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Abstract

We consider some graph type hypersurfaces in a semi-Euclidean space $\mathbb{R}^{n+1}_q$ and give conditions of the dimension $n+1$ and the index q when a hypersurface is lightlike, totally geodesic and minimal.

Key Words: graph, lightlike hypersurface, minimal, semi-Euclidean space, totally geodesic.

1. Introduction

The theory of submanifolds is one of the most important topics of differential geometry. While the geometry of Riemannian or semi-riemannian (i.e., non-degenerate) submanifolds is fully developed, the study of lightlike (i.e., degenerate) submanifolds is relatively new and in a developing stage (see [2]). A typical example of lightlike submanifold is the lightcone in $\mathbb{R}^3_1$, and the surface of revolution with degenerate metric in $\mathbb{R}^3_1$ is the ligtcone. This fact indicates that non-trivial lightlike submanifolds will be given in $\mathbb{R}^n_q$ of higher dimension n and greater index q.

The purpose of this paper is to consider a lightlike hypersurface M in $\mathbb{R}^n_q$ under the condition that M is "graph type", i.e., M is locally defined by a function of coordinats of $\mathbb{R}^n_q$.

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After preliminaries of section 2, in section 3, we consider the lightcone of $\mathbb{R}^3_1$. Section 4 is devoted to study graph type lightlike hypersurfaces in $\mathbb{R}^{n+1}_q$. In section 5, we consider non-degenerate graph type hypersurfaces in $\mathbb{R}^{n+1}_q$ using a similar technique of section 4.

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2. Preliminaries

Let $\mathbb{R}^{n+1}_q$ be the (n+1)-dimensional semi-Euclidean space of index q with the natural metric $\overline{g}$. So, for $x = (x^1, \dots, x^{n+1}) \in \mathbb{R}^{n+1}_q, \overline{g}(x, x) = -\sum_{i=1}^{q}(x^i)^2 + \sum_{j=q+1}^{n+1}(x^j)^2$. By $\overline{\nabla}$, we denote the covariant derivative of $\mathbb{R}^{n+1}_q$ given by $\overline{g}$.

We consider a hypersurface M of $\mathbb{R}^{n+1}_q$ with the metric g induced by $\overline{g}$.

When g is non-degenerate, a function $B : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)^{\perp}$ such that

$$B(X, Y) = nor(\overline{\nabla} \times Y)$$

is called the second fundamental form of M. Let $\{e_1, \dots, e_n\}$ be a local frame of M. Then the mean curvature vector field H of M is defined by setting

$$H = \frac{1}{n}\sum_{i=1}^{n} g(e_i, e_i)B_{ii},$$

where $B_{ii} = B(e_i, e_i)$ and g is the metric of M induced by $\overline{g}$.

A hypersurface M of $\mathbb{R}^{n+1}_q$ is totally geodesic provided its second fundamental form vanishes, i.e., $B = 0$. If the mean curvature vector field H of M vanishes, then M is called minimal.

Next we recall the definition of a lightlike hypersurface [2], i.e., the case when g is degenerate.

Let p be a point of a hypersurface M. We consider

$$T_pM^{\perp} = \{V_p \in T_p\mathbb{R}^{n+1}_q \mid \overline{g}(V_p, X_p) = 0, \forall X_p \in T_pM\}.$$

The radical (or null space) $RadT_pM$ is defined by

$$RadT_pM = T_pM \cap T_pM^{\perp}.$$

If $RadT_pM \neq \phi$ at any point $p \in M$, then M is called a lightlike hypersurface of $\mathbb{R}^{n+1}_q$. In this case, $TM^\perp = \cup_{p \epsilon M} T_pM^\perp$ is a distribution on M and $RadTM = TM^\perp$.

A complementary vector bundle $S(TM)$ of $TM^\perp$ in TM is called a screen distribution on M. For $S(TM)$, there exists a unique vector bundle $tr(TM)$ of rank 1 over M, such that for any non-zero section ξ of $TM^\perp$, there exists a unique section N of $tr(TM)$ satisfying

$$\overline{g}(N, \xi) = 1, \overline{g}(N, N) = \overline{g}(N, W) = 0, \forall W \in \Gamma(S(TM)).$$

Therefore we have decompositions

$$\begin{aligned} TM &= RadTM + S(TM), \\ T\mathbb{R}^{n+1}_q &= TM \oplus tr(TM) \\ &= S(TM) \perp (RadTM) \oplus tr(TM)) \\ &= S(TM) \perp S(TM)^\perp. \end{aligned}$$

A typical example of a lightlike hypersurface of $\mathbb{R}^{n+1}_q$ is the lightcone.

