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ABDULLAH MAĞDEN

EKREM KADIOĞLU

ARİF A. SALIMOV

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A. Mağden, E. Kadıoğlu and A.A. Salimov

Abstract

The purpose of the present paper is to study, using the Tachibana operator, the complete lifts of affinor structures along a pure cross-section of the tensor bundle and to investigate their transfers. The results obtained are to some extent similar to results previously established for tangent (cotangent) bundles [1]. However there are various important differences and it appears that the problem of lifting affinor structures to the tensor bundle on the pure cross-section presents difficulties which are not encountered in the case of the tangent (cotangent) bundle.

Key words and phrases. Tensor, bundle, affinor, complete lift, pure cross-section, Tachibana operator

1. Introduction

Let M_n be a differentiable manifold of class C^∞ and finite dimension n, and let $T_q^p(M_n), p+q>0$ be the bundle over M_n of tensors of type (p,q): $T_q^p(M_n) = \bigcup_{P\in M_n} T_q^p(P)$, where $T_q^p(P)$ denotes the tensor(vector) spaces of tensors of type (p,q) at $P \in M_n$.

We list below notations used in this paper.

i. $\pi : T_q^p(M_n) \mapsto M_n$ is the projection $T_q^p(M_n)$ onto M_n.

*ii.*The indices $i, j, \cdots$ run from 1 to n , the indices $\overline{i}, \overline{j}, \cdots$ from $n+1$ to $n+n^{p+q} = dim\, T_q^p(M_n)$ and the indices $I = (i, \overline{i})$, $J = (j, \overline{j})$, ... from 1 to $n+n^{p+q}$. The so-called Einsteins summation convention is used.

iii. $\mathfrak{F}(M)$ is the ring of real-valued C^∞ functions on M_n. $\mathfrak{T}_q^p(M_n)$ is the module over $\mathfrak{F}(M)$ of C^∞tensor fields of type (p,q).

iv. Vector fields in M_n are denoted by $V, W, ...$. The Lie derivation with respect to V is denoted by L_V. Affinor fields (tensor fields of type $(1,1)$) are denoted by φ, ψ, ...

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Denoting by x^j the local coordinates of $P = \pi(\widetilde{P})$ ($\widetilde{P} \in T^p_q(M_n)$) in a neighborhood $U \subset M_n$ and if we make $(x^j, t^{i_1\cdots i_p}_{j_1\cdots j_q}) = (x^j, x^{\overline{j}})$ correspond to the point $\widetilde{P} \in \pi^{-1}(U)$, we can introduce a system of local coordinates $(x^j, x^{\overline{j}})$ in a neighborhood $\pi^{-1}(U) \subset T^p_q(M_n)$, where $t^{i_1\cdots i_p}_{j_1\cdots j_q} \overset{def}{=} x^{\overline{j}}$ are components of $t \in T^p_q(P)$ with respect to the natural frame ∂_i.

If $\alpha \in \mathfrak{T}^q_p(M_n)$, it is regarded, in a natural way(by contraction), as a function in $T^p_q(M_n)$, which we denote by $\imath\alpha$. If α has the local expression $\alpha = \alpha^{j_1\cdots j_q}_{i_1\cdots i_p}\partial_{j_1} \otimes \cdots \otimes \partial_{j_q} \otimes dx^{i_1} \otimes \cdots \otimes dx^{i_p}$ in a coordinate neighborhood $U(x^i) \subset M_n$, then $\imath\alpha$ has the local expression

$$\imath\alpha = \alpha(t) = \alpha^{j_1\cdots j_q}_{i_1\cdots i_p} t^{i_1\cdots i_p}_{j_1\cdots j_q}$$

with respect to the coordinates $(x^j, x^{\overline{j}})$ in $\pi^{-1}(U)$.

Suppose that $A \in \mathfrak{T}^p_q(M_n)$. We define the vertical lift ${}^VA \in \mathfrak{T}^1_0(T^p_q(M_n))$ of A to $T^p_q(M_n)$ (see [2]) by

$${}^VA(\imath\alpha) = \alpha(A) \circ \pi =^V (\alpha(A))$$

where ${}^V(\alpha(A))$ is the vertical lift of the function $\alpha(A) \in \mathfrak{F}(M_n)$. The vertical lift ${}^VA$ of A to $T^p_q(M_n)$ has components

$${}^VA = \begin{pmatrix} {}^VA^j \\ {}^VA^{\overline{j}} \end{pmatrix} = \begin{pmatrix} 0 \\ A^{i_1\cdots i_p}_{j_1\cdots j_q} \end{pmatrix} \tag{1.1}$$

with respect to the coordinates $(x^j, x^{\overline{j}})$ in $T^p_q(M_n)$.

