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## On the Asymptotics of Fourier Coefficients for the Potential in Hill's Equation

*Haskız Coşkun*

### Abstract

We consider Hill's equation  $y'' + (\lambda - q)y = 0$  where  $q \in L^1[0, \pi]$ . We show that if  $l_n$ —the length of the  $n$ -th instability interval—is of order  $O(n^{-k})$  then the real Fourier coefficients  $a_n, b_n$  of  $q$  are of the same order for  $(k = 1, 2, 3)$ , which in turn implies that  $q^{(k-2)}$ , the  $(k-2)$ th derivative of  $q$ , is absolutely continuous almost everywhere for  $k = 2, 3$ .

### 1. Introduction

We consider the differential equation

$$y''(t) + (\lambda - q(t))y(t) = 0 \quad (1)$$

on  $[0, \pi]$  where  $\lambda$  is a real parameter,  $q(t)$  is integrable over the interval  $[0, \pi]$  which may be extended to the real line by periodicity. We associate three types of boundary conditions with (1) over  $[0, \pi]$ :

- periodic boundary conditions

$$y(0) = y(\pi), y'(0) = y'(\pi), \quad (2)$$

- semi-periodic boundary conditions

$$y(0) = -y(\pi), y'(0) = -y'(\pi), \quad (3)$$

- auxiliary boundary conditions

$$y(\tau) = y(\tau + \pi) = 0, \quad (4)$$

where  $0 \leq \tau < \pi$ .

Let  $\lambda_n (n = 1, 2, \dots)$  denote the periodic eigenvalues of the problem (1) and (2), and  $\mu_n (n = 1, 2, \dots)$  the semi-periodic eigenvalues of the problem (1) and (3), and  $\Lambda_n(\tau), (n = 1, 2, \dots)$  the auxiliary eigenvalues of (1) with condition (4). It is well known that , see for example [4],

$$\lambda_0 < \mu_0 \leq \mu_1 < \lambda_1 \leq \lambda_2 < \mu_2 \leq \mu_3 < \dots$$

We define the instability intervals of (1) as follows:

$I_0 = (-\infty, \lambda_0), I_{2m+1} = (\mu_{2m}, \mu_{2m+1}), I_{2m+2} = (\lambda_{2m+1}, \lambda_{2m+2})$ , and for  $n \geq 1$  their lengths by  $l_n$ . It is shown in [10] that (1) with (4) is equivalent to the Dirichlet problem

$$y''(t) + (\lambda - q(t + \tau))y(t) = 0, \quad (5)$$

$$y(0) = y(\pi) = 0. \quad (6)$$

Some asymptotic estimates for the eigenvalues and instability intervals of (1) are provided in [6],[7], respectively. We suppose without loss of generality that

$$\int_0^\pi q(t)dt = 0$$

and let  $a_n, b_n$  denote the real Fourier coefficients of  $q$  on  $[0, \pi]$ , i.e,

$$a_n = \frac{2}{\pi} \int_0^\pi q(t) \cos(2nt) dt, b_n = \frac{2}{\pi} \int_0^\pi q(t) \sin(2nt) dt. \quad (7)$$

As a result of the needs of modern physics, inverse problems became a hot research area. One of the earliest such problems formulated and solved by Ambarzumian[1]. In 1929, he considered the following question:

$$y''(t) + (\lambda - q(t))y(t) = 0$$

and

$$y'' + \lambda y = 0$$

subject to the boundary conditions

$$y'(0) = y'(\pi) = 0$$

with the same eigenvalues. What can be said about  $q(z)$ ? Ambarzumian's answer was that  $q(z) = 0$ .

Borg[4] considered the general problem of what can be said about  $q(z)$  from a knowledge of spectrum. Similar problems have also been investigated by Hochstadt[10] and Ungar[12]. Now, we state a result proven independently by Hochstadt and Ungar.