Let $F : D \to \mathbb{R}$ be a smooth function, where D is an open set of $\mathbb{R}^{n+1}_q$. Then a graph

$$M = \{(x^1, \ldots, X^{n+1}) \in \mathbb{R}^{n+1}_q \mid x^1 = F(x^2, \ldots, x^{n+1})\},$$

is a hypersurface of $\mathbb{R}^{n+1}_q$. It is well known that this hypersurface is lightlike if and only if F is a solution of the differential equation

$$1 + \sum_{i=2}^{q} \left(\frac{\partial F}{\partial x^i}\right)^2 = \sum_{j=q+1}^{n+1} \left(\frac{\partial F}{\partial x^j}\right)^2. \tag{2.2}$$

3. Lightlike Surfaces of Revolution in $\mathbb{R}^3_1$

First we consider lightlike surfaces of revolution with timelike rotation axis. We shall take x^1x^3 plane as the plane of a r egular curve C and x^3 (timelike) axis as the rotation axis. For a suitable interval (a, b), the curve C is given by

$$x^3 = F(u), u \in (a, b).$$

Then a surface of revolution is obtained as a map

$$X(u,v) = (u \cos v, u \sin v, F(u)),$$

hence,

$$x^1 = x^1, x^2 = x^2, x^3 = F(\sqrt{(x^1)^2 + (x^2)^2}).$$

By differentiating the funcition F, it follows that

$$\frac{\partial F}{\partial x^1} = F' \frac{x^1}{\sqrt{(x^1)^2 + (x^2)^2}},$$

$$\frac{\partial F}{\partial x^2} = F' \frac{x^1}{\sqrt{(x^1)^2 + (x^2)^2}}.$$

Substituting these equations into (2.1), we have

$$\left(\frac{\partial F}{\partial x^1}\right)^2 + \left(\frac{\partial F}{\partial x^2}\right)^2 = (F')^2 = 1.$$

Therefore, we obtain

Theorem 3.1. *If M be a lightlike surface of revolution in $\mathbb{R}^3_1$ with timelike axis, then M is the lightcone in $\mathbb{R}^3_1$.*

Next we consider lightlike surfaces of revolution with spacelike axis. Let C be a regular plane curve on x^2x^3 plane with

$$x^2 = F(u), u \in (a,b).$$

If we set x^2 axis as the rotation axis, then a surface of revolution is given as a map

$$X(u,v) = (u \cos v, F(u), u \sin v),$$

hence,

$$x^1 = x^1, x^2 = x^2, x^3 = F^{-1}(x^2)\ \sin\,(\cos^{-1}\ \frac{x^1}{F^{-1}(x^2)}).$$

By differentiating x^3, it follows that

$$\frac{\partial x^3}{\partial x^1} = \frac{x^1}{F^{-1}(x^2)}\frac{1}{\sqrt{1 - \frac{(x^1)^2}{F^{-1}(x^2)^2}}},$$

$$\frac{\partial x^3}{\partial x^2} = (F^{-1})'(x^2)\ \sin\,(\cos^{-1}\frac{x^1}{F^{-1}(x^2)}) - \frac{(x^1)^2(F^{-1})'(x^2)}{F^{-1}(x^2)^2}\frac{1}{\sqrt{1 - \frac{(x^1)^2}{F^{-1}(x^2)^2}}}.$$

Substituting these equations into (2.1), we have

$$\frac{(x^1)^2}{(F^{-1}(x^2))^2 - (x^1)^2} + (F^{-1})'(x^2)^2 + (F^{-1})'(x^2)^2\frac{(x^1)^2}{F^{-1}(x^2)^2} + \frac{(x^1)^4(F^{-1})'(x^2)^2}{F^{-1}(x^2)^2((F^{-1}(x^2))^2 - (x^1)^2)} = 1,$$

so that

$$(F^{-1}(x^2))^2(2(x^1)^2 + ((F^{-1})'(x^2))^4 - ((F^{-1})(x^2))^2) = 0.$$

Differentiating this equation by x^1 repeatedly, we obtain

$$(F^{-1})'(x^2) = 0,$$

which means that the function F^{-1} is a constant function. This is the contradiction. Hence we have

