 2.2. It is given by

$$(7.5) \qquad \begin{aligned} S(\gamma, e) &= +1 \text{ (resp. } -1) \text{ if } c \neq 0 \text{ and } \left(\begin{smallmatrix} c & d \\ 0 & 1 \end{smallmatrix}\right)(e) = e_k \text{ (resp. } \overline{e}_k) \\ &\qquad \text{for some } k \in \mathbb{Z}, \\ &= 0 \text{ otherwise.} \end{aligned}$$

This may also be shown directly from the definitions.

The function $r(g)$ as well as the spectral norms $\|g\|_v$ for $v \in X(\mathcal{T})$ have been calculated *loc. cit.* 2.18. On the principal axis, the result for $E_{q-1} = [1]^{-1} \cdot g$ is:

$$(7.6) \qquad \begin{aligned} r(E_{q-1})(e_k) &= 0 \text{ (for } k \geq 0) \text{ and } 1-q \text{ (for } k<0), \\ \log_q \|E_{q-1}\|_{v_k} &= 0 \text{ (for } k \geq 0) \text{ and } -k(q-1) \text{ (for } k<0), \end{aligned}$$

where $\|E_{q-1}\|_{v_k} = |E_{q-1}(z)|$ $(z \in \mathcal{F}_k)$ as long as $k \neq 0$.

Again, this may be obtained directly from combining (3.11), (6.11), (7.4), and (7.5). Note that (6.13) gives a corresponding result for $\|j\|$, up to scaling. The missing scalar factor is determined *loc. cit.* 2.13, which yields

$$(7.7) \qquad \log_q \|j\|_{v_k} = q^{|k|+1} \text{ with } \|j\|_{v_k} = |j(z)| \ \ (z \in \mathcal{F}_k) \text{ if } k \neq 0.$$

7.8 Proposition. *Let k be divisible by $q-1$. Then $r(E_k) = \frac{k}{q-1} r(E_{q-1})$.*

Proof. Put $\varphi = r(E_k) - \frac{k}{q-1} r(E_{q-1})$. As a function on $Y(\mathcal{T})$, it is alternating and harmonic. The harmonicity in $v \notin \Gamma v_0$ holds for both terms individually ((6.11) + (7.2)), whereas harmonicity in $v \in \Gamma v_0$ results from (6.11) and (7.3). On the other hand, (7.4) yields that φ is Γ-invariant. Both properties together force that $\varphi = 0$, taking (6.4) into account. □

7.9 Corollary. *The meromorphic modular function $E_k/E_{q-1}^{k/(q-1)}$ has constant absolute value 1 on $\Omega - \Gamma\mathcal{F}_0$.*

Proof. Let $e = (o(e), t(e))$ be an edge of $\mathcal{T}$. Then $\log_q |E_k(z)|$ factors over λ and interpolates linearly the values $\log_q \|E_k\|_{o(e)}$ and $\log_q \|E_k\|_{t(e)}$. Now the assertion follows from (6.9), (7.6) and (7.8). □

(7.10) Much more can be said about the arithmetic of zeroes of the special Eisenstein series $g_k = \text{const.}\ E_{q^k-1}$. As has already been noted in (4.6), their roots on Ω are simple. We list here some of their properties known so far.

Let $A_k(X,Y) \in A[X,Y]$ be the polynomial producing g_k:

$$A_k(g_{\text{new}}, \Delta_{\text{new}}) = g_k,$$

and define further

$$\begin{aligned} \chi(k) &= 0 \text{ for even and } \chi(k) = 1 \text{ for odd } k \\ \lambda(k) &= \tfrac{q^{k-1}+(-1)^k}{q+1} \\ \mu(k) &= \tfrac{q^k - q^{\chi(k)}}{q^2-1} \\ \varphi_k &= \tfrac{A_k(X,Y)}{X^{\chi(k)}Y^{\mu(k)}}. \end{aligned}$$

Then φ_k is a monic polynomial of degree $\mu(k)$ in $\frac{X^{q+1}}{Y}$, and satisfies (as is easily derived from the results of section 4):

(7.11) For each non-elliptic $z \in \Omega$,

$$\varphi_k(j(z)) = 0 \Leftrightarrow g_k(z) = 0,$$

(7.12) $\varphi_0 = \varphi_1 = 1$, $\varphi_k(X) = X^{\lambda(k)}\varphi_{k-1}(X) - [k-1]\varphi_{k-2}(X)$ $(k \geq 2)$.