We define the complete lift ${}^cV = \bar{L}_V \in \mathfrak{T}^1_0(T^p_q(M_n))$ of $V \in \mathfrak{T}^1_0(M_n)$ to $T^p_q(M_n)$ [2] by

$${}^cV(\imath\alpha) = \imath(L_V\alpha)\ ,\ \ \alpha \in \mathfrak{T}^q_p(M_n).$$

The complete lift ${}^cV$ of $V \in \mathfrak{T}^1_0(M_n)$ to $T^p_q(M_n)$ has components

$${}^cV^j = V^j,\ {}^cV^{\overline{j}} = \sum_{\mu=1}^{p} t^{i_1\cdots s\cdots i_p}_{j_1\cdots j_q}\,\partial_s V^{i_\mu} - \sum_{\lambda=1}^{q} t^{i_1\cdots i_p}_{j_1\cdots s\cdots j_q}\,\partial_{j_\lambda} V^s \tag{1.2}$$

with respect to the coordinates $(x^j, x^{\bar{j}})$ in $T_q^p(M_n)$.

Suppose that there is given a tensor field $\xi \in \mathfrak{T}_q^p(M_n)$. Then the correspondence $x \mapsto \xi_x$, ξ_x being the value of ξ at $x \in M_n$, determines a mapping $\sigma_\xi : M_n \mapsto T_q^p(M_n)$, such that $\pi \circ \sigma_\xi = id_{M_n}$, and the n dimensional submanifold $\sigma_\xi(M_n)$ of $T_q^p(M_n)$ is called the cross-section determined by ξ. If the tensor field ξ has the local components $\xi_{k_1 \cdots k_q}^{l_1 \cdots l_p}(x^k)$, the cross-section $\sigma_\xi(M_n)$ is locally expressed by

$$\begin{cases} x^k &= x^k \\ x^{\overline{k}} &= \xi_{k_1 \cdots k_q}^{l_1 \cdots l_p}(x^k) \end{cases} \tag{1.3}$$

with respect to the coordinates $(x^k, x^{\overline{k}})$ in $T_q^p(M_n)$. Differentiating (1.3) by x^j, we see that the n tangent vector fields B_j to $\sigma_\xi(M_n)$ have components

$$(B_j^K) = (\frac{\partial x^K}{\partial x^j}) = \begin{pmatrix} \delta_j^k \\ \partial_j \ \xi_{k_1 \cdots k_q}^{l_1 \cdots l_p} \end{pmatrix}, \tag{1.4}$$

with respect to the natural frame $\{\partial_k, \partial_{\overline{k}}\}$ in $T_q^p(M_n)$.

On the other hand, the fibre is locally expressed by

$$\begin{cases} x^k &= const, \\ t_{k_1 \cdots k_q}^{l_1 \cdots l_p} &= t_{k_1 \cdots k_q}^{l_1 \cdots l_p}, \end{cases}$$

$t_{k_1 \cdots k_q}^{l_1 \cdots l_p}$ being consider as parameters. Thus, on differentiating with respect to $x^{\overline{j}} = t_{j_1 \cdots j_q}^{i_1 \cdots i_p}$, we see that the n^{p+q} tangent vector fields $C_{\overline{j}}$ to the fibre have components

$$(C_{\overline{j}}^K) = (\frac{\partial x^K}{\partial x^{\overline{j}}}) = \begin{pmatrix} 0 \\ \delta_{k_1}^{j_1} \cdots \delta_{k_q}^{j_q} \delta_{i_1}^{l_1} \cdots \delta_{i_p}^{l_p} \end{pmatrix} \tag{1.5}$$

with respect to the natural frame $\{\partial_k, \partial_{\overline{k}}\}$ in $T_q^p(M_n)$.

We consider in $\pi^{-1}(U) \subset T_q^p(M_n)$, $n + n^{p+q}$ local vector fields B_j and $C_{\overline{j}}$ along $\sigma_\xi(M_n)$. They form a local family of frames $\{B_j, C_{\overline{j}}\}$ along $\sigma_\xi(M_n)$, which is called the adapted $(B, C)-$frame of $\sigma_\xi(M_n)$ in $\pi^{-1}(U)$. Taking account of (1.2), we can prove that, the complete lift ${}^cV$ has along $\sigma_\xi(M_n)$ components of the form

$${}^cV = \begin{pmatrix} {}^c\widetilde{V}^j \\ {}^c\widetilde{V}^{\overline{j}} \end{pmatrix} = \begin{pmatrix} V^j \\ -(L_V \xi)_{j_1 \cdots j_q}^{i_1 \cdots i_p} \end{pmatrix} \tag{1.6}$$

with respect to the adapted (B,C)-frame [3], where $(L_V\xi)^{i_1\cdots i_p}_{j_1\cdots j_q}$ are local components of $L_V\xi$ in M_n.