Theorem 3.2. *There is no lightlike surface of revolution with spacelike axis in $\mathbb{R}^3_1$.*

4. Some Graph Type Lightlike Hypersurfaces

In the sequel, we consider a graph type hypersurface (M, g) in $(\mathbb{R}^{n+1}_q, \overline{g})$ of following form

$$x^{n+1} = x^1 + \ldots + x^k + F(x^{k+1} + \ldots + x^n),$$

where F is a non-trivial smooth function.

This section is devoted to give an existence condition of the lightlike hypersurface (M, g) by the index of the semi-Euclidean space $\mathbb{R}^{n+1}_q$.

A local frame $(e_1, \ldots, e_n)$ of M is given as

$$\begin{aligned} e_i &= \tfrac{\partial}{\partial x^i} + \tfrac{\partial}{\partial x^{n+1}}(i = 1, \ldots k), \\ e_i &= \tfrac{\partial}{\partial x^i} + F' \tfrac{\partial}{\partial x^{n+1}}(i = k+1, \ldots n), \end{aligned} \tag{4.1}$$

where $F' = \frac{\partial F}{\partial x^j}(j = k+1, \ldots, n)$.

We remark tha for $F(x^{k+1} + \ldots + x^n)$, $\frac{\partial F}{\partial x^i} = \frac{\partial F}{\partial x^j} = F'(i, j = k+1, \ldots n)$.

When the index q satisfies $q \leq k$, we consider a vector field

$$\upsilon = \sum_{i=1}^{q} \frac{\partial}{\partial x^i} - \sum_{i=q+1}^{k} \frac{\partial}{\partial x^i} - \sum_{i=k+1}^{n} F' \frac{\partial}{\partial x^i} + \frac{\partial}{\partial x^{n+1}} \tag{4.2}$$

on $\mathbb{R}^{n+1}_q$. From (4.1), it follows that

$$\begin{aligned} &\overline{g}(e_i, \upsilon) = 0(i = 1, \ldots, n), \\ &\overline{g}(\upsilon, \upsilon) = k - 2q + 1 + (n-k)(F')^2. \end{aligned} \tag{4.3}$$

Hence if M is lightlike, we have $n = k = 2q-1$ and $\upsilon = \sum_{i=1}^{q} e_i - \sum_{i=q+1}^{n} e_i \in \Gamma(RadTM)$.

When the index q satisfies $q > k$, we consider a vector field

$$\upsilon = \sum_{i=1}^{k} \frac{\partial}{\partial x^i} + \sum_{i=k+1}^{q} F' \frac{\partial}{\partial x^i} + \sum_{i=q+1}^{n} F' \frac{\partial}{\partial x^i} + \frac{\partial}{\partial x^{n+1}} \tag{4.4}$$

on $\mathbb{R}^{n+1}_q$. Then

$$\begin{aligned} &\overline{g}(e_i, \upsilon) = 0(i = 1, \ldots, n), \\ &\overline{g}(\upsilon, \upsilon) = 1 - k + (n - 2q + k)(F')^2. \end{aligned} \tag{4.3}$$

Hence if M is lightlike, we obtain $2q = n+1$ and $\upsilon = \sum_{i=1}^{q} e_i - \sum_{i=q+1}^{n} e_i \in \Gamma(RadTM)$.

Therefore we have

Theorem 4.1. *If the semi-Euclidean space $\mathbb{R}^{n+1}_q$ has a graph type lightlike hypersurface*

$$x^{n+1} = x^1 + \ldots + x^k + F(x^{k+1} + \ldots + x^n),$$

where F is a non-trivial smooth function, the dimension of the ambient space is two times of the index of it, that is $\mathbb{R}^{2q}_q$, and the graph type hypersurfaces reduce to

$$x^{2q} = x^1 + \ldots + x^{2q-1}$$

or

$$x^{2q} = x^1 + F(x^2 + \ldots + x^{2q-1}).$$

Remarks.

