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A Generalized Trapezoid Inequality for Functions of Bounded Variation

P. Cerone, S. S. Dragomir, C. E. M. Pearce

Abstract

We establish a generalization of a recent trapezoid inequality for functions of bounded variation. A number of special cases are considered. Applications are made to quadrature formulæ, probability theory, special means and the estimation of the beta function.

Key Words: Trapezoid inequality, bounded variation, numerical integration, beta function.

1. Introduction

In [1], Dragomir proved the following trapezoid inequality for functions of bounded variation. Here and subsequently in the paper, if f is of bounded variation on $[a, b]$, we denote its total variation on that interval by $\bigvee_a^b(f)$.

Theorem A. *Let $f : [a, b] \rightarrow \mathbf{R}$ be of bounded variation on $[a, b]$. Then*

$$\left| \int_a^b f(t) dt - \frac{f(a) + f(b)}{2} (b - a) \right| \leq \frac{1}{2} (b - a) \bigvee_a^b(f). \quad (1.1)$$

The constant $1/2$ is best possible.

We introduce the notation $I_n : a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ for a division of the interval $[a, b]$, with $h_i := x_{i+1} - x_i$ ($0 \leq i < n$) and $\nu(h) := \max \{h_i \mid i = 0, \dots, n-1\}$ for the norm of the division. Then we may deduce from Theorem A that

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$$\int_a^b f(t) dt = T(f, I_n) + R(f, I_n), \quad (1.2)$$

where

$$T(f, I_n) := \frac{1}{2} \sum_{i=0}^{n-1} [f(x_i) + f(x_{i+1})] h_i, \quad (1.3)$$

and that the remainder term satisfies

$$|R(f, I_n)| \leq \frac{1}{2} \nu(h) \bigvee_a^b(f). \quad (1.4)$$

Here the constant $1/2$ is also best possible.

The main aim of this paper is to compare $\int_a^b f(t) dt$ with

$$f(a)(x-a) + f(b)(b-x),$$

where $x \in [a, b]$ is a free parameter. The choice $x = (a+b)/2$ gives the trapezoid estimate

$$\frac{f(a) + f(b)}{2} (b-a)$$

for mappings of bounded variation.

In Section 2 we derive our basic estimate, which provides an upper bound for the difference between $\int_a^b f(t) dt$ and the estimate proposed above for the case when f is a function of bounded variation. We examine the important special cases when f has a continuous derivative or is Lipschitz, monotone or convex. In Section 3 these results are applied to the estimation of the error term in some quadrature formulæ and in Section 4 to some estimates in probability theory, in particular, that of the mean $E(X)$ of a random variable X . Section 5 uses particular choices of f to obtain some apparently new inequalities subsisting amongst various well-known means of a pair of positive numbers. Finally a further special choice is taken in Section 6 to address the estimation of Euler's beta function.

For a compendious treatment of other inequalities of trapezoid type, see [2] and the references therein.

2. Some Integral Inequalities

We start with a basic integral inequality for mappings of bounded variation. For convenience we set

$$J(x) := \int_a^b f(t) dt - f(a)(x-a) - f(b)(b-x).$$

Theorem 1 *Let $f : [a, b] \rightarrow \mathbf{R}$ be a mapping of bounded variation. Then*

$$|J(x)| \leq \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b(f) \quad (2.1)$$

for all $x \in [a, b]$. The constant $1/2$ is best possible.

Proof. By the integration by parts formula for a Riemann–Stieltjes integral, we have

$$\int_a^b (x-t) df(t) = (x-t)f(t)|_a^b + \int_a^b f(t) dt,$$

whence we derive the identity

$$\int_a^b f(t) dt = (b-x)f(b) + (x-a)f(a) + \int_a^b (x-t) df(t) \quad (2.2)$$

for all $x \in [a, b]$.

