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Efficient Presentations for Some Direct Products of Groups

Bilal Vatansever and David M. Gill

Abstract

In this paper we give efficient presentations for $A_4 \times D_n$, where n is odd number, or n is even number and $(n,3)=1$. We also give efficient presentations for $A_5 \times D_n$ where n is an even or odd number.

1. Introduction

Let G be finite group with a presentation on n generators and r relations. The deficiency of the presentation is $n - r$. A group H of maximal order with the properties that there is a subgroup A with $A \leq Z(H) \cap H'$ and $H/A \cong G$ is called a covering group of G . In general, H is not unique but A is unique and is called the Schur multiplier $M(G)$ of G . For details see [1, 7, 15].

Schur [9] showed that any presentation for G with n generators requires at least $n + \text{rank}(M(G))$ relations. If G has a presentation with n generators and precisely $n + \text{rank}(M(G))$ relations we say that G is efficient. Not all groups are efficient and examples of soluble groups with trivial multipliers which are not efficient were given by Swan [10] and inefficient groups have been found by Wotherspoon [16]. Further details of such groups are given in [1], [14], [15].

For the finite field $GF(p)$, for a prime p , let $SL(2, p)$ denote the group of 2×2 matrices of determinant 1 over the field $GF(p)$. Define $PSL(2, p) = SL(2, p) / \{\pm I\}$, where I is the 2×2 identity matrix.

For any group G we shall use G' and $Z(G)$ to denote the derived group of G and the center of G , respectively. We also use the notation A_4 and A_5 to denote, respectively, the alternating groups of degree four and five. Let D_n denote the dihedral group of order $2n$.

Questions concerning the efficiency of direct products have been of considerable interest for a number of years. The first questions concerning the efficiency of direct products were posed by Wiegold in [15]. In particular his questions were whether $PSL(2, 5) \times PSL(2, 5)$ and $SL(2, 5) \times SL(2, 5)$ are efficient.

The Schur-Künneth formula [6] gives the Schur multiplier of a direct product:

$$M(G \times H) = M(G) \times M(H) \times (G \otimes H).$$

The first of these questions was answered by Kenne in [8]. He showed that $PSL(2, 5) \times PSL(2, 5)$ is efficient. The second question was answered by Campbell et al. [2]. In [4] C. M. Campbell, E. F. Robertson and P. D. Williams have obtained efficient presentations for certain direct products involving field of the same characteristic. Some work on direct product of groups $PSL(2, p^{n_i})$ for a fixed prime p and different n_i 's is done and also some efficient presentations for $PSL(2, q_1) \times PSL(2, q_2)$, q_1, q_2 prime power, are given by Vatansever in [11]. In [12] the efficiency of the group $PSL(2, Z_n) \times PSL(2, Z_m)$, for certain n, m is given. In [5] D. M. Gill has obtained efficient presentations of direct products of familiar groups. In [13] the efficiency of the group $PSL(2, 7) \times PSL(2, 3^2)$ is given.

In this paper we consider the problem of giving efficient presentations for the direct products $A_4 \times D_n$ and $A_5 \times D_n$.

From the Schur-Künneth formula we have:

- i) If n is odd then the rank of $M(A_k \times D_n)$ is 1 where $k = 4, 5$.
- ii) If n is even then the rank of $M(A_k \times D_n)$ is 2 where $k = 4, 5$.

2. The direct products $A_4 \times D_n$ and $A_5 \times D_n$.

Theorem 1. *When n is odd, a presentation of $(2, 3, 2r + 1) \times D_n$ is*

$$\langle x, y \mid (xy)^{2r+1}, x^{n+1}y^3, x^2yx^2y^5, y^3 = xy^3x \rangle.$$

Proof. Take $(2, 3, 2r + 1)$ as $\langle a, b \mid a^2, b^3, (ab)^{2r+1} \rangle$ and D_n as $\langle c, d \mid c^2, d^n, (cd)^2 \rangle$. Define $x = ad$, and $y = bc$, so $x^n = a$, $x^{n+1} = d$, $y^3 = c$, and $y^{-2} = b$. Therefore the direct product of these two groups has presentation

$$\langle x, y \mid x^{2n}, y^6, (x^ny^{-2})^{2r+1}, (y^3x^{n+1})^2, x^ny^3 = y^3x^n, x^2y^2 = y^2x^2 \rangle.$$

Firstly $(y^3x^{n+1})^2 = y^5x^{n+1}yx^{n+1}$ (using $x^2y^2 = y^2x^2$), for which we have $y = x^{n+1}yx^{n+1}$ (using $y^6 = 1$) and $y = x^{2n+2}yx^{2n+2} = x^2yx^2$ (using $x^{2n} = 1$).