2. Complete Lifts of The Affinor field to The Tensor Bundle Along a Pure Cross- Section

Let $\varphi\in\mathfrak{T}^1_1(M_n)$. Making use of the Jacobian matrix

$$(\frac{\partial x^{I'}}{\partial x^I})=\begin{pmatrix}\frac{\partial x^{i'}}{\partial x i} & \frac{\partial x^{i'}}{\partial x^{\bar{i}}}\\ \frac{\partial x^{\bar{i}'}}{\partial x i} & \frac{\partial x^{\bar{i}'}}{\partial x^{\bar{i}}}\end{pmatrix}=\begin{pmatrix}A^{i'}_i & 0\\ t^{(j)}_{(k)}\partial_i(A^{(k)}_{(i')}A^{(j')}_{(j)}) & A^{(i)}_{(i')}A^{(j')}_{(j)}\end{pmatrix},$$

of the coordinate transformation in $T^p_q(M_n)$: $x^{i'}=x^{i'}(x^i)$, $x^{\bar{i}'}=t^{(j')}_{(i')}=A^{(i)}_{(i')}A^{(j')}_{(j)}t^{(j)}_{(i)}=A^{(i)}_{(i')}A^{(j')}_{(j)}x^{\bar{i}}$ $\quad(t^{(j)}_{(i)}=t^{j_1\cdots j_p}_{i_1\cdots i_q}$, $A^{(i)}_{(i')}=A^{i_1}_{i'_1}\cdots A^{i_q}_{i'_q}$, $A^i_{i'}=\frac{\partial x^i}{\partial x^{i'}}$, $A^{(j')}_{(j)}=A^{j'_1}_{j_1}\cdots A^{j'_p}_{j_p}$, $A^{j'}_j=\frac{\partial x^{j'}}{\partial x^j}$) we can define a vector field $\gamma\varphi\in\mathfrak{T}^1_0(T^p_q(M_n))$:

$$\gamma\varphi=((\gamma\varphi)^I)=\begin{pmatrix}0\\ \begin{cases}-\sum_{b=2}^p t^{l_1\cdots m\cdots l_p}_{k_1\cdots k_q}\varphi^{l_b}_m,\ p>0\\ t^{l_1\cdots l_p}_{mk_2\cdots k_q}\varphi^m_{k_1}-\sum_{b=1}^p t^{l_1\cdots m\cdots l_p}_{k_1\cdots k_q}\varphi^{l_b}_m,\ q>0\end{cases}\end{pmatrix},$$

where $\varphi^m_{i_1}$ are local components of φ in M_n. Clearly, we have $(\gamma\varphi)(^Vf)=0$ for any $f\in\mathfrak{F}(M_n)$, so that $\gamma\varphi$ is a vertical vector field. We can easily verify that the vertical vector field $\gamma\varphi$ has along $\sigma_\xi(M_n)$ components

$$\gamma\varphi=((\widetilde{\gamma\varphi})^I)=\begin{pmatrix}0\\ \begin{cases}-\sum_{b=2}^p \xi^{l_1\cdots m\cdots l_p}_{k_1\cdots k_q}\varphi^{l_b}_m,\ p>0\\ \xi^{l_1\cdots l_p}_{mk_2\cdots k_q}\varphi^m_{k_1}-\sum_{b=1}^p \xi^{l_1\cdots m\cdots l_p}_{k_1\cdots k_q}\varphi^{l_b}_m,\ q>0\end{cases}\end{pmatrix}\tag{2.1}$$

with respect to the adapted (B,C)-frame.

A tensor field $\xi\in\mathfrak{T}^p_q(M_n)$ is called pure with respect to the affinor φ-structure ($\varphi\in\mathfrak{T}^1_1(M_n)$) [4], if

$$\varphi^{i_1}_r\xi^{ri_2\cdots i_p}_{j_1\cdots j_q}=\cdots=\varphi^{i_p}_r\xi^{i_1\cdots i_{p-1}r}_{j_1\cdots j_q}=\varphi^r_{j_1}\xi^{i_1\cdots i_p}_{rj_2\cdots j_q}=\cdots=\varphi^r_{j_q}\xi^{i_1\cdots i_p}_{j_1\cdots j_{q-1}r}=\underset{\varphi}{\overset{*}{\xi}}{}^{i_1\cdots i_p}_{j_1\cdots j_q}.$$

In particular, vector(covector) fields will be considered to be pure.