**Theorem 1.1** [10] *If  $q(z)$  is real and integrable, and if all finite instability intervals vanish then  $q(z) = 0$  almost everywhere.*

In this paper we assume that all finite instability intervals are  $O(n^{-k})$  and show that  $a_n, b_n = O(n^{-k})$ , from which we deduce that  $q^{(k-2)}$  is absolutely continuous a.e. ( $k = 2, 3$ ).

Central to our analysis is the following theorem of Hochstadt [10] which involves the auxiliary eigenvalues of (1) considered on the interval  $[\tau, \tau + \pi]$  where  $0 \leq \tau < \pi$  with boundary conditions

$$y(\tau) = y(\tau + \pi) = 0.$$

**Theorem 1.2** [10] *The ranges of  $\Lambda_{2m}(\tau)$  and  $\Lambda_{2m+1}(\tau)$ , as functions of  $\tau$  are  $[\mu_{2m}, \mu_{2m+1}]$  and  $[\lambda_{2m+1}, \lambda_{2m+2}]$  respectively.*

**Remark:** We make use of the Theorem 1.2 in the sense that if all finite instability intervals are  $O(n^{-k})$  then  $\Lambda_n(\tau_2) - \Lambda_n(\tau_1) = O(n^{-k})$  for any  $\tau_1, \tau_2 \in [0, \pi]$ .

We also state a sequence of Lemmas which will be used in proving our results. Let

$$c_n = \frac{1}{\pi} \int_0^\pi q(z) e^{-2inz} dz \quad (8)$$

be the Fourier coefficient of  $q(z)$  over  $[0, \pi]$ .

**Lemma 1.1** [10] *Let  $q(z)$  be periodic with period  $\pi$ , integrable over  $[0, \pi]$  and such that*

$$c_n = O\left(\frac{1}{n^2}\right)$$

*as  $n \rightarrow \infty$ . Then  $q(z)$  is absolutely continuous almost everywhere.*

**Lemma 1.2** *Let  $q(z)$  be periodic with period  $\pi$ , integrable over  $[0, \pi]$  and such that*

$$c_n = O\left(\frac{1}{n^3}\right)$$

*as  $n \rightarrow \infty$ . Then  $q'(z)$  is absolutely continuous almost everywhere.*

**Proof.** The proof of Lemma 1.1 goes through.

**Lemma 1.3** [9] *For  $k = 1, 2, 3, \dots$ ,  $\tau \leq t \leq \tau + \pi$*

$$\Theta(t, \tau) - \Theta_k(t, \tau) = o(\Lambda^{-k/2})$$

*as  $\Lambda \rightarrow \infty$ .*

**Lemma 1.4** [2] *For  $q$  integrable and for any  $x_1, x_2$  such that  $\tau \leq x_1 < x_2 \leq \tau + \pi$*

$$\int_{x_1}^{x_2} q(t) \sin(2\Lambda^{1/2}t) dt = o(1)$$

*as  $\Lambda \rightarrow \infty$ .*

Now, we introduce the function  $\Theta(t, \Lambda, \tau)$ , the so-called modified Prüfer transformation of [2], which is defined for any given solution of (1) as

$$\tan \Theta(t, \Lambda, \tau) = \frac{\Lambda^{1/2} y(t, \tau)}{y'(t, \tau)},$$

for  $\tau \leq t \leq \tau + \pi$ . This fixes  $\Theta$  to within additive multiples of  $\pi$ . For definiteness we assume that  $0 \leq \Theta(t, \tau) \leq \pi$  and observe that the boundary conditions (4) correspond to

$$\Theta(t, \tau) = 0, \Theta(t, \tau + \pi) = (n + 1)\pi, \quad (9)$$

similarly the boundary conditions (6) correspond to

$$\Theta(t, 0) = 0, \Theta(t, \pi) = (n + 1)\pi. \quad (10)$$

From now on, we suppress the dependence of  $\Theta$  on  $\Lambda$  and write  $\Theta(t, \tau)$  instead of  $\Theta(t, \Lambda, \tau)$ . Under the Prüfer transformation the differential equation corresponding to (1) can be written as

$$\Theta'(t, \tau) = \Lambda^{1/2} - \frac{1}{2}\Lambda^{-1/2}q(t) + \frac{1}{2}\Lambda^{-1/2}q(t)\cos(2\Theta(t, \tau)), \quad (11)$$

and from (11)

$$\Theta(t, \tau) = (\Lambda^{1/2})(t - \tau) - \frac{1}{2}\Lambda^{-1/2} \int_{\tau}^t q(s)ds + \frac{1}{2}\Lambda^{-1/2}q(t) \int_{\tau}^t q(s)\cos(2\Theta(s, \tau))ds. \quad (12)$$