E.g., $\varphi_2(X) = X - [1]$, $\varphi_3(X) = X^q - [1]X^{q-1} - [2]$.

7.13 Theorem ([1] [2]). *All the roots x of φ_k are simple and satisfy $|x| = q^q$.*

Note that (7.2) together with (7.7) yields only $|j(z)| \leq q^q$ for zeroes z of g_k. Further:

7.14 Theorem (G. Cornelissen, in preparation). *φ_k is irreducible.*

It is conjectured that $\mathrm{Gal}(\varphi_k)$ is always the full symmetric group if $k \geq 4$. This has been proved in some cases by Cornelissen (q odd, k satisfying certain congruence conditions), and relates to intriguing questions about the j-invariants of zeroes of classical Eisenstein series, see [11] section 8. About the distribution of zeroes of φ_k in $\mathcal{F}_0$, we have the following result.

7.15 Theorem ([11] 8.5). *For each $z_0 \in \mathbb{F}_{q^{k+1}} - \mathbb{F}_q$, there exists a unique zero $z \in \mathcal{F}_0$ of g_k that satisfies $|z - z_0| < 1$, and these are all the zeroes of g_k in $\mathcal{F}_0$ and even in $\mathcal{F}$.*

The following definition has been proposed by Cornelissen [2]. Put

$$\mathcal{D}_k = \prod_{0<i<k} [i]^{\mu(k)-\mu(i)-\chi(k-i)} \in A.$$

Then $\mathcal{D}_k$ is close (and presumably identical) to the discriminant of φ_k.

7.16 Theorem ([11] 7.14 + 8.11). *Let* $\operatorname{disc}(\varphi_k)$ *be the discriminant. Then*

(i) $|\operatorname{disc}(\varphi_k)| = |\mathcal{D}_k|$.

(ii) $\operatorname{disc}(\varphi_k)$ *and* $\mathcal{D}_k$ *have the same prime divisors, namely the primes* $\mathfrak{p}$ *of* A *of degree less than* k (*with the exception of the case* $q = 2$, $k = 3$, *where the prime* $\mathfrak{p} = (T^2+T+1)$ *divides neither* $\operatorname{disc}(\varphi_k)$ *nor* $\mathcal{D}_k$).

We don't know, however, whether these primes appear in $\operatorname{disc}(\varphi_k)$ with the exponents prescribed by $\mathcal{D}_k$.

8. The lattice functions α_k.

In this section, the only one in which we give detailed proofs, we investigate the functions α_k introduced in (3.7). We first determine their zeroes in $\mathcal{F}$, which happen to be located in the $\mathcal{F}_i$ with $0 \le i \le k-1$. Applying contour integration, we get the growth of $|\alpha_k|$ along the principal axis of $\mathcal{T}$.

(8.1) Let ϕ be the Drinfeld module associated with the generic lattice $\Lambda = Az + A$, where $z \in \Omega$. We write the attached exponential function e_Λ as

$$e_\Lambda(w) = \sum_{k \ge 0} \alpha_k(z) w^{q^k}.$$

First note that

$$\alpha_k(\infty) := \lim_{|z|_i \to \infty} \alpha_k(z) = \alpha_k(A) = \frac{\overline{\pi}^{q^k-1}}{D_k}, \tag{8.2}$$

which results from (2.14) and simple non-archimedean estimates. From the functional equation $e_\Lambda(Tw) = Te_\Lambda(w) + g(z)e_\Lambda(w)^q + \Delta(z)e_\Lambda(w)^{q^2}$, we derive

$$[k]\alpha_k = g\alpha_{k-1}^q + \Delta\alpha_{k-2}^{q^2} \tag{8.3}$$

for $k \geq 1$ with $\alpha_k = 0$ for $k < 0$ and $\alpha_0 = 1$. If $C_k(X, Y)$ is the unique polynomial such that $C_k(g, \Delta) = \alpha_k$, we get the corresponding recursion $[k]C_k = XC_{k-1}^q + YC_{k-2}^{q^2}$. Similar to (7.10), we define

$$\xi_k = \frac{C_k(X, Y)}{X^{(q^k-1)/(q-1)}}. \tag{8.4}$$

Then ξ_k is a polynomial in $\frac{Y}{X^{q+1}}$ with coefficients in K (note here the change from $\frac{X^{q+1}}{Y}$ to its reciprocal!). The next three properties are easily verified.

