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Intrinsic Equations for a Relaxed Elastic Line on an Oriented Hypersurface in the Minkowski Space \mathbb{R}_1^n

N. Gürbüz and A. Görgülü

Abstract

We give the intrinsic equations for a relaxed elastic line on an oriented surface in \mathbb{R}_1^3 ([1],[2]).

In this paper, we derived the intrinsic equations for a relaxed elastic line on an oriented time-like hypersurface and space-like hypersurface in the Minkowski space \mathbb{R}_1^n and give additional results about relaxed elastic lines on various timelike and spacelike hypersurface in the Minkowski space \mathbb{R}_1^n .

Key Words: Elastic line, Minkowski space.

1. Introduction

In this section, we give some fundamental definitions and theorems.

Definition 1.1. Let α denote an arc on a connected oriented hypersurface M in \mathbb{R}_1^n parametrized by arc length s , $0 \leq s \leq l$. Let $k_1(s)$ be the curvature of the first curvature of $\alpha(s)$. The first total square curvature K of α in \mathbb{R}_1^n is defined by

$$K = \int_0^l k_1^2 ds. \quad (1.1)$$

Definition 1.2. The arc α is called a relaxed elastic line if it is an extremal for the variational problem of minimizing the value of K within the family of all arcs of length l on M having the same initial point and initial direction as α in the Minkowski space \mathbb{R}_1^n .

Definition 1.3. On an $n \times n$ matrix, the following conditions are equivalent:

(1) $g \in O_\nu(n)$

(2) $g^t = \varepsilon g^{-1} \varepsilon$

(3) The columns(rows) of g form an orthonormal basis for \mathbb{R}_ν^n (first ν vectors are timelike).

(4) g carries one (hence every) orthonormal basis for \mathbb{R}_ν^n to an orthonormal basis [3].

Definition 1.4. Let M be a pseudo-Euclidean hypersurface in \mathbb{R}_1^n and a curve α which lies on M . Apart from the Frenet vector field system $\{V_1, \overline{V_2}, \overline{V_3}, \dots, \overline{V_{n-1}}, \overline{V_n}\}$, there is also exist a second orthonormal vector field system $\{V_1, \dots, V_{n-1}, N\}$ at every point of the curve α . At a point $\alpha(s)$ of α , let $V_1(s) = \alpha'(s)$ denote the unit tangent vector to α , let $N(s)$ denote the unit hypersurface normal to M . $\{V_1, \dots, V_{n-1}, N\}$ gives a basis for all vectors at $\alpha(s)$ and $\{V_1, \dots, V_{n-1}, N\}$ gives a basis for the vectors tangent to M at $\alpha(s)$. Let II denote the second fundamental form of M . The orthonormal system $\{V_1, \dots, V_{n-1}, N\}$ is called natural frame field for hypersurface strip (α, M) .

Definition 1.5. Let M be a pseudo-Euclidean hypersurface in \mathbb{R}_1^n and a curve α be a curve on M . Then, for each i , $1 \leq i \leq n - 1$, the function

$$k_{ig} : I \subset \mathbb{R} \rightarrow \mathbb{R}$$

defined for $s \in I$ by

$$k_{ig}(s) = \langle V_i'(s), V_{i+1}(s) \rangle$$

is called the i^{th} geodesic curvature function of the curve α and $k_{ig}(s)$ is called the i^{th} geodesic curvature of the curve α at $\alpha(s)$ in \mathbb{R}_1^n .

Theorem 1.1. Let M be a pseudo-Euclidean hypersurface in \mathbb{R}_1^n and α denote an arc on M . The derivative formulas of orthonormal vector field system $\{V_1, \dots, V_{n-1}, N\}$ is

$$\begin{bmatrix} V'_1 \\ V'_2 \\ \cdot \\ \cdot \\ \cdot \\ V'_{n-1} \\ N' \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_2 k_{1g} & 0 & \dots & 0 & \varepsilon_n a_1 \\ -\varepsilon_1 k_{1g} & 0 & \varepsilon_3 k_{2g} & \dots & 0 & \varepsilon_n a_2 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 0 & \varepsilon_n a_{n-1} \\ -\varepsilon_1 a_1 & -\varepsilon_2 a_2 & -\varepsilon_3 a_3 & \dots & -\varepsilon_{(n-1)} a_{n-1} & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \cdot \\ \cdot \\ \cdot \\ V_{n-1} \\ N \end{bmatrix}, \tag{1.2}$$

where k_{ig} is the i^{th} geodesic curvature funtion,

$$a_i = II(V_1, V_i), 1 \leq i \leq n - 1$$

and

$$\langle V_1, V_1 \rangle = \varepsilon_1, \langle V_2, V_2 \rangle = \varepsilon_2, \dots, \langle N, N \rangle = \varepsilon_n.$$