We define a sequence of approximating functions for (12) as follows:

$$\begin{aligned} \Theta_1(t, \tau) &:= (\Lambda^{1/2})(t - \tau) - \frac{1}{2}\Lambda^{-1/2} \int_{\tau}^t q(s)ds, \\ \Theta_{k+1}(t, \tau) &:= \Theta_k(t, \tau) + \frac{1}{2}\Lambda^{-1/2} \int_{\tau}^t q(s)\cos(2\Theta_k(s, \tau))ds \end{aligned} \quad (13)$$

for  $k = 1, 2, \dots$ , and  $\tau \leq t \leq \tau + \pi$ .

## 2. The Results

**Theorem 2.1** *For any integer  $k$ , the auxiliary eigenvalues of (1), as functions of  $\tau$ , satisfy*

$$(n + 1)\pi = \Lambda_n^{1/2}(\tau)\pi + \frac{1}{2}\Lambda^{-1/2} \int_0^{\pi} q(t + \tau)\cos(2\Theta_k(t, \tau))dt + O(n^{-(k+1)}) \quad (14)$$

as  $n \rightarrow \infty$ .

**Proof.** We consider the differential equation (5) with the boundary conditions (6). From (11) we get

$$\Theta'(t, \tau) = \Lambda^{1/2} - \frac{1}{2}\Lambda^{-1/2}q(t + \tau) + \frac{1}{2}\Lambda^{-1/2}q(t + \tau)\cos(2\Theta(t, \tau)). \quad (15)$$

We also know from Lemma 1.3 that

$$\Theta(t, \tau) - \Theta_k(t, \tau) = o(\Lambda^{-k/2}), \quad (16)$$

so that

$$\cos(2\Theta(t, \tau)) = \cos(2\Theta_k(t, \tau)) + O(\Lambda^{-k/2}). \quad (17)$$

Substituting (17) into (15) we obtain

$$\Theta'(t, \tau) = \Lambda^{1/2} - \frac{1}{2}\Lambda^{-1/2}q(t + \tau) + \frac{1}{2}\Lambda^{-1/2}q(t + \tau)\cos(2\Theta_k(t, \tau)) + O(\Lambda^{-\frac{(k+1)}{2}}). \quad (18)$$

Integrating (18) with respect to  $\tau$  on  $[0, \pi]$  and using (10) we complete the proof.

**Corollary 2.1** *As  $n \rightarrow \infty$  the auxiliary eigenvalues of (1), as functions of  $\tau$ , satisfy*

$$\Lambda_n^{1/2}(\tau) = (n + 1) + \frac{1}{(n + 1)}F_1(n, \tau) + O(n^{-2}) \quad (19)$$

where

$$F_1(n, \tau) = -\frac{1}{2\pi} \int_0^\pi q(t + \tau)\cos(2(n + 1)t)dt. \quad (20)$$

**Proof.** From (13), we see that

$$\cos(2\theta_1(t, \tau)) = \cos(2\Lambda^{1/2}t) + O(\Lambda^{-1/2}). \quad (21)$$

Substituting (21) into (14) and using reversion we complete the proof.