Let $\overset{*}{\mathfrak{T}}{}^{p}_{q}(M_n)$ denotes a module of all the tensor fields $\xi \in \mathfrak{T}^p_q(M_n)$ which are pure with respect to φ. We consider the Tachibana operator on the module $\overset{*}{\mathfrak{T}}{}^{p}_{q}(M_n)$ [4]:

$$(\Phi_\varphi \xi)^{i_1\cdots i_p}_{kj_1\cdots j_q} = \varphi^m_k \partial_m \xi^{i_1\cdots i_p}_{j_1\cdots j_q} - \partial_k \underset{\varphi}{\overset{*}{\xi}}{}^{i_1\cdots i_p}_{j_1\cdots j_q} + \sum_{a=1}^{q} (\partial_{j_a}\varphi^r_k)\xi^{i_1\cdots i_p}_{j_1\cdots r \cdots j_q} +$$

$$+ \sum_{b=1}^{p} (\partial_k \varphi^{i_b}_r - \partial_r \varphi^{i_b}_k)\xi^{i_1\cdots r \cdots i_p}_{j_1\cdots j_q}. \tag{2.2}$$

where $\Phi_\varphi \xi \in \mathfrak{T}^p_{q+1}(M_n)$. After some calculations we have, from (2.2):

$$V^k(\Phi_\varphi \xi)^{i_1\cdots i_p}_{kj_1\cdots j_q} = \mathcal{L}_{\varphi V}\xi^{i_1\cdots i_p}_{j_1\cdots j_q} - \mathcal{L}_V \underset{\varphi}{\overset{*}{\xi}}{}^{i_1\cdots i_p}_{j_1\cdots j_q} + \sum_{b=1}^{p} (\mathcal{L}_V \varphi^{i_b}_r)\xi^{i_1\cdots r \cdots i_p}_{j_1\cdots j_q} \tag{2.3}$$

for any $V \in \mathfrak{T}^1_0(M_n)$ with local components V^k.

Suppose that $A \in \mathfrak{T}^p_q(M_n)$ with local components $A^{j_1\cdots j_p}_{i_1\cdots i_q}$ in $U(x^i) \subset M_n$. From (1.1),(1.4),(1.5) and ${}^V A = {}^V \widetilde{A}{}^i B_i + {}^V \widetilde{A}{}^{\bar{i}} C_{\bar{i}}$, we easily obtain ${}^V \widetilde{A}{}^i = 0$, ${}^V \widetilde{A}{}^{\bar{i}} = {}^V A^{\bar{i}} = A^{j_1\cdots j_p}_{i_1\cdots i_q}$. Thus the vertical lift ${}^V A$ also has components of the form (1.1) with respect to the adapted $(B,C)-$frame of $\sigma_\xi(M_n)$.

Now, we consider a pure cross-section $\sigma^\varphi_\xi(M_n)$ determined by $\xi \in \overset{*}{\mathfrak{T}}{}^{p}_{q}(M_n)$.

We define a tensor field ${}^c\varphi \in \mathfrak{T}^1_1(T^p_q(M_n))$ along the pure cross-section $\sigma^\varphi_\xi(M_n)$ by

$$\begin{cases} {}^c\varphi({}^cV) = & {}^c(\varphi(V)) + \gamma(L_V\varphi),\ \forall V \in \mathfrak{T}^1_0(M_n), \\ {}^c\varphi({}^VA) = & {}^V(\varphi(A)),\ \forall A \in \mathfrak{T}^p_q(M_n), \end{cases} \tag{2.4}$$

where $\varphi(A) \in \mathfrak{T}^p_q(M_n)$ and call ${}^c\varphi$ the complete lift of $\varphi \in \mathfrak{T}^1_1(M_n)$ to $T^p_q(M_n)$ along $\sigma^\varphi_\xi(M_n)$.

Let ${}^c\widetilde{\varphi}{}^K_L$ be components of ${}^c\varphi$ with respect to the adapted $(B,C)-$frame of the pure cross-section $\sigma^\varphi_\xi(M_n)$. From (2.4) we have

$$\begin{cases} {}^c\widetilde{\varphi}{}^K_L\, {}^c\widetilde{V}{}^L = {}^c(\widetilde{\varphi(V)})^K + \gamma(\widetilde{L_V\varphi})^K\ , & (i) \\ {}^c\widetilde{\varphi}{}^K_L\, {}^V\widetilde{A}{}^L = {}^V(\widetilde{\varphi(A)})^K\,, & (ii) \end{cases} \tag{2.5}$$

where(see (2.1))

$$\gamma(L_V\tilde{\varphi})^K) = \left(\begin{array}{c} 0 \\ \begin{cases} -\sum_{b=2}^{p} \xi_{k_1\cdots k_q}^{l_1\cdots m\cdots l_p}(L_V\varphi)_m^{l_b},\ p>0 \\ \xi_{mk_2\cdots k_q}^{l_1\cdots l_p}(L_V\varphi)_{k_1}^m - \sum_{b=1}^{p}\xi_{k_1\cdots k_q}^{l_1\cdots m\cdots l_p}(L_V\varphi)_m^{l_b},\ q>0 \end{cases} \end{array} \right),$$

$$({}^V(\widetilde{\varphi(A)})^K) = \left(\begin{array}{c} 0 \\ \begin{cases} \varphi_m^{l_1} A_{k_1\cdots k_q}^{ml_2\cdots l_p},\ p>0 \\ \varphi_{k_1}^m A_{mk_2\cdots k_q}^{l_1\cdots l_p},\ q>0 \end{cases} \end{array} \right).$$

