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Some Graph Type Hypersurfaces in a Semi-Euclidean Space

Ikawa Toshihiko, Honda Kyoko

Abstract

We consider some graph type hypersurfaces in a semi-Euclidean space \mathbb{R}_q^{n+1} and give conditions of the dimension $n + 1$ and the index q when a hypersurface is lightlike, totally geodesic and minimal.

Key Words: graph, lightlike hypersurface, minimal, semi-Euclidean space, totally geodesic.

1. Introduction

The theory of submanifolds is one of the most important topics of differential geometry. While the geometry of Riemannian or semi-riemannian (i.e., non-degenerate) submanifolds is fully developed, the study of lightlike (i.e., degenerate) submanifolds is relatively new and in a developing stage (see [2]). A typical example of lightlike submanifold is the lightcone in \mathbb{R}_1^3 , and the surface of revolution with degenerate metric in \mathbb{R}_1^3 is the lightcone. This fact indicates that non-trivial lightlike submanifolds will be given in \mathbb{R}_q^n of higher dimension n and greater index q .

The purpose of this paper is to consider a lightlike hypersurface M in \mathbb{R}_q^n under the condition that M is “graph type”, i.e., M is locally defined by a function of coordinates of \mathbb{R}_q^n .

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After preliminaries of section 2, in section 3, we consider the lightcone of \mathbb{R}_1^3 . Section 4 is devoted to study graph type lightlike hypersurfaces in \mathbb{R}_q^{n+1} . In section 5, we consider non-degenerate graph type hypersurfaces in \mathbb{R}_q^{n+1} using a similar technique of section 4.

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2. Preliminaries

Let \mathbb{R}_q^{n+1} be the $(n+1)$ -dimensional semi-Euclidean space of index q with the natural metric \bar{g} . So, for $x = (x^1, \dots, x^{n+1}) \in \mathbb{R}_q^{n+1}$, $\bar{g}(x, x) = -\sum_{i=1}^q (x^i)^2 + \sum_{j=q+1}^{n+1} (x^j)^2$. By $\bar{\nabla}$, we denote the covariant derivative of \mathbb{R}_q^{n+1} given by \bar{g} .

We consider a hypersurface M of \mathbb{R}_q^{n+1} with the metric g induced by \bar{g} .

When g is non-degenerate, a function $B : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)^\perp$ such that

$$B(X, Y) = \text{nor}(\bar{\nabla} \times Y)$$

is called the second fundamental form of M . Let $\{e_1, \dots, e_n\}$ be a local frame of M . Then the mean curvature vector field H of M is defined by setting

$$H = \frac{1}{n} \sum_{i=1}^n g(e_i, e_i) B_{ii},$$

where $B_{ii} = B(e_i, e_i)$ and g is the metric of M induced by \bar{g} .

A hypersurface M of \mathbb{R}_q^{n+1} is totally geodesic provided its second fundamental form vanishes, i.e., $B = 0$. If the mean curvature vector field H of M vanishes, then M is called minimal.

Next we recall the definition of a lightlike hypersurface [2], i.e., the case when g is degenerate.

Let p be a point of a hypersurface M . We consider

$$T_p M^\perp = \{V_p \in T_p \mathbb{R}_q^{n+1} \mid \bar{g}(V_p, X_p) = 0, \forall X_p \in T_p M\}.$$

The radical (or null space) $Rad T_p M$ is defined by

$$Rad T_p M = T_p M \cap T_p M^\perp.$$

If $RadT_pM \neq \phi$ at any point $p \in M$, then M is called a lightlike hypersurface of \mathbb{R}_q^{n+1} . In this case, $TM^\perp = \cup_{p \in M} T_pM^\perp$ is a distribution on M and $RadTM = TM^\perp$.