**Theorem 2.2** *Let  $q(t)$  be real-valued integrable function on  $[0, \pi]$ . If  $l_n = O(n^{-1})$ , then  $a_n, b_n = O(n^{-1})$  as  $n \rightarrow \infty$ .*

**Proof.** It is easily seen that

$$F_1(n, \tau) = -\frac{1}{4}[\cos(2(n+1)\tau)a_{n+1} + \sin(2(n+1)\tau)b_{n+1}], \quad (22)$$

where  $a_{n+1}$  and  $b_{n+1}$  are defined in (7), and  $F_1(n, \tau)$  is given by (20). From Corollary 2.1, for any  $\tau_1, \tau_2 \in [0, \pi)$

$$\begin{aligned} \Lambda_n^{1/2}(\tau_2) - \Lambda_n^{1/2}(\tau_1) &= \frac{1}{4(n+1)} \{[\cos(2(n+1)\tau_1) - \cos(2(n+1)\tau_2)]a_{n+1} \\ &\quad + [\sin(2(n+1)\tau_1) - \sin(2(n+1)\tau_2)]b_{n+1}\} + O(n^{-2}) \\ &= \frac{1}{2(n+1)} \{ \sin((n+1)(\tau_1 + \tau_2)) \sin((n+1)(\tau_2 - \tau_1))a_{n+1} \\ &\quad + \cos((n+1)(\tau_1 + \tau_2)) \sin((n+1)(\tau_1 - \tau_2))b_{n+1} \} + O(n^{-2}) \\ &= \frac{1}{2(n+1)} \sin((n+1)(\tau_2 - \tau_1)) \{ \sin((n+1)(\tau_1 + \tau_2))a_{n+1} \\ &\quad - \cos((n+1)(\tau_1 + \tau_2))b_{n+1} \} + O(n^{-2}). \end{aligned} \quad (23)$$

On the other hand, from the assumption that  $l_n = O(n^{-1})$  and Lemma 1.4.

$$\Lambda_n(\tau_2) - \Lambda_n(\tau_1) = O(n^{-1})$$

and hence of

$$\Lambda_n^{1/2}(\tau_2) - \Lambda_n^{1/2}(\tau_1) = \frac{\Lambda_n(\tau_2) - \Lambda_n(\tau_1)}{\Lambda_n^{1/2}(\tau_2) + \Lambda_n^{1/2}(\tau_1)} = O(n^{-2}). \quad (24)$$

The result follows from (23) and (24).

**Corollary 2.2** *As  $n \rightarrow \infty$  the auxiliary eigenvalues of (1), as functions of  $\tau$ , satisfy*

$$\Lambda_n^{1/2}(\tau) = (n+1) + \frac{1}{(n+1)}F_1(n, \tau) + \frac{1}{(n+1)^2}F_2(n, \tau) + O(n^{-3}), \quad (25)$$

where  $F_1(n, \tau)$  is given by (20) and

$$F_2(n, \tau) = -\frac{1}{2\pi} \int_0^\pi q(t+\tau) \left( \int_0^t q(s+\tau) ds \right) \sin(2(n+1)t) dt$$



$$+ \frac{1}{2\pi} \int_0^\pi q(t + \tau) \left( \int_0^t q(s + \tau) \cos(2(n+1)s) ds \right) \sin(2(n+1)t) dt. \quad (26)$$

**Proof.** From (13) we observe that for  $k = 1$

$$\begin{aligned} \cos(2\Theta_2(t, \tau)) &= \cos(2\Lambda^{1/2}t) + \Lambda^{-1/2} \sin(2\Lambda^{1/2}t) \left( \int_0^t q(s + \tau) ds \right) \\ &\quad - \Lambda^{-1/2} \sin(2\Lambda^{1/2}t) \left( \int_0^t q(s + \tau) \cos(2\Lambda^{1/2}s) ds \right) + O(\Lambda^{-1}). \end{aligned} \quad (27)$$

Substituting (27) into (14) and using reversion we complete the proof.

**Theorem 2.3** *Let  $q(t)$  be real-valued integrable function on  $[0, \pi]$ . If  $l_n = O(n^{-2})$  then  $a_n, b_n = O(n^{-2})$  as  $n \rightarrow \infty$ .*

**Proof.** Since  $l_n = O(n^{-2})$  by assumption, it is  $O(n^{-1})$  as well. Hence  $a_n, b_n = O(n^{-1})$  by Theorem 2.2. From this and Lemma 1.4 we observe that

$$F_2(n, \tau) = O(n^{-1}).$$

Therefore, (25) reduces to

$$\Lambda_n^{1/2}(\tau) = (n+1) + \frac{1}{(n+1)} F_1(n, \tau) + O(n^{-3}).$$

By a similar argument in Theorem 2.2, we complete the proof.