If $g, v : [a, b] \rightarrow \mathbf{R}$ are such that g is continuous and v of bounded variation on $[a, b]$, then $\int_a^b g(t) dv(t)$ exists and

$$\left| \int_a^b g(t) dv(t) \right| \leq \sup_{t \in [a, b]} |g(t)| \bigvee_a^b(v).$$

Thus

$$\left| \int_a^b (x-t) df(t) \right| \leq \sup_{t \in [a, b]} |x-t| \bigvee_a^b(f). \quad (2.3)$$

As

$$\sup_{t \in [a, b]} |x - t| = \max \{x - a, b - x\} = \frac{1}{2}(b - a) + \left| x - \frac{a + b}{2} \right|,$$

(2.1) follows from (2.3) and (2.2).

Now suppose that (2.1) holds with a constant $c > 0$, that is,

$$|J(x)| \leq \left[c(b - a) + \left| x - \frac{a + b}{2} \right| \right] \bigvee_a^b(f)$$

for all $x \in [a, b]$. For $x = (a + b)/2$, we get

$$\left| \int_a^b f(t) dt - \frac{f(a) + f(b)}{2}(b - a) \right| \leq c(b - a) \bigvee_a^b(f). \quad (2.4)$$

Define $f : [a, b] \rightarrow \mathbf{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x = a \\ 1 & \text{if } x \in (a, b) \\ 0 & \text{if } x = b. \end{cases}$$

Then f is of bounded variation on $[a, b]$ and

$$\int_a^b f(x) dx = b - a, \quad \bigvee_a^b(f) = 2.$$

For this choice of f , (2.4) provides

$$b - a \leq 2c(b - a)$$

or $c \geq 1/2$, concluding the proof. □

Remark 1 a) The choice $x = b$ supplies the “left rectangle” inequality

$$\left| \int_a^b f(x) dx - f(a)(b-a) \right| \leq (b-a) \bigvee_a^b(f).$$

b) Setting $x = a$ yields the “right rectangle” inequality

$$\left| \int_a^b f(x) dx - f(b)(b-a) \right| \leq (b-a) \bigvee_a^b(f).$$

c) For $x = (a+b)/2$ we obtain the known “trapezoid” inequality (1.1). This is the best possible inequality we can derive from (2.1) in the sense that the constant $1/2$ is best possible.

Further standard assumptions about f lead to useful corollaries.

Corollary 1 Suppose $f \in C^{(1)}[a, b]$. Then

$$|J(x)| \leq \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \|f'\|_1$$

for all $x \in [a, b]$. Here as subsequently $\|\cdot\|_1$ is the L_1 -norm

$$\|f'\|_1 := \int_a^b |f'(t)| dt.$$

Corollary 2 Let $f : [a, b] \rightarrow \mathbf{R}$ be a Lipschitzian mapping with the constant $L > 0$. Then

$$|J(x)| \leq \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] (b-a)L$$

for all $x \in [a, b]$.

Proof. As f is L -Lipschitzian on $[a, b]$, it is also of bounded variation. If $Div[a, b]$ denotes the family of divisions on $[a, b]$, then

$$\begin{aligned} \bigvee_a^b(f) &= \sup_{I_n \in Div[a, b]} \sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)| \\ &\leq L \sup_{I_n \in Div[a, b]} |x_{i+1} - x_i| \\ &= (b - a) L, \end{aligned}$$

and the desired result is proved. \square

Corollary 3 Let $f : [a, b] \rightarrow \mathbf{R}$ be a monotone mapping on $[a, b]$. Then

$$|J(x)| \leq \left[\frac{1}{2}(b - a) + \left| x - \frac{a + b}{2} \right| \right] |f(b) - f(a)|$$

for all $x \in [a, b]$.

For $f : [a, b] \rightarrow \mathbf{R}$ convex on $[a, b]$, we have the Hermite–Hadamard inequality

$$f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}.$$

The above results enable us to place bounds on the difference between the two sides of the second inequality. Thus if f is convex and of bounded variation on $[a, b]$, (1.1) provides

$$0 \leq \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(t) dt \leq \frac{1}{2} \bigvee_a^b(f).$$

If f is convex and Lipschitzian with the constant L on $[a, b]$, then Corollary 2.3 yields

$$0 \leq \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(t) dt \leq \frac{1}{2}(b - a)L.$$