A complementary vector bundle $S(TM)$ of TM^\perp in TM is called a screen distribution on M . For $S(TM)$, there exists a unique vector bundle $tr(TM)$ of rank 1 over M , such that for any non-zero section ξ of TM^\perp , there exists a unique section N of $tr(TM)$ satisfying

$$\bar{g}(N, \xi) = 1, \bar{g}(N, N) = \bar{g}(N, W) = 0, \forall W \in \Gamma(S(TM)).$$

Therefore we have decompositions

$$\begin{aligned} TM &= RadTM + S(TM), \\ T\mathbb{R}_q^{n+1} &= TM \oplus tr(TM) \\ &= S(TM) \perp (RadTM) \oplus tr(TM) \\ &= S(TM) \perp S(TM)^\perp. \end{aligned}$$

A typical example of a lightlike hypersurface of \mathbb{R}_q^{n+1} is the lightcone.

Let $F : D \rightarrow \mathbb{R}$ be a smooth function, where D is an open set of \mathbb{R}_q^{n+1} . Then a graph

$$M = \{(x^1, \dots, X^{n+1}) \in \mathbb{R}_q^{n+1} \mid x^1 = F(x^2, \dots, x^{n+1})\},$$

is a hypersurface of \mathbb{R}_q^{n+1} . It is well known that this hypersurface is lightlike if and only if F is a solution of the differential equation

$$1 + \sum_{i=2}^q \left(\frac{\partial F}{\partial x^i} \right)^2 = \sum_{j=q+1}^{n+1} \left(\frac{\partial F}{\partial x^j} \right)^2. \quad (2.2)$$

3. Lightlike Surfaces of Revolution in \mathbb{R}_1^3

First we consider lightlike surfaces of revolution with timelike rotation axis. We shall take x^1x^3 plane as the plane of a regular curve C and x^3 (timelike) axis as the rotation axis. For a suitable interval (a, b) , the curve C is given by

$$x^3 = F(u), u \in (a, b).$$

Then a surface of revolution is obtained as a map

$$X(u, v) = (u \cos v, u \sin v, F(u)),$$

hence,

$$x^1 = x^1, x^2 = x^2, x^3 = F(\sqrt{(x^1)^2 + (x^2)^2}).$$

By differentiating the function F , it follows that

$$\frac{\partial F}{\partial x^1} = F' \frac{x^1}{\sqrt{(x^1)^2 + (x^2)^2}},$$

$$\frac{\partial F}{\partial x^2} = F' \frac{x^2}{\sqrt{(x^1)^2 + (x^2)^2}}.$$

Substituting these equations into (2.1), we have

$$\left(\frac{\partial F}{\partial x^1}\right)^2 + \left(\frac{\partial F}{\partial x^2}\right)^2 = (F')^2 = 1.$$

Therefore, we obtain

Theorem 3.1. *If M be a lightlike surface of revolution in \mathbb{R}_1^3 with timelike axis, then M is the lightcone in \mathbb{R}_1^3 .*

Next we consider lightlike surfaces of revolution with spacelike axis. Let C be a regular plane curve on x^2x^3 plane with

$$x^2 = F(u), u \in (a, b).$$

If we set x^2 axis as the rotation axis, then a surface of revolution is given as a map

$$X(u, v) = (u \cos v, F(u), u \sin v),$$

hence,

$$x^1 = x^1, x^2 = x^2, x^3 = F^{-1}(x^2) \sin(\cos^{-1} \frac{x^1}{F^{-1}(x^2)}).$$

By differentiating x^3 , it follows that

$$\frac{\partial x^3}{\partial x^1} = \frac{x^1}{F^{-1}(x^2)} \frac{1}{\sqrt{1 - \frac{(x^1)^2}{F^{-1}(x^2)^2}}},$$

$$\frac{\partial x^3}{\partial x^2} = (F^{-1})'(x^2) \sin(\cos^{-1} \frac{x^1}{F^{-1}(x^2)}) - \frac{(x^1)^2 (F^{-1})'(x^2)}{F^{-1}(x^2)^2} \frac{1}{\sqrt{1 - \frac{(x^1)^2}{F^{-1}(x^2)^2}}}.$$