**Corollary 2.3** *Let  $q(t)$  be a real-valued integrable function on  $[0, \pi]$ . If  $l_n = O(n^{-2})$  as  $n \rightarrow \infty$  then  $q(t)$  is absolutely continuous almost everywhere.*

**Proof.** It follows from Theorem 2.3 and Lemma 1.1

**Corollary 2.4** *As  $n \rightarrow \infty$  the auxiliary eigenvalues of (1), as functions of  $\tau$ , satisfy*

$$\Lambda_n^{1/2}(\tau) = (n+1) + \frac{1}{(n+1)} F_1(n, \tau) + \frac{1}{(n+1)^2} F_2(n, \tau) + \frac{1}{(n+1)^3} F_3(n, \tau) + O(n^{-4}), \quad (28)$$

where  $F_1(n, \tau)$  is given by (20),  $F_2(n, \tau)$  is given by (26) and

$$\begin{aligned}
 F_3(n, \tau) = & -\frac{1}{4(n+1)^3\pi} \int_0^\pi q(t+\tau) \left[ \int_0^t q(s+\tau) ds - \int_0^t q(s+\tau) \cos(2(n+1)s) ds \right]^2 \\
 & \times \cos(2(n+1)t) dt \\
 & + \frac{1}{2(n+1)^3\pi} \int_0^\pi q(t+\tau) \left( \int_0^t q(s+\tau) \left( \int_0^s q(l+\tau) dl \right) \sin(2(n+1)s) ds \right) \\
 & \times \sin(2(n+1)t) dt. \tag{29}
 \end{aligned}$$

**Proof.** From (13) for  $k = 2$  we have

$$\begin{aligned}
 \cos(2\Theta_3(t, \tau)) = & \cos(2\Lambda^{1/2}t) + \Lambda^{-1/2} \sin(2\Lambda^{1/2}t) \left( \int_0^t q(s+\tau) ds \right) \\
 & - \Lambda^{-1/2} \sin(2\Lambda^{1/2}t) \left( \int_0^t q(s+\tau) \cos(2\Lambda^{1/2}s) ds \right) \\
 & - \frac{1}{2} \Lambda^{-1} \left( \int_0^t q(s+\tau) ds - \int_0^t q(s+\tau) \cos(2\Lambda^{1/2}s) ds \right)^2 \cos(2\Lambda^{1/2}t) \\
 & - \Lambda^{-1} \left( \int_0^t q(s+\tau) \left( \int_0^s q(l+\tau) dl \right) \sin(2\Lambda^{1/2}s) ds \right) \sin(2\Lambda^{1/2}t) \\
 & + O(\Lambda^{-3/2}). \tag{30}
 \end{aligned}$$

Substituting (30) into (14), and using reversion we complete the proof.

**Theorem 2.4** *Let  $q(t)$  be real-valued integrable function on  $[0, \pi]$ . If  $l_n = O(n^{-3})$  then  $a_n, b_n = O(n^{-3})$  as  $n \rightarrow \infty$ .*

**Proof.** Since  $l_n = O(n^{-3})$  by assumption, it is  $O(n^{-2})$  as well. Hence  $a_n, b_n = O(n^{-2})$  by Theorem 2.3. Therefore, the terms involving  $F_2(n, \tau)$  and  $F_3(n, \tau)$  in (28) are included in the error term by Lemma 1.4. By a similar argument in Theorem 2.2 we complete the proof.

**Corollary 2.5** *Let  $q(t)$  be a real-valued function on  $[0, \pi]$ . If  $l_n = O(n^{-3})$  then  $q'(t)$  is absolutely continuous almost everywhere.*

**Proof.** It follows from Theorem 2.4 and Lemma 1.2.

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