2. Obtaining the Equations

Now, assume that α lies in a coordinate patch $(u_1, \dots, u_{n-1}) \rightarrow x(u_1, \dots, u_{n-1})$ of M and let $x_{u_1} = \frac{\partial x}{\partial u_1}, x_{u_2} = \frac{\partial x}{\partial u_2}, \dots, x_{u_{n-1}} = \frac{\partial x}{\partial u_{n-1}}$. Then α is expressed as

$$\alpha(s) = x(u_1(s), u_2(s), u_3(s), \dots, u_{n-1}(s)), \quad 0 \leq s \leq l$$

with

$$V_1(s) = \alpha'(s) = x_{u_1} \frac{du_1}{ds} + x_{u_2} \frac{du_2}{ds} + \dots + x_{u_n} \frac{du_n}{ds}$$

and

$$V_2(s) = p_1(s)x_{u_1} + p_2(s)x_{u_2} + \dots + p_{n-1}(s)x_{u_{n-1}}$$

for suitable scalar functions $p_1(s), p_2(s), \dots, p_{n-1}(s)$.

Next, we must define variational fields for our problem. In order to obtain variational arcs of length l , it is generally necessary to extend α to an arc α^* defined for $0 \leq s \leq l^*$, with $l^* > l$, but sufficiently close to l so that α^* lies in the coordinate patch. Let $\mu(s)$, $0 \leq s \leq l^*$, be a scalar function of class C^{n-1} , not vanishing identically. Define

$$\eta_1(s) = \mu(s)p_1^*(s) \ , \ \eta_2(s) = \mu(s)p_2^*(s), \dots, \eta_{n-1}(s) = \mu(s)p_{n-1}^*(s).$$

Then, along α

$$\eta_1(s)x_{u_1} + \eta_2(s)x_{u_2} + \dots + \eta_{n-1}(s)x_{u_{n-1}} = \mu(s)V_2(s). \quad (2.1)$$

Assume also that

$$\mu(0) = 0, \mu'(0) = 0. \quad (2.2)$$

Now define

$$\beta(\sigma; t) = x(u_1(\sigma) + t\eta_1(\sigma), \dots, u_{n-1}(\sigma) + t\eta_{n-1}(\sigma)), \quad (2.3)$$

for $0 \leq \sigma \leq l^*$. For $|t| < \varepsilon$ (where $\varepsilon > 0$ depends upon the choice of α^* and of μ), the point $\beta(\sigma; t)$ lies in the coordinate patch. For fixed t , $\beta(\sigma; t)$ gives an arc with the same initial point and initial direction as α , because of (2.2). For $t = 0$, $\beta(\sigma; 0)$ is the same as α^* and σ is arc length. For $t \neq 0$, the parameter σ is not arc length in general.

For fixed t , $|t| < \varepsilon$, let $L^*(t)$ denote the length of the arc $\beta(\sigma; t)$, $0 \leq \sigma \leq l^*$. Then

$$L^*(t) = \int_0^{l^*} \sqrt{\left| \left\langle \frac{\partial \beta}{\partial \sigma}(\sigma; t), \frac{\partial \beta}{\partial \sigma}(\sigma; t) \right\rangle \right|} d\sigma \quad (2.4)$$

with

$$L^*(0) = l^* > l. \quad (2.5)$$

It is clear from (2.3) and (2.4) that $L^*(t)$ is continuous. In particular, it follows from (2.5) that

$$L^*(t) > \frac{l + l^*}{2} > l, \quad (|t| < \varepsilon_*) \quad (2.6)$$

for a suitable ε_* satisfying $0 < \varepsilon_* \leq \varepsilon$. Because of (2.6), we can restrict $\beta(\sigma; t)$, $0 \leq |t| < \varepsilon_*$, to an arc of length l by restricting the parameter σ to an interval $0 \leq \sigma \leq \lambda(t) \leq l^*$, by requiring

$$\int_0^{\lambda(t)} \sqrt{\left| \left\langle \frac{\partial \beta}{\partial \sigma}, \frac{\partial \beta}{\partial \sigma} \right\rangle \right|} d\sigma = l. \quad (2.7)$$

Note that $\lambda(0) = l$. The function $\lambda(t)$ need not be determined explicitly, but we shall need

$$\left. \frac{d\lambda}{dt} \right|_{t=0} = \varepsilon_1 \int_0^l \mu k_{1g} ds. \quad (2.8)$$

The proof of (2.8) and of other results below will depend on calculations from (2.3) such as

$$\left. \frac{\partial \beta}{\partial \sigma} \right|_{t=0} = V_1, \quad 0 \leq \sigma \leq l \quad (2.9)$$

which gives

$$\left. \frac{\partial^2 \beta}{\partial \sigma^2} \right|_{t=0} = V_1' = \varepsilon_2 k_{1g} V_2 + \varepsilon_n a_1 N. \quad (2.10)$$