If f is convex and monotonic on $[a, b]$, then by Corollary 2.4

$$0 \leq \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{1}{2} |f(b) - f(a)|.$$

Finally, if $f \in C^{(1)}[a, b]$ and convex, then by Corollary 2.2

$$0 \leq \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{1}{2} \|f'\|_1.$$

3. Applications to Quadrature Formulæ

We now introduce the intermediate points $\xi_i \in [x_i, x_{i+1}]$ ($i = 0, \dots, n-1$) in the division I_n of $[a, b]$ and define

$$T_P(f, I_n, \xi) := \sum_{i=0}^{n-1} [(\xi_i - x_i) f(x_i) + (x_{i+1} - \xi_i) f(x_{i+1})].$$

We have the following result concerning the approximation by T_P of the integral $\int_a^b f(x) dx$.

Theorem 2 *Let $f : [a, b] \rightarrow \mathbf{R}$ be of bounded variation on $[a, b]$. Then*

$$\int_a^b f(x) dx = T_P(f, I_n, \xi) + R_P(f, I_n, \xi), \tag{3.1}$$

with remainder term satisfying

$$|R_P(f, I_n, \xi)| \leq \left[\frac{1}{2} \nu(h) + \max_{0 \leq i < n} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \bigvee_a^b(f) \leq \nu(h) \bigvee_a^b(f). \tag{3.2}$$

The constant $1/2$ is best possible.

Proof. Application of Theorem 1 to the intervals $[x_i, x_{i+1}]$ ($i = 0, \dots, n-1$) gives

$$\left| \int_{x_i}^{x_{i+1}} f(t) dt - [f(x_i)(\xi_i - x_i) + f(x_{i+1})(x_{i+1} - \xi_i)] \right|$$

$$\leq \left[\frac{1}{2}h_i + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \bigvee_{x_i}^{x_{i+1}}(f)$$

for all $i \in \{0, \dots, n-1\}$.

By this and the generalized triangle inequality,

$$\begin{aligned} |R_P(f, I_n, \xi)| &\leq \sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} f(t) dt - [f(x_i)(\xi_i - x_i) + f(x_{i+1})(x_{i+1} - \xi_i)] \right| \\ &\leq \sum_{i=0}^{n-1} \left[\frac{1}{2}h_i + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \bigvee_{x_i}^{x_{i+1}}(f) \\ &\leq \max_{0 \leq i < n} \left[\frac{1}{2}h_i + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \sum_{i=0}^{n-1} \bigvee_{x_i}^{x_{i+1}}(f) \\ &\leq \left[\frac{1}{2}\nu(h) + \max_{0 \leq i < n} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \bigvee_a^b(f) \end{aligned}$$

and the first inequality in (3.2) is proved.

For the second, we observe that

$$\left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \leq \frac{1}{2}h_i, \quad i = 0, \dots, n-1$$

so that

$$\max_{0 \leq i < n} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \leq \frac{1}{2}\nu(h),$$

proving the theorem. □

Remark 2 a) Choosing $\xi_i = x_{i+1}$ ($i = 0, \dots, n-1$) provides

$$\int_a^b f(x) dx = D_L(f, I_n) + R_L(f, I_n).$$

Here $D_L(f, I_n)$ is constructed from the left rectangle rule

$$D_L(f, I_n) = \sum_{i=0}^{n-1} f(x_i) h_i$$

and the remainder satisfies

$$|R_L(f, I_n)| \leq \nu(h) \bigvee_a^b(f).$$

b) Taking $\xi_i = x_i$ ($i = 0, \dots, n-1$) gives

$$\int_a^b f(x) dx = D_R(f, I_n) + R_R(f, I_n),$$

where $D_R(f, I_n)$ is built from the right rectangle rule

$$D_R(f, I_n) = \sum_{i=0}^{n-1} f(x_{i+1}) h_i$$

and the remainder term satisfies

$$|R_R(f, I_n)| \leq \nu(h) \bigvee_a^b(f).$$

c) Finally, if we choose $\xi_i = (x_i + x_{i+1})/2$, we get (1.2) with (1.3) and (1.4).