For the graph type hypersurface

$$x^{2q} = x^1 + \ldots + x^{2q-1},$$

υ is a local section of $RadTM$ and a local section of $tr(TM)$ is given by

$$V = -\sum_{i=1}^{2q-1} \frac{\partial}{\partial x^i} + \frac{\partial}{\partial x^{2q}}.$$

Therefore,

$$\begin{aligned} S(TM)^{\perp} &= Span\{\upsilon, V\}, \\ S(TM) &= Span\{e_1 - e_2, e_1 - e_3, \ldots, e_1 - e_q, e_{q+1}, \ldots, e_{2q-1}\}. \end{aligned}$$

Moreover, we have decompositions

$$\begin{aligned} S(TM) &= S(TM)^- \oplus S(TM)^+, \\ S(TM)^- &= Span\left\{\tfrac{\partial}{\partial x^1} - \tfrac{\partial}{\partial x^2}, \tfrac{\partial}{\partial x^1} - \tfrac{\partial}{\partial x^3}, \ldots, \tfrac{\partial}{\partial x^1} - \tfrac{\partial}{\partial x^q}\right\}, \\ S(TM)^+ &= Span\left\{\tfrac{\partial}{\partial x^{q+1}} + F'\tfrac{\partial}{\partial x^{2q}}, \tfrac{\partial}{\partial x^{q+2}} + F'\tfrac{\partial}{\partial x^{2q}}, \ldots, \tfrac{\partial}{\partial x^{2q-1}} + F'\tfrac{\partial}{\partial x^{2q}}\right\}. \end{aligned}$$

For the graph type hypersurface

$$x^{2q} = x^1 + F(x^2 + \ldots + x^{2q-1}),$$

υ is a local section of $RadTM$ and

$$V = -\frac{\partial}{\partial x^1} - \sum_{i=2}^{2q-1} F' \frac{\partial}{\partial x^i} + \frac{\partial}{\partial x^{2q}},$$

is a local section of $tr(TM)$, so

$$\begin{aligned}
S(TM)^{\perp} &= Span\{\upsilon, V\}, \\
S(TM) &= Span\{F'e_1 - e_2, F'e_1 - e_3, \ldots, F'e_1 - e_q,\ e_{q+1}, \ldots, e_{2q-1}\}, \\
S(TM) &= S(TM)^{-} \oplus S(TM)^{+}, \\
S(TM)^{-} &= Span\{F'\tfrac{\partial}{\partial x^1} - \tfrac{\partial}{\partial x^2}, F'\tfrac{\partial}{\partial x^1} - \tfrac{\partial}{\partial x^3}, \ldots, F'\tfrac{\partial}{\partial x^1} - \tfrac{\partial}{\partial x^q}\}, \\
S(TM)^{+} &= Span\{\tfrac{\partial}{\partial x^{q+1}} + \tfrac{\partial}{\partial x^{2q}}, \tfrac{\partial}{\partial x^{q+2}} + \tfrac{\partial}{\partial x^{2q}}, \ldots, \tfrac{\partial}{\partial x^{2q-1}} + \tfrac{\partial}{\partial x^{2q}}\}.
\end{aligned}$$

5. Some Graph Type Non-Degenerate Hypersurfaces

In this section, we consider the non-degenerate case of graph type hypersurface

$$x^{n+1} = x^1 + \ldots + x^k + F(x^{k+1} + \ldots + x^n).$$

To simplify the presentation, we define two (n,m)-matrix ($S(n,m)$ and $S(f)(n,m)$ by setting:

Each element of $S(n,m)$ is 1,

Each element of $S(f)(n,m)$ is f,

where f means a function, respectively, and we write (n,n)-unit matrix as $E(n)$. An (n,n)-matrix S, is defined as

$$S = \begin{bmatrix} S(q,q) & S(q,k-q) & S(F')(q,n-k) \\ S(k-q,q) & S(k-q,k-q) & S(F')(k-q,n-k) \\ S(F')(n-k,q) & S(F')(n-k,k-q) & S((F')^2)(n-k,n-k) \end{bmatrix}. \tag{5.1}$$

When the hypersurface M is non-degenerate, the vector υ defined by (4.2) or (4.4) is a normal vector of M. Then the second fundamental form B_{ij} is defined by

$$B_{ij} = \overline{g}(\overline{\nabla}_{ej} e_i, \frac{\upsilon}{\sqrt{\mid \upsilon \mid}})$$

Case $q \leq k$.