Also, it follows from (2.1) that

$$\left. \frac{\partial \beta}{\partial t} \right|_{t=0} = \mu V_2. \quad (2.11)$$

Using (2.1), the second differentiation of (2.11) gives

$$\left. \frac{\partial^2 \beta}{\partial t \partial \sigma} \right|_{t=0} = -\varepsilon_1 \mu k_{1g} V_1 + \mu' V_2 + \varepsilon_3 \mu k_{2g} V_3 + \varepsilon_n \mu a_2 N \quad (2.12)$$

and the third differentiation of (2.11) gives

$$\begin{aligned}
 \left. \frac{\partial^3 \beta}{\partial t \partial \sigma^2} \right|_{t=0} &= (-2\varepsilon_1 \mu' k_{1g} - \varepsilon_1 \mu k'_{1g} - \varepsilon_1 \varepsilon_n \mu a_1 a_2) V_1 \\
 &+ (\mu'' - \varepsilon_1 \varepsilon_2 \mu k_{1g}^2 - \varepsilon_2 \varepsilon_3 \mu k_{2g}^2 - \varepsilon_2 \varepsilon_n \mu a_2^2) V_2 \\
 &+ (2\varepsilon_3 \mu' k_{2g} + \varepsilon_3 \mu k'_{2g} - \varepsilon_3 \varepsilon_n \mu a_2 a_3) V_3 \\
 &+ (\varepsilon_3 \varepsilon_4 \mu k_{2g} k_{3g} - \varepsilon_4 \varepsilon_n \mu a_2 a_4) V_4 \\
 &- (\varepsilon_5 \varepsilon_n \mu a_2 a_5 V_5 + \varepsilon_6 \varepsilon_n \mu a_2 a_6 V_6 + \dots + \varepsilon_{n-1} \varepsilon_n \mu a_2 a_{n-1} V_{n-1}) \\
 &+ (-\varepsilon_1 \varepsilon_n \mu k_{1g} a_1 + 2\varepsilon_n \mu' a_2 + \varepsilon_3 \varepsilon_n \mu k_{2g} a_3 + \varepsilon_n \mu a_2') N.
 \end{aligned} \tag{2.13}$$

To prove (2.8), differentiate (2.7) with respect to t , remembering that l is constant, and evaluate at $t=0$ using (2.9) and (2.12), with $\lambda(0) = l$.

$$\frac{d\lambda}{dt} \Big|_{t=0} \sqrt{\left\langle \left\langle \frac{\partial \beta}{\partial \sigma} \Big|_{t=0}, \frac{\partial \beta}{\partial \sigma} \Big|_{t=0} \right\rangle \right\rangle} + \int_0^l \left\langle \frac{\partial \beta}{\partial \sigma} \Big|_{t=0}, \frac{\partial^2 \beta}{\partial \sigma \partial t} \Big|_{t=0} \right\rangle \frac{\sqrt{\left\langle \left\langle \frac{\partial \beta}{\partial \sigma} \Big|_{t=0}, \frac{\partial \beta}{\partial \sigma} \Big|_{t=0} \right\rangle \right\rangle}}{\left\langle \frac{\partial \beta}{\partial \sigma} \Big|_{t=0}, \frac{\partial \beta}{\partial \sigma} \Big|_{t=0} \right\rangle} ds = 0$$

Now, let $K(t)$ denote the total square curvature of the arc $\beta(\sigma; t)$, $0 \leq \sigma \leq \lambda(t)$, $|t| < \varepsilon_*$. Since σ is not generally arc length for $t \neq 0$, the total square curvature is ,

$$K(t) = \int_0^{\lambda(t)} \frac{\left| \left\langle \frac{\partial \beta}{\partial \sigma}(\sigma, t) \wedge \frac{\partial^2 \beta}{\partial \sigma^2}(\sigma, t), \frac{\partial \beta}{\partial \sigma}(\sigma, t) \wedge \frac{\partial^2 \beta}{\partial \sigma^2}(\sigma, t) \right\rangle \right|}{\left| \left\langle \frac{\partial \beta}{\partial \sigma}(\sigma, t), \frac{\partial \beta}{\partial \sigma}(\sigma, t) \right\rangle \right|^3} \left| \left\langle \frac{\partial \beta}{\partial \sigma}(\sigma, t), \frac{\partial \beta}{\partial \sigma}(\sigma, t) \right\rangle \right|^{1/2} d\sigma.$$