Substituting these equations into (2.1), we have

$$\frac{(x^1)^2}{(F^{-1}(x^2))^2 - (x^1)^2} + (F^{-1})'(x^2)^2 + (F^{-1})'(x^2)^2 \frac{(x^1)^2}{F^{-1}(x^2)^2} + \frac{(x^1)^4 (F^{-1})'(x^2)^2}{F^{-1}(x^2)^2 ((F^{-1}(x^2))^2 - (x^1)^2)} = 1,$$

so that

$$(F^{-1}(x^2))^2 (2(x^1)^2 + ((F^{-1})'(x^2))^4 - ((F^{-1})'(x^2))^2) = 0.$$

Differentiating this equation by x^1 repeatedly, we obtain

$$(F^{-1})'(x^2) = 0,$$

which means that the function F^{-1} is a constant function. This is the contradiction. Hence we have

Theorem 3.2. *There is no lightlike surface of revolution with spacelike axis in \mathbb{R}_1^3 .*

4. Some Graph Type Lightlike Hypersurfaces

In the sequel, we consider a graph type hypersurface (M, g) in $(\mathbb{R}_q^{n+1}, \bar{g})$ of following form

$$x^{n+1} = x^1 + \dots + x^k + F(x^{k+1} + \dots + x^n),$$

where F is a non-trivial smooth function.

This section is devoted to give an existence condition of the lightlike hypersurface (M, g) by the index of the semi-Euclidean space \mathbb{R}_q^{n+1} .

A local frame (e_1, \dots, e_n) of M is given as

$$\begin{aligned} e_i &= \frac{\partial}{\partial x^i} + \frac{\partial}{\partial x^{n+1}} (i = 1, \dots, k), \\ e_i &= \frac{\partial}{\partial x^i} + F' \frac{\partial}{\partial x^{n+1}} (i = k+1, \dots, n), \end{aligned} \quad (4.1)$$

where $F' = \frac{\partial F}{\partial x^j} (j = k+1, \dots, n)$.

We remark that for $F(x^{k+1} + \dots + x^n)$, $\frac{\partial F}{\partial x^i} = \frac{\partial F}{\partial x^j} = F' (i, j = k+1, \dots, n)$.

When the index q satisfies $q \leq k$, we consider a vector field

$$v = \sum_{i=1}^q \frac{\partial}{\partial x^i} - \sum_{i=q+1}^k \frac{\partial}{\partial x^i} - \sum_{i=k+1}^n F' \frac{\partial}{\partial x^i} + \frac{\partial}{\partial x^{n+1}} \quad (4.2)$$

on \mathbb{R}_q^{n+1} . From (4.1), it follows that

$$\begin{aligned} \bar{g}(e_i, v) &= 0 (i = 1, \dots, n), \\ \bar{g}(v, v) &= k - 2q + 1 + (n - k)(F')^2. \end{aligned} \quad (4.3)$$

Hence if M is lightlike, we have $n = k = 2q - 1$ and $v = \sum_{i=1}^q e_i - \sum_{i=q+1}^n e_i \in \Gamma(\text{Rad}TM)$.

When the index q satisfies $q > k$, we consider a vector field

$$v = \sum_{i=1}^k \frac{\partial}{\partial x^i} + \sum_{i=k+1}^q F' \frac{\partial}{\partial x^i} + \sum_{i=q+1}^n F' \frac{\partial}{\partial x^i} + \frac{\partial}{\partial x^{n+1}} \quad (4.4)$$

on \mathbb{R}_q^{n+1} . Then

$$\begin{aligned} \bar{g}(e_i, v) &= 0 (i = 1, \dots, n), \\ \bar{g}(v, v) &= 1 - k + (n - 2q + k)(F')^2. \end{aligned} \quad (4.3)$$

Hence if M is lightlike, we obtain $2q = n + 1$ and $v = \sum_{i=1}^q e_i - \sum_{i=q+1}^n e_i \in \Gamma(\text{Rad}TM)$.

Therefore we have

Theorem 4.1. *If the semi-Euclidean space \mathbb{R}_q^{n+1} has a graph type lightlike hypersurface*

$$x^{n+1} = x^1 + \dots + x^k + F(x^{k+1} + \dots + x^n),$$

where F is a non-trivial smooth function, the dimension of the ambient space is two times of the index of it, that is \mathbb{R}_q^{2q} , and the graph type hypersurfaces reduce to

$$x^{2q} = x^1 + \dots + x^{2q-1}$$

or

$$x^{2q} = x^1 + F(x^2 + \dots + x^{2q-1}).$$

Remarks.