In this case, the second fundamental form B_{ij} deduces

$$B_{ij} = \begin{cases} 0 & (i = 1, \dots, k \ or \ j = 1, \dots, k) \\ \frac{F''}{\sqrt{|\upsilon|}} & (i, j = k+1, \dots, n). \end{cases}$$

That is,

$$(B_{ij}) = A_1 \begin{bmatrix} 0 & 0 \\ 0 & S(n-k, n-k) \end{bmatrix}, \tag{5.2}$$

where

$A_1 = \frac{F''}{\sqrt{|\upsilon|}}$.

The fundamental tensors g_{ij} and g^{ij} of M with respect to the local frame $(e_1, \dots, e_n)$ is represented as

$$\begin{aligned} (g_{ij}) &= S + J_1, \\ (g^{ij}) &= -\tfrac{1}{|\upsilon|^2} J_1 S J_1 + J_1, \end{aligned}$$

where

$$J_1 = \begin{bmatrix} -E(q) & 0 & \\ 0 & E(k-q) & 0 \\ 0 & 0 & E(n-k) \end{bmatrix}.$$

Hence, from (5.2), it follows that

$$(g^{ik} B_{kj}) = \frac{1}{\mid \upsilon \mid} A_1 (n-k) \begin{bmatrix} 0 & 0 & S(F')(q, n-k) \\ 0 & 0 & -S(F')(k-q, n-k) \\ 0 & 0 & -S((F')^2)(n-k, n-k) \end{bmatrix} + A_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & S(n-k, n-k) \end{bmatrix},$$

so that

$$g^{ik}B_{ki} = -\frac{(n-k)^2(n-2q+1)F'F''}{\mid v \mid^{3/2}}.$$

Therefore the mean curvature H of M is

$$H = -\frac{(n-k)(n-2q+1)F''}{n \mid v \mid^{3/2}}. \tag{5.3}$$

Case $q > k$.

In this case, B_{ij} deduces

$$B_{ij} = \begin{cases} 0 & (i = 1, \ldots, k \; or \; j = 1, \ldots, k) \\ \frac{F''}{\sqrt{|v|}} & (i, j = k+1, \ldots, n). \end{cases}$$

Hence

$$(B_{ij}) = A_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & S(q-k, q-k) & S(q-k, n-k) \\ 0 & S(q-k, q-k) & S(q-k, n-k) \end{bmatrix} \tag{5.4}$$

where $A_2 = \frac{F''}{\sqrt{|v|}}$.

Since

$$\begin{aligned} (g_{ij}) &= S_2 + J_2, \\ (g^{ij}) &= -\tfrac{1}{|v|^2} J_2 S_2 J_2 + J_2, \end{aligned}$$

where

$$J_2 = \begin{bmatrix} -E(k) & 0 & \\ 0 & -E(q-k) & 0 \\ 0 & 0 & E(n-q) \end{bmatrix},$$

we have

$$(g^{ik}B_{kj}) = \frac{(1-k)}{\mid v \mid} A_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & -S(q-k, q-k) & -S(q-k, n-k) \\ 0 & S(n-q, q-k) & S(n-q, n-q) \end{bmatrix}.$$

Therefore the mean curvature H of M is

$$H = -\frac{(1-k)(n-2q+k)^2 F' F''}{n \mid v \mid^{3/2}} (n-k)((1-k)+(n-q)F'^2)F'. \tag{5.5}$$

From (5.2), (5.3), (5.4) and (5.5), we have

Theorem 5.1. *Suppose M is a non-degenerate graph type hypersurface*

$$x^{n+1} = x^1 + \ldots + x^k + F(x^{k+1} + \ldots + x^n)$$

in the semi-Euclidean space $\mathbb{R}_q^{n+1}$, where F is a non-trivial smooth function. If M is minimal then

(a) M is

$$x^{n+1} = x^1 + \ldots + x^n$$

in $\mathbb{R}_q^{n+1}$. This case, M reduces to totally geodesic.

(b) M is

$$x^{n+1} = x^1 + \ldots + x^{2q-1} + F(x^{2q} + \ldots + x^n)$$

in $\mathbb{R}_q^{n+1}$.

(c) M is

$$x^{q+1} = x^1 + F(x^2 + \ldots + x^q)$$

in $\mathbb{R}_q^{q+1}$.

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