First, consider the case where $K=k$. In this case, (i) of (2.5) reduces to

$$^c\tilde{\varphi}\,{}_l^k\,{}^c\tilde{V}^l + {}^c\tilde{\varphi}\,{}_{\bar{l}}^k\,{}^c\tilde{V}^{\bar{l}} = {}^c(\widetilde{\varphi(V)})^k = (\varphi(V))^k = \varphi_l^k V^l. \tag{2.6}$$

Since the right-hand side of (2.6) are functions depending only on the base coordinates x^i, the left-hand side of (2.6) are too. Then, since ${}^c\tilde{V}^{\bar{l}}$ depend on fibre coordinates, from (2.6) we obtain

$$^c\tilde{\varphi}\,{}_{\bar{l}}^k = 0. \tag{2.7}$$

From (2.6) and (2.7), we have ${}^c\tilde{\varphi}_l^k\,{}^cV^l = {}^c\tilde{\varphi}_l^k V^l = \varphi_l^k V^l$, V^i being arbitrary, which implies

$$^c\tilde{\varphi}_l^k = \varphi_l^k. \tag{2.8}$$

When $K=k$, (ii) of (2.5) can be rewritten, by virtue of (1.1), (2.7) and (2.8), as $0=0$. When $K=\bar{k}$, (ii) of (2.5) reduces to

$$^c\tilde{\varphi}_l^{\bar{k}}\,{}^V\tilde{A}^l + {}^c\tilde{\varphi}_{\bar{l}}^{\bar{k}}\,{}^V\tilde{A}^{\bar{l}} = {}^V(\widetilde{\varphi(A)})^{\bar{k}}$$

or

$$^c\tilde{\varphi}_{\bar{l}}^{\bar{k}} A_{r_1\cdots r_q}^{s_1\cdots s_p} = \varphi_m^{l_1} A_{k_1\cdots k_q}^{ml_2\cdots l_p} = \varphi_{s_1}^{l_1}\delta_{s_2}^{l_2}\cdots\delta_{s_p}^{l_p}\delta_{k_1}^{r_1}\cdots\delta_{k_q}^{r_q} A_{r_1\cdots r_q}^{s_1\cdots s_p}, p>0$$

for all $A\in\mathfrak{T}_q^p(M_n)$, which implies

$$^c\tilde{\varphi}_{\bar{l}}^{\bar{k}} = \varphi_{s_1}^{l_1}\delta_{s_2}^{l_2}\cdots\delta_{s_p}^{l_p}\delta_{k_1}^{r_1}\cdots\delta_{k_q}^{r_q}\ ,\ p>0\,,$$

where δ^r_k is the Kronecker symbol, $x^{\overline{l}} = t^{s_1\cdots s_p}_{r_1\cdots r_q}$, $x^{\overline{k}} = t^{l_1\cdots l_p}_{k_1\cdots k_q}$.

By similar devices, we have

$$^c\widetilde{\varphi}^{\overline{k}}_{\overline{l}} = \delta^{l_1}_{s_1}\cdots\delta^{l_p}_{s_p}\varphi^{r_1}_{k_1}\delta^{r_2}_{k_2}\cdots\delta^{r_q}_{k_q} \quad , \; q>0.$$

We shall investigate components $^c\widetilde{\varphi}^{\overline{k}}_{l}$. Suppose for example that $p=0$ and $q=2$. In this case, when $K=\overline{k}$, (i) of (2.5) reduces to

$$^c\widetilde{\varphi}^{\bar{k}}_{l}\,{}^c\widetilde{V}^{l} + {}^c\widetilde{\varphi}^{\bar{k}}_{\bar{l}}\,{}^c\widetilde{V}^{\bar{l}} = {}^c(\varphi\widetilde{(V)})^{\bar{k}} + \xi_{lk_2}(L_V\varphi)^l_{k_1}.$$

or

$$^c\widetilde{\varphi}^{\bar{k}}_{l}\,{}^c\widetilde{V}^{l} + \varphi^{r_1}_{k_1}\delta^{r_2}_{k_2}\,{}^c\widetilde{V}^{\bar{l}} - \xi_{lk_2}(L_V\varphi)^l_{k_1} = {}^c(\varphi\widetilde{(V)})^{\bar{k}}. \tag{2.9}$$