Corollary 4 Let $f : [a, b] \rightarrow \mathbf{R}$ be Lipschitzian with constant $L > 0$. Then we have (3.1) and the remainder satisfies

$$|R_T(f, I_n, \xi)| \leq L \left[\frac{1}{2} \nu(h) + \max_{0 \leq i < n} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \leq L \nu(h).$$

Corollary 5 Let $f : [a, b] \rightarrow \mathbf{R}$ be monotone on $[a, b]$. Then we have the quadrature formula (3.1) and the remainder satisfies

$$\begin{aligned} |R_T(f, I_n, \xi)| &\leq \left[\frac{1}{2} \nu(h) + \max_{0 \leq i < n} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] |f(b) - f(a)| \\ &\leq \nu(h) |f(b) - f(a)|. \end{aligned}$$

4. Applications to Probability

Proposition 1 *Let $f : [a, b] \rightarrow \mathbf{R}$ be a probability density function of bounded variation on $[a, b]$ and $F : [a, b] \rightarrow \mathbf{R}$ the corresponding distribution function*

$$F(x) = \int_a^x f(t) dt, \quad x \in [a, b].$$

Then

$$|F(x) - [f(a)(y-a) + f(x)(x-y)]| \leq \left[\frac{1}{2}(x-a) + \left| y - \frac{a+x}{2} \right| \right] \bigvee_a^x(f) \quad (4.1)$$

for all $a \leq y \leq x$. In particular, choosing $y = (a+x)/2$ gives

$$\left| F(x) - \frac{f(a) + f(x)}{2}(x-a) \right| \leq \frac{1}{2}(x-a) \bigvee_a^x(f) \quad (4.2)$$

for all $x \in [a, b]$. The constant $1/2$ in (4.1) and (4.2) is best possible.

Proof. The result is immediate from Theorem 1. □

The following approximation holds for the expectation of a random variable.

Proposition 2 *Let X be a random variable having distribution function F and expectation $E(X)$. Then*

$$\left| E(X) - \sum_{i=0}^{n-1} F(x_i)(\xi_{i+1} - \xi_i) - \xi_{n-1} \right| \leq \frac{1}{2}\nu(h) + \max_{0 \leq i < n} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right|. \quad (4.3)$$

Proof. We apply Theorem 2 to F to get

$$\begin{aligned} & \left| \int_a^b F(t) dt - \sum_{i=0}^{n-1} F(x_i)(\xi_i - x_i) - \sum_{i=0}^{n-1} F(x_{i+1})(x_{i+1} - \xi_i) \right| \\ & \leq \left[\frac{1}{2}\nu(h) + \max_{0 \leq i < n} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \bigvee_a^b(F). \end{aligned} \quad (4.4)$$

But

$$\bigvee_a^b (F) = F(b) - F(a) = 1$$

and

$$\int_a^b F(t) dt = tF(t)|_a^b - \int_a^b tf(t) dt = bF(b) - aF(a) - E(X) = b - E(X).$$

By (4.4),

$$\begin{aligned} & \left| b - E(X) - F(a)(\xi_0 - a) - \sum_{i=1}^{n-1} F(x_i)(\xi_i - x_i) \right. \\ & \quad \left. - \sum_{i=0}^{n-2} F(x_{i+1})(x_{i+1} - \xi_i) - F(b)(b - \xi_{n-1}) \right| \\ & \leq \frac{1}{2}\nu(h) + \max_{0 \leq i < n} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \end{aligned}$$

or

$$\left| -E(X) - \sum_{i=1}^{n-1} F(x_i)(\xi_i - \xi_{i-1}) + \xi_{n-1} \right| \leq \frac{1}{2}\nu(f) + \max_{0 \leq i < n} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right|$$

and the proposition is proved. \square

Remark 3 a) Suppose the division is reduced to the endpoints, that is, $a = x_0 < x_1 = b$ and $\xi_1 = \xi \in [a, b]$. Then by (4.3)

$$|E(X) - \xi| \leq \frac{1}{2}(b - a) + \left| \xi - \frac{a + b}{2} \right|$$

for all $\xi \in [a, b]$.

b) Suppose $a = x_0 < x < x_2 = b$ and $\xi \in [a, x], \mu \in [x, b]$.