(8.5) For each non-elliptic $z \in \Omega$,

$$\xi_k(j^{-1}(z)) = 0 \Leftrightarrow \alpha_k(z) = 0.$$

The quantity $j(z)$ with a zero z of α_k is briefly called a *j-zero of α_k*.

$$[k]\xi_k(X) = \xi_{k-1}(X)^q + X\xi_{k-2}(X)^{q^2} \tag{8.6}$$

for $k \geq 1$ with $\xi_0 = 1$, $\xi_k = 0$ for $k < 0$. The first few of the ξ_k are $\xi_1(X) = \frac{1}{[1]}$, $\xi_2(X) = \frac{1}{[2]}(\frac{1}{[1]^q} + X)$, $\xi_3(X) = \frac{1}{[3]}(\frac{1}{[2]^q[1]^{q^2}} + \frac{1}{[1]^{q^2}} + \frac{1}{[2]^q}X^q)$.

$$\deg\ \xi_k = \mu(k). \tag{8.7}$$

(See (7.10); use (8.6) and induction.)

8.8 Proposition. *$\xi_k(X)$ is separable.*

Proof. From (8.6), $[k]\xi_k'(X) = \xi_{k-2}^{q^2}(X)$. Hence a multiple root x of ξ_k is also a root of ξ_{k-2}, hence a root of ξ_{k-1}. From (8.6) applied to $k-1$, x is also a root of ξ_{k-3}, thus a multiple root of ξ_{k-1}. Now use induction. □

8.9 Corollary. (i) *$\alpha_k(z)$ vanishes in elliptic points if and only if k is odd.*

(ii) *All the zeroes of α_k on Ω are simple.*

Proof. The modular form α_k has weight $q^k - 1$ and doesn't vanish at infinity (8.2). Hence (i) and the simplicity of zeroes in elliptic points follows from (8.5), (8.7) and (3.10). Assertion (ii) follows from (8.8) since $j : \Omega \longrightarrow C$ is unramified off elliptic points. □

Now write $\xi_k(X) = \sum_n \xi_{k,n} X^n$, and let $E(k) \subset \{0, 1, \ldots, \mu(k)\}$ be the set of subscripts n such that $\xi_{k,n}$ doesn't vanish.

8.10 Proposition. (i) *We have $E(0) = E(1) = \{0\}$ and the recursion $(k \geq 2)$*

$$E(k) = qE(k-1) \,\dot\cup\, q^2 E(k-1) + 1 = E(k-1) \,\dot\cup\, E(k-2) + q^{k-2}.$$

(ii) $E(k) = \{0\} \cup \{q^{i_1} + \cdots + q^{i_s} \mid 0 \leq i_1 < \cdots < i_s \leq k-2,\ |i_{t+1} - i_t| \geq 2\ \forall\ t < s\}$
(iii) *If $n = \sum_{1 \leq t \leq s} q^{i_t} \in E(k)$ then $v(\xi_{k,n}) = (k-s)q^k$ (recall that v is the valuation on K_∞).*
(iv) *Consider the Newton polygon of $\xi_k(X)$ as e.g. defined in* [17]. *Its vertices $(n, v(\xi_{k,n}))$ are given by the following table:*

k even:

n	0	1	$1+q^2$	$1+q^2+q^4$	$\cdots$	$1+q^2+\cdots+q^{k-2}$
$v(\xi_{k,n})$	kq^k	$(k-1)q^k$	$(k-2)q^k$	$(k-3)q^k$	$\cdots$	$\frac{k}{2}q^k$

k odd:

0	1	$1+q^2$	$\cdots$	$1+q^2+\cdots+q^{k-3}$	$q+q^3+\cdots+q^{k-2}$
kq^k	$(k-1)q^k$	$(k-2)q^k$	$\cdots$	$\frac{k+1}{2}q^k$	$\frac{k+1}{2}q^k$

Proof. (i) The first equality results immediately from (8.6). Note that there is no cancellation in (8.6) since terms coming from ξ_{k-1} (resp. ξ_{k-2}) have subscript congruent to 0 (resp. 1) modulo q. Therefore $\#(E(k)) = F_k$, the k-th Fibonacci number, $F_0 = F_1 = 1$, $F_k = F_{k-1} + F_{k-2}$ $(k \geq 2)$.