For the graph type hypersurface

$$x^{2q} = x^1 + \dots + x^{2q-1},$$

v is a local section of $RadTM$ and a local section of $tr(TM)$ is given by

$$V = - \sum_{i=1}^{2q-1} \frac{\partial}{\partial x^i} + \frac{\partial}{\partial x^{2q}}.$$

Therefore,

$$\begin{aligned} S(TM)^\perp &= Span\{v, V\}, \\ S(TM) &= Span\{e_1 - e_2, e_1 - e_3, \dots, e_1 - e_q, e_{q+1}, \dots, e_{2q-1}\}. \end{aligned}$$

Moreover, we have decompositions

$$\begin{aligned} S(TM) &= S(TM)^- \oplus S(TM)^+, \\ S(TM)^- &= Span \left\{ \frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^3}, \dots, \frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^q} \right\}, \\ S(TM)^+ &= Span \left\{ \frac{\partial}{\partial x^{q+1}} + F' \frac{\partial}{\partial x^{2q}}, \frac{\partial}{\partial x^{q+2}} + F' \frac{\partial}{\partial x^{2q}}, \dots, \frac{\partial}{\partial x^{2q-1}} + F' \frac{\partial}{\partial x^{2q}} \right\}. \end{aligned}$$

For the graph type hypersurface

$$x^{2q} = x^1 + F(x^2 + \dots + x^{2q-1}),$$

v is a local section of $RadTM$ and

$$V = -\frac{\partial}{\partial x^1} - \sum_{i=2}^{2q-1} F' \frac{\partial}{\partial x^i} + \frac{\partial}{\partial x^{2q}},$$

is a local section of $tr(TM)$, so

$$\begin{aligned} S(TM)^\perp &= Span\{v, V\}, \\ S(TM) &= Span\{F'e_1 - e_2, F'e_1 - e_3, \dots, F'e_1 - e_q, e_{q+1}, \dots, e_{2q-1}\}, \\ S(TM) &= S(TM)^- \oplus S(TM)^+, \\ S(TM)^- &= Span\{F' \frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^2}, F' \frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^3}, \dots, F' \frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^q}\}, \\ S(TM)^+ &= Span\{\frac{\partial}{\partial x^{q+1}} + \frac{\partial}{\partial x^{2q}}, \frac{\partial}{\partial x^{q+2}} + \frac{\partial}{\partial x^{2q}}, \dots, \frac{\partial}{\partial x^{2q-1}} + \frac{\partial}{\partial x^{2q}}\}. \end{aligned}$$

5. Some Graph Type Non-Degenerate Hypersurfaces

In this section, we consider the non-degenerate case of graph type hypersurface

$$x^{n+1} = x^1 + \dots + x^k + F(x^{k+1} + \dots + x^n).$$

To simplify the presentation, we define two (n, m) -matrix $S(n, m)$ and $S(f)(n, m)$ by setting:

Each element of $S(n, m)$ is 1,

Each element of $S(f)(n, m)$ is f ,

where f means a function, respectively, and we write (n, n) -unit matrix as $E(n)$. An (n, n) -matrix S , is defined as

$$S = \begin{bmatrix} S(q, q) & S(q, k-q) & S(F')(q, n-k) \\ S(k-q, q) & S(k-q, k-q) & S(F')(k-q, n-k) \\ S(F')(n-k, q) & S(F')(n-k, k-q) & S((F')^2)(n-k, n-k) \end{bmatrix}. \quad (5.1)$$

When the hypersurface M is non-degenerate, the vector v defined by (4.2) or (4.4) is a normal vector of M . Then the second fundamental form B_{ij} is defined by

$$B_{ij} = \bar{g}(\bar{\nabla}_{e_j} e_i, \frac{v}{\sqrt{|v|}})$$

Case $q \leq k$.