Then by (4.3)

$$\begin{aligned} & |E(X) - F(x)(\xi - \mu) - \mu| \\ & \leq \frac{1}{2} \max\{|x - a|, |b - x|\} + \max\left\{\left|\xi - \frac{a+x}{2}\right|, \left|\mu - \frac{x+b}{2}\right|\right\} \\ & = \frac{1}{2}(b - a) + \left|x - \frac{a+b}{2}\right| + \max\left\{\left|\xi - \frac{a+x}{2}\right|, \left|\mu - \frac{x+b}{2}\right|\right\} \end{aligned}$$

for all $a \leq \xi \leq x \leq \mu \leq b$.

In particular, if $\xi = (a + x)/2$ and $\mu = (x + b)/2$, then

$$\left|E(X) - \frac{1}{2}F(x)(a - b) - \frac{x+b}{2}\right| \leq \frac{1}{2}(b - a) + \left|x - \frac{a+b}{2}\right|$$

for all $x \in [a, b]$.

5. Applications to Special Means

We now derive some results for various well-known means. For $a, b \geq 0$ we have the arithmetic mean

$$A = A(a, b) := (a + b)/2$$

and the geometric mean

$$G = G(a, b) := \sqrt{ab}.$$

For $a, b > 0$ we have the harmonic mean

$$H = H(a, b) := 2/(a^{-1} + b^{-1}),$$

the logarithmic mean

$$L = L(a, b) := \begin{cases} a & \text{if } a = b \\ \frac{b - a}{\ln b - \ln a} & \text{if } a \neq b, \end{cases}$$

the identric mean

$$I := I(a, b) = \begin{cases} a & \text{if } a = b \\ \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}} & \text{if } a \neq b \end{cases}$$

and for $p \in \mathbf{R} \setminus \{-1, 0\}$, the p -logarithmic mean

$$L_p = L_p(a, b) := \begin{cases} \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{1/p} & \text{if } a \neq b; \\ a & \text{if } a = b. \end{cases}$$

It is well-known that with $L_{-1} := L$ and $L_0 := I$, the net (L_p) is monotone nondecreasing in $p \in \mathbf{R}$. In particular, we have the inequalities

$$H \leq G \leq L \leq I \leq A.$$

In what follows we establish some rather more involved inequalities for the above means by the use of (2.1), which we express in the equivalent form

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{bf(b) - af(a)}{b-a} + x \cdot \frac{f(b) - f(a)}{b-a} \right| \quad (5.1) \\ & \leq \left[\frac{1}{2}(b-a) + |x-A| \right] \frac{1}{b-a} \bigvee_a^b(f). \end{aligned}$$

Define $f : [a, b] \subset (0, \infty) \rightarrow \mathbf{R}$ by $f(x) = x^p$, $p \in \mathbf{R} \setminus \{-1, 0\}$. Then

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(t) dt &= L_p^p(a, b), \\ \frac{bf(b) - af(a)}{b-a} &= (p+1) L_p^p(a, b), \\ \frac{f(b) - f(a)}{b-a} &= p L_{p-1}^{p-1}(a, b), \end{aligned}$$

$$\frac{1}{b-a} \bigvee_a^b (f) = \frac{1}{b-a} \int_a^b |f'(t)| dt = |p| L_{p-1}^{p-1}(a, b).$$

We deduce from (5.1) that

$$\left| L_p^p - (p+1) L_p^p + px L_{p-1}^{p-1} \right| \leq \left[\frac{1}{2} (b-a) + |x-A| \right] |p| L_{p-1}^{p-1},$$

which is equivalent to

$$\left| x L_{p-1}^{p-1} - L_p^p \right| \leq \left[\frac{1}{2} (b-a) + |x-A| \right] L_{p-1}^{p-1}, \quad x \in [a, b].$$