For the second equation, we note that the two sets $E(k-1)$ and $E(k-2)+q^{k-2}$ are always disjoint since deg $\xi_{k-1} = \mu(k-1) < q^{k-2}$. Hence $E(k)$ and $E(k-1) \dot{\cup} E(k-2) + q^{k-2}$ have the same cardinality F_k and it suffices to show an inclusion between them. This is postponed for the moment.

(ii) Let M_k be the number of subsets $\{i_1, i_2, \dots, i_s\}$ of $\{0, 1, \dots, k-2\}$ satisfying the stated conditions. Then $M_0 = M_1 = 0$, $M_2 = 1$, and a moment's thought shows that $M_k = M_{k-1} + M_{k-2} + 1$ for $k \geq 3$. Thus $M_k = F_k - 1$, and both sets in (ii) have the same cardinality. Using again (8.6) and induction, we see that $E(k)$ is contained in the right hand side, and we have identity.

(i) (continued) It is obvious from (ii) that $E(k-1) \dot{\cup} E(k-2) + q^{k-2}$ is contained in $E(k)$, and the two sets therefore agree.

(iii) From (8.6) we have

$$\xi_{k,n} = \frac{1}{[k]}(\xi_{k-1,n/q}^{q} + \xi_{k-2,(n-1)/q^2}^{q^2}),$$

where $\xi_{k,m} = 0$ if m is not an integer. Now $v([k]) = -q^k$, and the assertion follows as usual from induction on k.

(iv) This is clear from (iii). □

Now we are able to describe the location of the j-zeroes of α_k.

8.11 Theorem. *For $0 \leq i < \frac{k-\chi(k)}{2}$, there are precisely q^{2i} j-zeroes x of α_k that satisfy $\log_q |x| = q^{k-2i}$. For k odd, there are $\lambda(k) = \frac{q^{k-1}-1}{q+1}$ further j-zeroes x with $\log_q |x| = 0$ and the j-zero $x = 0$. These are all the j-zeroes of α_k.*

Proof. Immediate from (8.10) (iv), (8.9) and the properties of the Newton polygon. □

Recall from (6.6) and (7.7) that for $k \geq 0$, $\mathcal{F}_k = \lambda^{-1}(v_k) = \{z \in C \mid |z| = |z|_i = q^k\}$, and this equals $\{z \in \mathcal{F} \mid \log_q |j(z)| = q^{k+1}\}$ for $k > 0$ and $\{z \in \mathcal{F} \mid \log_q |j(z)| \leq q^q\}$ for $k = 0$. Furthermore, the group Γ_k of (6.5) acts on $\mathcal{F}_k$ and actually, $\Gamma_k = \{\gamma \in \Gamma \mid \gamma\mathcal{F}_k \cap \mathcal{F}_k \neq \emptyset\}$. Hence the quotient map $j : \Omega \longrightarrow C$ restricted to $\mathcal{F}_k$ identifies precisely $\frac{\#(\Gamma_k)}{q-1} = (q-1)q^{k+1}$ elements of $\mathcal{F}_k$ if $k > 0$ and $\frac{\#(\Gamma_0)}{q-1} = q^3 - q$ elements of $\mathcal{F}_0$, provided they are non-elliptic. Also, the $q^2 - q$ elliptic points in $\mathcal{F}_0$ are identified. Taking this into account, we get the following distribution of zeroes of α_k on $\mathcal{F}$.

8.12 Corollary. *All the zeroes of α_k in $\mathcal{F}$ lie in some $\mathcal{F}_i$. If k is even (odd), α_k has precisely $(q-1)q^k$ zeroes in $\mathcal{F}_1, \mathcal{F}_3, \mathcal{F}_5, \ldots, \mathcal{F}_{k-1}$ (in $\mathcal{F}_0, \mathcal{F}_2, \mathcal{F}_4, \ldots, \mathcal{F}_{k-1}$) each, and no further zeroes in $\mathcal{F}$.* $\square$

We use this information to determine or at least to estimate $|\alpha_k(z)|$ on $\mathcal{F}$.