In this case, the second fundamental form B_{ij} deduces

$$B_{ij} = \begin{cases} 0 & (i = 1, \dots, k \text{ or } j = 1, \dots, k) \\ \frac{F''}{\sqrt{|v|}} & (i, j = k + 1, \dots, n). \end{cases}$$

That is,

$$(B_{ij}) = A_1 \begin{bmatrix} 0 & 0 \\ 0 & S(n-k, n-k) \end{bmatrix}, \quad (5.2)$$

where

$$A_1 = \frac{F''}{\sqrt{|v|}}.$$

The fundamental tensors g_{ij} and g^{ij} of M with respect to the local frame (e_1, \dots, e_n) is represented as

$$\begin{aligned} (g_{ij}) &= S + J_1, \\ (g^{ij}) &= -\frac{1}{|v|^2} J_1 S J_1 + J_1, \end{aligned}$$

where

$$J_1 = \begin{bmatrix} -E(q) & 0 & \\ 0 & E(k-q) & 0 \\ 0 & 0 & E(n-k) \end{bmatrix}.$$

Hence, from (5.2), it follows that

$$(g^{ik} B_{kj}) = \frac{1}{|v|} A_1(n-k) \begin{bmatrix} 0 & 0 & S(F')(q, n-k) \\ 0 & 0 & -S(F')(k-q, n-k) \\ 0 & 0 & -S((F')^2)(n-k, n-k) \end{bmatrix} + A_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & S(n-k, n-k) \end{bmatrix},$$

so that

$$g^{ik}B_{ki} = -\frac{(n-k)^2(n-2q+1)F'F''}{|v|^{3/2}}.$$

Therefore the mean curvature H of M is

$$H = -\frac{(n-k)(n-2q+1)F''}{n|v|^{3/2}}. \quad (5.3)$$

Case $q > k$.

In this case, B_{ij} deduces

$$B_{ij} = \begin{cases} 0 & (i = 1, \dots, k \text{ or } j = 1, \dots, k) \\ \frac{F''}{\sqrt{|v|}} & (i, j = k+1, \dots, n). \end{cases}$$

Hence

$$(B_{ij}) = A_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & S(q-k, q-k) & S(q-k, n-k) \\ 0 & S(q-k, q-k) & S(q-k, n-k) \end{bmatrix} \quad (5.4)$$

where $A_2 = \frac{F''}{\sqrt{|v|}}$.

Since

$$\begin{aligned} (g_{ij}) &= S_2 + J_2, \\ (g^{ij}) &= -\frac{1}{|v|^2}J_2S_2J_2 + J_2, \end{aligned}$$

where

$$J_2 = \begin{bmatrix} -E(k) & 0 & \\ 0 & -E(q-k) & 0 \\ 0 & 0 & E(n-q) \end{bmatrix},$$

we have

$$(g^{ik}B_{kj}) = \frac{(1-k)}{|v|} A_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & -S(q-k, q-k) & -S(q-k, n-k) \\ 0 & S(n-q, q-k) & S(n-q, n-q) \end{bmatrix}.$$

Therefore the mean curvature H of M is

$$H = -\frac{(1-k)(n-2q+k)^2 F' F''}{n |v|^{3/2}} (n-k)((1-k) + (n-q)F'^2) F'. \quad (5.5)$$

From (5.2), (5.3), (5.4) and (5.5), we have

Theorem 5.1. *Suppose M is a non-degenerate graph type hypersurface*

$$x^{n+1} = x^1 + \dots + x^k + F(x^{k+1} + \dots + x^n)$$

in the semi-Euclidean space \mathbb{R}_q^{n+1} , where F is a non-trivial smooth function. If M is minimal then

(a) M is

$$x^{n+1} = x^1 + \dots + x^n$$

in \mathbb{R}_q^{n+1} . This case, M reduces to totally geodesic.

(b) M is

$$x^{n+1} = x^1 + \dots + x^{2q-1} + F(x^{2q} + \dots + x^n)$$

in \mathbb{R}_q^{n+1} .

(c) M is

$$x^{q+1} = x^1 + F(x^2 + \dots + x^q)$$

in \mathbb{R}_q^{q+1} .

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