So add $y = x^2yx^2$ and note that $x^2y^2 = y^2x^2$ is then redundant.

Also,

$$\begin{aligned} (y^3x^{n+1})^2 &= y^3xy^3x^{2n+1} && \text{(using } x^ny^3 = y^3x^n \text{) we have} \\ &y^3 = xy^3x. && \text{(using } x^{2n} = 1, y^6 = 1 \text{)} \end{aligned}$$

So replace $(y^3x^{n+1})^2 = 1$ by $y^3 = xy^3x$. Then replace $(x^ny^{-2})^{2r+1} = 1$ by $(x^ny)^{2r+1}y^3 = 1$. (Therefore $x^ny^3 = y^3x^n$ is redundant).

So we now have:

$$\langle x, y \mid x^{2n}, y^6, (x^ny)^{2r+1}y^3, y = x^2yx^2, y^3 = xy^3x \rangle.$$

$$\begin{aligned} \text{But } 1 &= (x^ny)^{2r+1}y^3 = (x^nyx^ny)^r x^ny^4 \\ &= (xyxy)^r x^ny^4 && \text{(using } y = x^2yx^2 \text{)} \\ &= (xy)^{2r} xx^{n-1}yy^3 \\ &= (xy)^{2r} xyx^{1-n}y^3 && \text{(using } y = x^2yx^2 \text{)} \\ &= (xy)^{2r+1}x^{1+n}y^3. \end{aligned}$$

Replace $(x^ny)^{2r+1}y^3 = 1$ by this.

But now $x^{2n} = 1$ is redundant as from this new relation we see that

$$\begin{aligned} x^{-2n-2} &= y^3(xy)^{2r+1}y^3(xy)^{2r+1} \\ &= y^6(x^{-1}y)^{2r+1}(xy)^{2r+1} && \text{(using } xy^3 = y^3x^{-1} \text{)} \\ &= x^{-1}y(x^{-1}yx^{-1}y)^r(xy)^{2r+1} \\ &= x^{-1}y(xyxy)^r(xy)^{2r+1} && \text{(using } x^{-1}y = xyx^2 \text{)} \\ &= x^{-2}(xy)^{4r+2} \\ &= x^{-2}y^{-3}x^{-1-n}y^{-3}x^{-1-n} && \text{(using } (xy)^{2r+1}x^{1+n}y^3 = 1 \text{)} \\ &= x^{-2}y^{-6} = x^{-2} && \text{(using } y^3 = xy^3x \text{)}. \end{aligned}$$

Consider

$$H = \langle x, y \mid (xy)^{2r+1}x^{1+n}y^3, x^2yx^2y^5, y^3 = xy^3x \rangle.$$

Now $(1 = x^2yx^2y^3y^2 = x^2y^4x^{-2}y^2)$ we have $x^2 = y^2x^2y^4$.

Similarly $(1 = y^3x^2yx^2y^2 = x^{-2}y^4x^2y^2)$ we have $x^2 = y^4x^2y^2$. Therefore $y^2x^2 = y^4x^2y^4 = x^2y^2$ and so $y^6 = 1$. Therefore H is a presentation of the direct product.

Corollary 2. *When n is odd, an efficient presentation of $A_4 \times D_n$ is*

$$\langle x, y \mid (xy)^3 x^{n+1} y^3, x^2 y x^2 y^5, y^3 = xy^3 x \rangle.$$

Corollary 3. *When n is odd, an efficient presentation of $A_5 \times D_n$ is*

$$\langle x, y \mid (xy)^5 x^{n+1} y^3, x^2 y x^2 y^5, y^3 = xy^3 x \rangle.$$

Theorem 4. *When n is even, a presentation of $(2, 3, 4r + 1) \times D_n$ is*

$$\langle x, y \mid x^3 y^2 x^3 = y^2, (xy^{4r})^2, y^{4r+1} x^2 y^{4r+1} = x^2, (xy^{4r+1})^n x^{2n} \rangle.$$

Proof. Take $(2, 3, 4r + 1)$ as $\langle a, b \mid a^2, b^3, (ab)^{4r+1} \rangle$ and D_n as $\langle c, d \mid c^2, d^n, (cd)^2 \rangle$. Let $x = bcd$ and $y = abcd^2$, so $x^3 = cd$, $x^4 = b$, $y^{4r+1} = (cd^2)^{4r+1} = (cd.dcd.d)^{2r}.cd^2 = (cd.c.d)^{2r}.cd^2 = cd^2$ and $y^{4r+2} = ab$.