From (2.3) we get

$$V^l(\Phi_\varphi\xi)_{lk_1k_2} = (L_{\varphi V}\xi)_{k_1k_2} - (L_V\underset{\varphi}{\overset{*}{\xi}})_{k_1k_2}$$

or

$$V^l(\Phi_\varphi\xi)_{lk_1k_2} + \varphi^l_{k_1}(L_V\xi)_{lk_2} + \xi_{lk_2}(L_V\varphi)^l_{k_1} = (L_{\varphi V}\xi)_{k_1k_2}, \tag{2.10}$$

for any $V\in\mathfrak{T}^1_0(M_n)$. Using (1.6), from (2.10) we have

$$V^l(\Phi_\varphi\xi)_{lk_1k_2} + \varphi^l_{k_1}(L_V\xi)_{lk_2} + \xi_{lk_2}(L_V\varphi)^l_{k_1} = V^l(\Phi_\varphi\xi)_{lk_1k_2} +$$

$$+\varphi^{r_1}_{k_1}\delta^{r_2}_{k_2}(L_V\xi)_{r_1r_2} + \xi_{lk_2}(L_V\varphi)^l_{k_1} = {}^cV^l(\Phi_\varphi\xi)_{lk_1k_2} -$$

$$-\varphi^{r_1}_{k_1}\delta^{r_2}_{k_2}\,{}^cV^{\bar{l}} + \xi_{lk_2}(L_V\varphi)^l_{k_1} = -{}^c(\varphi(V))^{\bar{l}}$$

or

$$(\Phi_\varphi\xi)_{lk_1k_2}\,{}^cV^l - \varphi^{r_1}_{k_1}\delta^{r_2}_{k_2}\,{}^cV^{\bar{l}} + \xi_{lk_2}(L_V\varphi)^l_{k_1} = -{}^c(\varphi(V))^{\bar{l}}, \tag{2.11}$$

Comparing (2.9) and (2.11), we get

$$^c\varphi^{\bar{k}}_{l} = -(\Phi_\varphi\xi)_{lk_1k_2}.$$

In general case, by similar devices, we can prove:

$$ {}^{c}\varphi_{l}^{\bar{k}} = -(\Phi_{\varphi}\xi)_{lk_1\cdots k_q}^{l_1\cdots l_p}. $$

Thus the complete lift ${}^{c}\varphi$ of φ has along the pure cross-section $\sigma_{\xi}(M_n)$ components

$$ {}^{c}\tilde{\varphi}_{l}^{k} = \varphi_{l}^{k}, \quad {}^{c}\tilde{\varphi}_{\bar{l}}^{k} = 0, {}^{c}\tilde{\varphi}_{l}^{\bar{k}} = -(\Phi_{\varphi}\xi)_{lk_1\cdots k_q}^{l_1\cdots l_p}, \tag{2.12} $$

$$ {}^{c}\tilde{\varphi}_{\bar{l}}^{\bar{k}} = \varphi_{s_1}^{l_1}\delta_{s_2}^{l_2}\cdots\delta_{s_p}^{l_p}\delta_{k_1}^{r_1}\cdots\delta_{k_q}^{r_q}, \quad p > 0 $$

$$ {}^{c}\tilde{\varphi}_{\bar{l}}^{\bar{k}} = \delta_{s_1}^{l_1}\cdots\delta_{s_p}^{l_p}\varphi_{k_1}^{r_1}\delta_{k_2}^{r_2}\cdots\delta_{k_q}^{r_q}, \quad q > 0 $$

with respect to the adapted $(B, C)-$frame of $\sigma_{\xi}(M_n)$, where $\Phi_{\varphi}\xi$ is the Tachibana operator.

3. Transfer of The Complete Lift of The Affinor Structure

Let M_n be a paracompact manifold with a Riemannian metric. We shall mean by the Riemannian metric a symmetric covariant tensor field g of degree 2 which is nondegenerate. If g is a pure tensor field, then a manifold M_n with an affinor φ-structure is called an almost B-manifold [5, p. 31] and this will be denoted V_n

Suppose that $T_q^p(V_n)$ and $T_{q+1}^{p-1}(V_n)$ are the tensor bundle of type (p,q) and $(p-1, q+1)$ over V_n, respectively. Clearly that $dimT_q^p(V_n) = dimT_{q+1}^{p-1}(V_n) = n + n^{p+q}$. Let the diffeomorphism $f : T_q^p(V_n) \to T_{q+1}^{p-1}(V_n)$, $y^I = y^I(x^J)$, $I, J = 1, ..., n + n^{p+q}$, be defined by a local expression such that

$$ \begin{cases} y^i = x^i = \delta_k^i x^k, \\ y^{\bar{i}} = t_{ij_1\cdots j_q}^{\cdot i_2\cdots i_p} = g_{im}t_{j_1\cdots j_q}^{mi_2\cdots i_p} = g_{il_1}t_{k_1\cdots k_q}^{l_1 l_2\cdots l_p}\delta_{l_2}^{i_2}\cdots\delta_{l_p}^{i_p}\delta_{j_1}^{k_1}\cdots\delta_{j_q}^{k_q} = \\ = g_{il_1}\delta_{l_2}^{i_2}\cdots\delta_{l_p}^{i_p}\delta_{j_1}^{k_1}\cdots\delta_{j_q}^{k_q}\, x^{\bar{k}}. \end{cases} $$