8.13 Corollary. *Let h_{k-1} be the half-line $(v_{k-1}, v_k, v_{k+1}, \ldots)$ in $\mathcal{T}(\mathbb{R})$. Then $|\alpha_k(z)|$ is constant on $\lambda^{-1}(h_{k-1} - \{v_{k-1}\})$ with $\log_q |\alpha_k(z)| = (\frac{q^k-1}{q-1})q - kq^k$, which also agrees with $\log_q \|\alpha_k\|_{v_{k-1}} = \log_q \|\alpha_k\|_{k-1}$.*

Proof. By the preceding corollary and (6.11), $r(\alpha_k)$ is harmonic at $v_k, v_{k+1}, \ldots$ In view of (6.14), we get

$$qr(\alpha_k)(e_{i-1}) = r(\alpha_k)(e_i) \quad \text{for } i \geq k.$$

Since $\alpha_k(\infty) \neq 0$ with $\log_q |\alpha_k(\infty)| = (\frac{q^k-1}{q-1})q - kq^k$ (see (8.2)), $r(\alpha_k) = 0$ on the edges of h_{k-1}, which gives the result. $\square$

The spectral norm $\|\alpha_k\|_i$ of α_k on $\mathcal{F}_i$ for $0 \leq i < k-1$ is given by

$$\log_q \|\alpha\|_{k-1} - \log_q \|\alpha_k\|_i = \int_{v_i}^{v_{k-1}} r(\alpha_k)(e)de, \tag{8.14}$$

with the obvious meaning of the integral. The relevant $r(\alpha_k)(e_j)$ may be calculated from (8.12), (6.11) and (6.14), see below.

(8.15) It is quite easy to determine $\|\alpha_k\|_0$ at least for even k. Let $z \in \mathcal{F}_0$. Then $|az+b| = \sup\{|a|, |b|\}$ for $a, b \in A$, hence the lattice $\Lambda_z = Az + A$ has precisely $q^{2(i+1)}$ elements of degree less or equal to i ($i \geq 0$), i.e., of absolute value $\leq q^i$. Since the elements of Λ_z are also the zeroes in w of

$$e_{\Lambda_z}(w) = \sum_{k \geq 0} \alpha_k(z) w^{q^k},$$

the Newton polygon of e_{Λ_z} has

one segment of length $q^2 - 1$ and slope 0, one segment of length $q^4 - q^2$ and slope 1, ...

one segment of length $q^{2(i+1)} - q^{2i}$ of slope i, ...

Summing up, its vertices are given by $(q^k, q^2(\frac{q^k-1}{q^2-1}) - \frac{k}{2}q^k)$ for $k = 0, 2, 4, \ldots$. Therefore:

(8.16) For even k, the absolute value $|\alpha_k(z)|$ is constant on $\mathcal{F}_0$ and satisfies $\log_q |\alpha_k(z)| = q^2(\frac{q^k-1}{q^2-1}) - \frac{k}{2}q^k$.

This method fails however for α_k with odd k. We will therefore pursue the approach in (8.14) to get a uniform description of $\|\alpha_k\|_0$ for even and odd k.

(8.17) The edges e_j with $-1 \leq j < k-1$ join v_{-1} to v_{k-1}. We let

$$r_{k,j} := r(\alpha_k)(e_j).$$

The basic relations between these numbers come from (6.11), (6.14), (8.12) and $r(\alpha_k)(\gamma e) = (q^k - 1)S(\gamma, e) + r(\alpha_k)(e)$ (see (7.4) and (7.5)), viz.,

$$(8.18)\qquad \begin{array}{lll} \textbf{\textit{k} even:} & qr_{k,-1} - r_{k,0} & = 0 \\ & qr_{k,0} - r_{k,1} & = -q^k(q-1) \\ & qr_{k,1} - r_{k,2} & = 0 \\ & \qquad\vdots & \\ & qr_{k,k-2} - r_{k,k-1} & = -q^k(q-1) \\ \textbf{\textit{k} odd:} & qr_{k,-1} - r_{k,0} & = -q^k(q-1) \\ & qr_{k,0} - r_{k,1} & = 0 \\ & qr_{k,1} - r_{k,2} & = -q^k(q-1) \\ & \qquad\vdots & \\ & qr_{k,k-2} - r_{k,k-1} & = -q^k(q-1). \end{array}$$