Therefore $d = x^{-3}y^{4r+1}$, $c = x^3y^{-4r-1}x^3$ and $a = y^{4r+2}x^{-4}$. Hence a presentation of the direct product is:

$$\langle x, y \mid (y^{4r+2}x^{-4})^2, x^6, y^{2(4r+1)}, (x^{-3}y^{4r+1})^n, x^2y^{4r+1} = y^{4r+1}x^2, x^3y^2 = y^2x^3 \rangle.$$

$$\begin{aligned} \text{The first relation is } 1 &= (y^{4r+2}x^{-4})^{-2} \\ &= x^{-6}(y^{4r+2}x^{-1})^{-2} && \text{(using } x^3y^2 = y^2x^3) \\ &= (xy^{4r})^2 && \text{(by } x^6 = 1, y^{2(4r+1)} = 1) \end{aligned}$$

$$\begin{aligned} \text{and the fourth is } 1 &= (x^{-3}y^{4r+1})^{-n} \\ &= (x^{-1}y^{4r+1})^{-n}x^{2n} && \text{(by } x^2y^{4r+1} = y^{4r+1}x^2) \\ &= (y^{4r+1}x)^n x^{2n}. && \text{(using } y^{2(4r+1)} = 1) \end{aligned}$$

So we have:

$$\langle x, y \mid (xy^{4r})^2, x^6, y^{2(4r+1)}, (xy^{4r+1})^n x^{2n}, x^2y^{4r+1} = y^{4r+1}x^2, x^3y^2 = y^2x^3 \rangle.$$

Now consider:

$$\langle x, y \mid x^3y^2x^3 = y^2, (xy^{4r})^2, y^{4r+1}x^2y^{4r+1} = x^2, (xy^{4r+1})^n x^{2n} \rangle.$$

First note that $x^3y^2x^3 = y^2$ implies $x^3y^4 = y^4x^3$.

Now for $(xy^{4r})^2 = 1$ we have $xy^{4r}x = y^{-4r}$ so

$$x^3y^{4r}x^3 = x^2y^{-4r}x^2$$

$$\text{we have } x^6y^{4r} = x^2y^{-4r}x^2. \quad \text{(using } x^3y^4 = y^4x^3)$$

$$\begin{aligned} \text{This is } x^6 &= x^2 y^{-4r} x^2 y^{-4r} \\ &= x^2 y \cdot y^{-4r-1} x^2 y^{-4r-1} \cdot y \\ &= x^2 y x^2 y. \end{aligned} \quad (\text{using } y^{4r+1} x^2 y^{4r+1} = x^2).$$

$$\begin{aligned} \text{Also, } x^4 y^{4r} x^4 &= x^3 y^{-4r} x^3 = x^6 y^{-4r}, \text{ but} \\ x^4 y^{4r} x^4 &= x^4 y^{-1} y^{4r+1} x^4 \\ &= x^4 y^{-1} x^4 y^{4r+1}. \end{aligned} \quad (\text{using } y^{4r+1} x^2 y^{4r+1} = x^2).$$

Equating these gives

$$yx^2 = x^4 y^{8r+1}. \quad (0.1)$$

We know $x^6 = x^2 y x^2 y$ so $x^6 = x^2 \cdot x^4 \cdot y^{8r+1} \cdot y$, i.e. $y^{8r+2} = 1$ and hence $y^{4r+1} x^2 y^{4r+1} = x^2$ shows that $x^2 y^{4r+1} = y^{4r+1} x^2$.

As $\gcd(4, 8r + 2) = 2$, $x^3 y^4 = y^4 x^3$ we have $x^3 y^2 = y^2 x^3$ and therefore from $x^3 y^2 x^3 = y^2$, we see that $x^6 = 1$ and we have a presentation of the direct product.