Since

$$ x^{\bar{k}} = t_{k_1\cdots k_q}^{l_1\cdots l_p}, $$

$$\frac{\partial y^{\overline{i}}}{\partial x^{\overline{k}}} = g_{il_1}\delta_{l_2}^{i_2}\cdots\delta_{l_p}^{i_p}\delta_{j_1}^{k_1}\cdots\delta_{j_q}^{k_q}\,,$$

$$0 = \frac{\partial y^{\overline{i}}}{\partial x^{k}} = \frac{\partial}{\partial x^{k}}(g_{im}t_{j_1\cdots j_q}^{mi_2\cdots i_p}) = (\partial_k g_{im})t_{j_1\cdots j_q}^{mi_2\cdots i_p},$$

we have

$$A = \left(\frac{\partial y^I}{\partial x^K}\right) = \begin{pmatrix} \frac{\partial y^i}{\partial x^k} & \frac{\partial y^i}{\partial x^{\overline{k}}} \\ \frac{\partial y^{\overline{i}}}{\partial x^k} & \frac{\partial y^{\overline{i}}}{\partial x^{\overline{k}}} \end{pmatrix} = \begin{pmatrix} \delta_k^i & 0 \\ 0 & g_{il_1}\delta_{l_2}^{i_2}\cdots\delta_{l_p}^{i_p}\delta_{j_1}^{k_1}\cdots\delta_{j_q}^{k_q} \end{pmatrix}.$$

The inverse of the mapping f is written as

$$x^l = y^l,\;\; x^{\overline{l}} = t_{r_1\cdots r_q}^{s_1\cdots s_p} = g^{s_1 m}t_{mr_1\cdots r_q}^{\cdot s_2\cdots s_p}.$$

Suppose that $y^{\overline{j}} = t_{ll_1\cdots l_q}^{\cdot k_2\cdots k_p}$, we have

$$A^{-1} = \left(\frac{\partial x^L}{\partial y^J}\right) = \begin{pmatrix} \delta_j^l & 0 \\ 0 & g^{s_1 l}\delta_{r_1}^{l_1}\cdots\delta_{r_q}^{l_q}\delta_{k_2}^{s_2}\cdots\delta_{k_p}^{s_p} \end{pmatrix},$$

which is the Jacobian matrix of inverse mapping f^{-1}.

Let us consider the pure cross-section $\xi_{j_1\cdots j_q}^{i_1\cdots i_p}(x)$ of $T_q^p(V_n)$. We can easily verify that the image $\xi_{ij_1\cdots j_q}^{\cdot i_2\cdots i_p}(y)$ of this cross-section under the diffeomorphism f is the pure cross-section in $T_{q+1}^{p-1}(V_n)$. In fact, we see that

$$\begin{aligned}\xi_{kj_1\cdots j_q}^{\cdot i_2\cdots i_p}\varphi_i^k &= (g_{km}\xi_{j_1\cdots j_q}^{mi_2\cdots i_p})\varphi_i^k = g_{ik}\xi_{j_1\cdots j_q}^{mi_2\cdots i_p}\varphi_m^k \\ &= g_{ik}\xi_{mj_2\cdots j_q}^{ki_2\cdots i_p}\varphi_{j_1}^m = \xi_{imj_2\cdots j_q}^{\cdot i_2\cdots i_p}\varphi_{j_1}^m.\end{aligned}$$

Theorem. Suppose that ${}^c\underset{1}{\varphi}$ and ${}^c\underset{2}{\varphi}$ denote the complete lift of the affinor φ-structure to $T_q^p(V_n)$ and $T_{q+1}^{p-1}(V_n)$ along the pure cross-sections $\xi_{j_1\cdots j_q}^{i_1\cdots i_p}(x)$ and $\xi_{ij_1\cdots j_q}^{\cdot i_2\cdots i_p}(y)$, respectively. If $\Phi_\varphi g = 0$, then ${}^c\underset{2}{\varphi}$ is transferred from ${}^c\underset{1}{\varphi}$ by means of the diffeomorphism f, where $\Phi_\varphi g$ denotes the Tachibana operator.