Furthermore, in both cases

$$-r_{k,0} = q^k - 1 + r_{k,-1}$$

since $\binom{0\,1}{1\,0}e_{-1} = \overline{e}_0$. Solving yields

$$
\begin{array}{llcl}
k\ \textbf{even:} & r_{k,-1} & = & -(\frac{q^k-1}{q+1}) =: \rho \\
& r_{k,0} & = & q\rho \\
& r_{k,1} & = & q^2\rho + q^k(q-1) \\
& r_{k,2} & = & q^3\rho + q^{k+1}(q-1) \\
& r_{k,3} & = & q^4\rho + q^{k+2}(q-1) + q^k(q-1) \\
& & \vdots & \\
& r_{k,k-1} & = & q^k\rho + (q^k + q^{k+2} + \cdots + q^{2k-2})(q-1)
\end{array}
$$

(8.19)

$$
\begin{array}{llcl}
k\ \textbf{odd:} & r_{k,-1} & = & -(\frac{q^{k+1}-1}{q+1}) \\
& r_{k,0} & = & -q(\frac{q^{k-1}-1}{q+1}) =: \rho \\
& r_{k,1} & = & q\rho \\
& r_{k,2} & = & q^2\rho + q^k(q-1) \\
& r_{k,3} & = & q^3\rho + q^{k+1}(q-1) \\
& & \vdots & \\
& r_{k,k-1} & = & q^{k-1}\rho + (q^k + q^{k+2} + \cdots + q^{2k-3})(q-1).
\end{array}
$$

Note that (8.13) prescribes that $r_{k,k-1} = 0$ in each case, which is easily checked from the above. Now we have all the ingredients (formulas (8.13) and (8.19)) of (8.14) to evaluate $\log_q \|\alpha_k\|_i$ for $i = 0, 1, \ldots, k-1$. We restrict to write down the result for $i = 0$, which comes out from a lengthy exercise in summing up geometric series.

8.20 Theorem. *Let $k' = k$ for even and $k' = k-1$ for odd $k \geq 1$. Then $\|\alpha_k\|_0$ is given by the formula*

$$\log_q \|\alpha_k\|_0 = q^2(\frac{q^{k'}-1}{q^2-1}) - \frac{k'}{2}q^k.$$

For even k, the spectral norm even equals the common absolute value $|\alpha_k(z)|$ for all $z \in \mathcal{F}_0$. $\square$

8.21 Remarks. (i) The statement for *even* k has already been given in (8.16). For *odd* k, the value for $\|\alpha_k\|_0$ is the upper bound for $|\alpha_k(z)|$ allowed by the Newton polygon for $e_{\Lambda_z}(w)$, see (8.15). That is, for almost all $z \in \mathcal{F}_0$, $|\alpha_k(z)|$ is as large as allowed by the Newton polygon. Note also that for $\lambda(z)$ in the interior of an edge $e \in Y(\mathcal{T})$, $\log_q |\alpha_k(z)|$ interpolates linearly the values of $\log_q \|\alpha_k\|_v$ on the extremities v of e. We therefore have complete control over the behavior of $|\alpha_k(z)|$ on $\mathcal{F}$.
(ii) In the dictionary given in section 5, e_Λ corresponds to the Weierstrass function $\wp_\Lambda$ of some lattice Λ in $\mathbb{C}$, and therefore its Laurent coefficients $\alpha_k(z)$ to the coefficients of $\wp_\Lambda$, which by (1.8) are the classical Eisenstein series E_k^{class}. Hence both our E_k and the α_k are analogues of E_k^{class}. Whereas the E_k are similar to the E_k^{class} regarding their congruences ([24] - results of section 4) or their zeroes ([19] - (7.2), (7.13)), the α_k differ significantly in that their zeroes tend to ∞ with k growing. It is worthwhile to study the arithmetical properties of their zeroes in the style of theorems (7.14) to (7.16), the field generated over K by their j-values, etc. Moreover, it would be desirable to extend the investigation to other classes of modular forms like e.g. the coefficient forms $l_k(T^i, z)$ and the Poincaré series $P_{k,m}$ mentioned in (3.7), or to the "Serre derivatives" introduced in (3.17) of such forms.

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