The choice $x = A$ yields

$$\left| A L_{p-1}^{p-1} - L_p^p \right| \leq \frac{1}{2} (b-a) L_{p-1}^{p-1}.$$

If instead we define $f : [a, b] \subset (0, \infty) \rightarrow \mathbf{R}$ by $f(x) = 1/x$, then

$$\frac{1}{b-a} \int_a^b f(t) dt = L^{-1}(a, b),$$

$$\frac{bf(b) - af(a)}{b-a} = 0,$$

$$\frac{f(b) - f(a)}{b-a} = -G^{-2}(a, b),$$

$$\frac{1}{b-a} \bigvee_a^b (f) = \frac{1}{b-a} \int_a^b |f'(t)| dt = G^{-2}(a, b).$$

From (5.1), we deduce that

$$\left| L^{-1} - xG^{-2} \right| \leq \left[\frac{1}{2} (b-a) + |x-A| \right] G^{-2},$$

or equivalently

$$|xL - G^2| \leq \frac{1}{2} [(b-a) + |x-A|] L, \quad x \in [a, b].$$

Choosing $x = A$, we get

$$0 \leq AL - G^2 \leq \frac{1}{2} (b-a).$$

Finally, define $f : [a, b] \subset (0, \infty) \rightarrow \mathbf{R}$ by $f(x) = \ln x$, so that

$$\frac{1}{b-a} \int_a^b f(t) dt = \ln I(a, b),$$

$$\frac{bf(b) - af(a)}{b-a} = \ln I(a, b) + 1,$$

$$\frac{f(b) - f(a)}{b-a} = L^{-1}(a, b),$$

$$\frac{1}{b-a} \int_a^b |f'(t)| dt = L^{-1}(a, b).$$

From (5.1), we deduce that

$$|x - L| \leq \frac{1}{2} (b-a) + |x - A|, \quad x \in [a, b].$$

With $x = A$, we get

$$0 \leq A - L \leq \frac{1}{2} (b-a).$$

6. Application to Euler's Beta Function

Let β be the Euler beta function given by

$$\beta(p, q) := \int_0^1 t^{p-1} (1-t)^{q-1} dt, \quad p, q > 0.$$

Proposition 3 *If $p, q > 1$, then*

$$\beta(p, q) = T(p, q, I_n, \xi) + R(p, q, I_n, \xi),$$

where

$$T(p, q, I_n, \xi) = \sum_{i=0}^{n-1} \left[(\xi_i - x_i) x_i^{p-1} (1 - x_i)^{q-1} + (x_{i+1} - \xi_i) x_{i+1}^{p-1} (1 - x_{i+1})^{q-1} \right]$$

and the remainder $R(p, q, I_n, \xi)$ satisfies

$$|R| \leq \left[\frac{1}{2} \nu(h) + \max \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \max(p-1, q-1) \beta(p-1, q-1).$$

Proof. For $p, q > 1$ define $f_{p,q} : (0, 1) \rightarrow \mathbf{R}$ by

$$f_{p,q}(t) = t^{p-1} (1-t)^{q-1}.$$

We have

$$f'_{p,q}(t) = [(p+q-2)t - q + 1] t^{p-2} (1-t)^{q-2},$$

so that

$$\begin{aligned} \bigvee_0^1(f_{p,q}) &= \int_0^1 |f'_{p,q}(t)| dt \\ &\leq \int_0^1 |(p+q-2)t - q + 1| t^{p-2} (1-t)^{q-2} dt \\ &\leq \max(q-1, p-1) \int_0^1 t^{p-2} (1-t)^{q-2} dt \\ &= \max(q-1, p-1) \beta(p-1, q-1) \end{aligned}$$

and the proposition is proved. □

Remark 4 The choice $\xi_i = (x_i + x_{i+1})/2$ yields

$$\beta(p, q) = T(p, q, I_n) + R(p, q, I_n)$$

where

$$T(p, q, I_n) := \frac{1}{2} \sum_{i=0}^{n-1} \left[x_i^{p-1} (1-x_i)^{q-1} + x_{i+1}^{p-1} (1-x_{i+1})^{q-1} \right] h_i$$

corresponds to the trapezoid rule and the remainder satisfies

$$|R(p, q, I_n)| \leq \frac{1}{2} \nu(h) \max(p-1, q-1) \beta(p-1, q-1)$$

for all $p, q > 1$.

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