Proof. Let $(\Phi_\varphi g)_{kij} \overset{\text{def}}{=} \underset{\varphi}{\Phi}{}_k g_{ij} = 0.$ If we take account of (2.12) and a fomula due to Tachibana [4]

$$\underset{\varphi}{\Phi}{}_j(g_{im}\xi^{mi_2\cdots i_p}_{j_1\cdots j_q}) = (\underset{\varphi}{\Phi}{}_j g_{im})\xi^{mi_2\cdots i_p}_{j_1\cdots j_q} + g_{im}\underset{\varphi}{\Phi}{}_j\xi^{mi_2\cdots i_p}_{j_1\cdots j_q} \ ,$$

then we have

$$\begin{aligned}
{}^c\underset{2}{\varphi} &= \left({}^c\underset{2}{\varphi}{}^I_J\right) \qquad (3.1)\\
&= \begin{pmatrix} \varphi^i_j & 0 \\ -\underset{\varphi}{\Phi}{}_j\xi^{\cdot i_2\cdots i_p}_{ij_1\cdots j_q} & \varphi^l_i\delta^{l_1}_{j_1}\cdots\delta^{l_q}_{j_q}\delta^{i_2}_{k_2}\cdots\delta^{i_p}_{k_p} \end{pmatrix}\\
&= \begin{pmatrix} \varphi^i_j & 0 \\ -(\underset{\varphi}{\Phi}{}_j g_{im})\xi^{mi_2\cdots i_p}_{j_1\cdots j_q} - g_{im}\underset{\varphi}{\Phi}{}_j\xi^{mi_2\cdots i_p}_{j_1\cdots j_q} & \varphi^l_i\delta^{l_1}_{j_1}\cdots\delta^{l_q}_{j_q}\delta^{i_2}_{k_2}\cdots\delta^{i_p}_{k_p} \end{pmatrix}\\
&= \begin{pmatrix} \varphi^i_j & 0 \\ -g_{im}\underset{\varphi}{\Phi}{}_j\xi^{mi_2\cdots i_p}_{j_1\cdots j_q} & \varphi^l_i\delta^{l_1}_{j_1}\cdots\delta^{l_q}_{j_q}\delta^{i_2}_{k_2}\cdots\delta^{i_p}_{k_p} \end{pmatrix}\\
&= \begin{pmatrix} \delta^i_k & 0 \\ 0 & g_{il_1}\delta^{i_2}_{l_2}\cdots\delta^{i_p}_{l_p}\delta^{k_1}_{j_1}\cdots\delta^{k_q}_{j_q} \end{pmatrix} \begin{pmatrix} \varphi^k_l & 0 \\ -(\Phi_\varphi\xi)^{l_1\cdots l_p}_{lk_1\cdots k_q} & \varphi^{l_1}_{s_1}\delta^{l_2}_{s_2}\cdots\delta^{l_p}_{s_p}\delta^{r_1}_{k_1}\cdots\delta^{r_q}_{k_q} \end{pmatrix}
\end{aligned}$$

$\begin{pmatrix} \delta^l_j & 0 \\ 0 & g^{s_1 l}\delta^{l_1}_{r_1}\cdots\delta^{l_q}_{r_q}\delta^{s_2}_{k_2}\cdots\delta^{s_p}_{k_p} \end{pmatrix} = A\,{}^c\underset{1}{\varphi}\,A^{-1}$, where $x^{\overline{l}} = t^{s_1\cdots s_p}_{r_1\cdots r_q}$, $x^{\overline{k}} = t^{l_1\cdots l_p}_{k_1\cdots k_q}$, $y^{\overline{i}} = t^{\cdot i_2\cdots i_p}_{ij_1\cdots j_q}$, $y^{\overline{j}} = t^{\cdot k_2\cdots k_p}_{ll_1\cdots l_q}$. To show (3.1), we have taken account of

$$g_{il_1}\delta^{i_2}_{l_2}\cdots\delta^{i_p}_{l_p}\delta^{k_1}_{j_1}\cdots\delta^{k_q}_{j_q}\varphi^{l_1}_{s_1}\delta^{l_2}_{s_2}\cdots\delta^{l_p}_{s_p}\delta^{r_1}_{k_1}\cdots\delta^{r_q}_{k_q}g^{s_1 l}\delta^{l_1}_{r_1}\cdots\delta^{l_q}_{r_q}\delta^{s_2}_{k_2}\cdots\delta^{s_p}_{k_p} =$$

$$= \varphi^l_i\delta^{l_1}_{j_1}\cdots\delta^{l_q}_{j_q}\delta^{i_2}_{k_2}\cdots\delta^{i_p}_{k_p}$$

and used that g_{ij} is the pure tensor field. □

Remark. In a manifold with affinor φ-structure, a pure tensor field g is called an almost analytic tensor field if $(\Phi_\varphi g)_{kij} = 0$ [6].

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A. MAĞDEN, E. KADIOĞLU,
A.A. SALIMOV
Atatürk University
Faculty of Arts and Sciences
Department of Mathematics
25240 Erzurum-TURKEY

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