□

Corollary 5. *An efficient presentation of $A_5 \times D_n$, where n is even, is*
 $\langle x, y \mid x^3 y^2 x^3 = y^2, (xy^4)^2, y^5 x^2 y^5 = x^2, (xy^5)^n x^2 \rangle.$

Theorem 6. *An efficient presentation of $A_4 \times D_n$, when n is even and $(n, 3) = 1$, is*
 $\langle x, y \mid x^6, y^{\varepsilon n-1} = x^3 y x^3, (xy)^2, x = y^3 x y^3 \rangle$

where $\varepsilon \equiv \frac{n}{2} \pmod{3}$ and $\varepsilon \in \{-1, 1\}$.

Proof. $M(A_4 \times D_n) = C_2 \times C_2$. Let $D_n = \langle a, b \mid a^2, b^n, (ab)^2 \rangle$ and $A_4 = \langle c, d \mid c^3, d^3, (cd)^2 \rangle$. First take $n = 6r + 2$. Let $x = ac$ and $y = bd$, hence $x^3 = a$, $x^{-2} = c$, $y^{-n} = d$ and using $(n = 6r + 2$ and $(n, 3) = 1$) we obtain $y^{n+1} = b$. So the direct product is

$$\langle x, y \mid x^6, y^{3n}, y^n = (x^3 y)^2, y^{n-2} = (x^2 y^{-1})^2, x^3 y^n = y^n x^3, x^2 y^3 = y^3 x^2 \rangle.$$

Note that $x^3 y^n = y^n x^3$ is redundant. (using $y^n = (x^3 y)^2$)

$$\begin{aligned} \text{Also we see that } 1 &= x^2 y^{-1} x^2 y^{-1} y^{2-n} \\ &= x^2 y^{-1} x^2 y \cdot y^{-n} \\ &= x^2 y^{-1} x^2 y (x^3 y)^{-2} \end{aligned} \quad (\text{using } y^n = (x^3 y)^2)$$

we have $(xy)^2 = 1$.

Therefore replace $y^{n-2} = (x^2 y^{-1})^2$ by $(xy)^2 = 1$.

Now $(y^{n-1})^3 = (x^3yx^3)^3$ (using $y^n = (x^3y)^2$)
 $= x^3y^3x^3$ (using $x^6 = 1$)
 we have $x^3y^3x^3y^3 = 1$ (using $y^{3n} = 1$)
 we obtain $x = y^3xy^3$. (using $x^2y^3 = y^3x^2, x^6 = 1$).

Add the relation $x = y^3xy^3$. However $x = y^3xy^3$ we have $x^2y^3 = y^3x^2$, so the later is redundant. So we see that the direct product can be presented by

$$\langle x, y \mid x^6, y^{3n}, y^n = (x^3y)^2, (xy)^2, x = y^3xy^3 \rangle.$$

Consider

$$\langle x, y \mid x^6, y^{n-1} = x^3yx^3, (xy)^2, x = y^3xy^3 \rangle.$$

As we just mentioned, $x = y^3xy^3$ we have $x^2y^3 = y^3x^2$.

And $(y^{n-1})^3 = (x^3yx^3)^3$
 $= x^3y^3x^3$. (using $x^6 = 1$)

Hence

$$y^{3n} = x^3y^3x^3y^3. \tag{0.2}$$

But, $x = y^3xy^3$ we have $x^3 = y^3x^3y^3$ (using $x^2y^3 = y^3x^2$)
 we obtain $x^3y^3x^3y^3 = 1$. (using $x^6 = 1$)

Therefore, from (0.2), we see that $y^{3n} = 1$.

Hence a presentation is one of $A_4 \times D_n$ when $n \equiv 2 \pmod{6}$ is

$$\langle x, y \mid x^6, y^{n-1} = x^3yx^3, (xy)^2, x = y^3xy^3 \rangle. \tag{0.3}$$

When $n \equiv -2 \pmod{6}$, consider $A_4 \times D_{-n}$. Note that $-n \equiv 2 \pmod{6}$ and so using (0.3) we see that

$$A_4 \times D_{-n} \cong \langle x, y \mid x^6, y^{(-n)-1} = x^3yx^3, x = y^3xy^3, (xy)^2 \rangle.$$

However as $D_n \cong \langle a, b \mid a^2, b^n, (ab)^2 \rangle = \langle a, b \mid a^2, b^{-n}, (ab)^2 \rangle \cong D_{-n}$ we see that $A_4 \times D_{-n} \cong A_4 \times D_n$ hence, when $n \equiv -2 \pmod{6}$,

$$A_4 \times D_n \cong \langle x, y \mid x^6, y^{-n-1} = x^3yx^3, x = y^3xy^3, (xy)^2 \rangle,$$

and the theorem is proved. \square

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