A necessary condition for α being extremal is that $K'(0) = 0$ for arbitrary μ satisfying (2.2). In calculating $K'(t)$, we give explicitly only those terms which do not vanish for $t = 0$. The omitted terms are those with a factor $\left\langle \frac{\partial^2 \beta}{\partial \sigma^2}, \frac{\partial \beta}{\partial \sigma} \right\rangle$, which vanishes at $t = 0$,

since $\langle V'_1, V_1 \rangle = 0$. Thus

$$\begin{aligned}
 K'(t) &= \frac{d\lambda}{dt} \left\{ \left| \left\langle \frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma} \right\rangle \right|^{-3/2} \left| - \left\langle \frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial^2\beta}{\partial\sigma^2} \right\rangle \right| \right\}_{\sigma=\lambda(t)} \\
 &\quad - 3 \int_0^{\lambda(t)} \left| \left\langle \frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma} \right\rangle \right|^{-5/2} \left\langle \frac{\partial^2\beta}{\partial t \partial\sigma}, \frac{\partial\beta}{\partial\sigma} \right\rangle \frac{\left| \left\langle \frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma} \right\rangle \right|}{\left\langle \frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma} \right\rangle} \left| \left\langle \frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial^2\beta}{\partial\sigma^2} \right\rangle \right| d\sigma \\
 &\quad + 2 \int_0^{\lambda(t)} \left| \left\langle \frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma} \right\rangle \right|^{-3/2} \left\langle \frac{\partial^3\beta}{\partial t \partial\sigma^2}, \frac{\partial^2\beta}{\partial\sigma^2} \right\rangle \frac{\left| \left\langle \frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial^2\beta}{\partial\sigma^2} \right\rangle \right|}{\left\langle \frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial^2\beta}{\partial\sigma^2} \right\rangle} d\sigma + \dots
 \end{aligned}$$

Using (2.8), (2.9), (2.12) and (2.10), we find

$$\begin{aligned}
 K'(0) &= \varepsilon_1 \int_0^l \mu k_{1g} ds \left\{ |\varepsilon_2 k_{1g}^2 + \varepsilon_n a_1^2| \right\}_{\sigma=\lambda(0)} \\
 &\quad + 2 \int_0^l k_{1g} (\mu'' - \varepsilon_1 \varepsilon_2 \mu k_{1g}^2 - \varepsilon_2 \varepsilon_3 \mu k_{2g}^2 - \varepsilon_2 \varepsilon_n \mu a_2^2) \frac{|\varepsilon_2 k_{1g}^2 + \varepsilon_n a_1^2|}{\varepsilon_2 k_{1g}^2 + \varepsilon_n a_1^2} ds \\
 &\quad + 2 \int_0^l a_1 (-\varepsilon_1 \varepsilon_n \mu k_{1g} a_1 + 2\varepsilon_n \mu' a_2 + \varepsilon_3 \varepsilon_n \mu k_{2g} a_3 + \varepsilon_n \mu a_2') \frac{|\varepsilon_2 k_{1g}^2 + \varepsilon_n a_1^2|}{\varepsilon_2 k_{1g}^2 + \varepsilon_n a_1^2} ds \\
 &\quad + 3\varepsilon_1 \int_0^l \mu k_{1g} |\varepsilon_2 k_{1g}^2 + \varepsilon_n a_1^2| ds.
 \end{aligned} \tag{2.14}$$

However, with integration by parts and (2.2),

$$2 \int_0^l \mu'' k_{1g} ds = 2\mu'(l)k_{1g}(l) - 2\mu(l)k'_{1g}(l) + 2 \int_0^l \mu k''_{1g} ds \tag{2.15}$$

and

$$4 \int_0^l \mu' a_1 a_2 ds = 4\mu(l)a_1(l)a_2(l) - 4 \int_0^l \mu a'_1 a_2 ds - 4 \int_0^l \mu a_1 a'_2 ds. \tag{2.16}$$

2.1. Intrinsic equations for a relaxed elastic line on a timelike hypersurface

If V_1 is timelike, V_2, V_3, \dots, V_{n-1} and N are spacelike then

$$\langle V_1, V_1 \rangle = \varepsilon_1 = -1, \quad \langle V_2, V_2 \rangle = \varepsilon_2 = 1, \dots, \langle N, N \rangle = \varepsilon_n = 1.$$

In the case of $k_{1g}^2 > a_1^2$,

$$|\varepsilon_2 k_{1g}^2 + \varepsilon_n a_1^2| = k_{1g}^2 + a_1^2. \tag{2.17}$$

Substituting (2.8), (2.9), (2.12), (2.13), (2.15), (2.16) and (2.17) in (2.14), we find

$$\begin{aligned} K'(0) = & \int_0^l \mu \{ 2k_{1g}'' - 2a_1 a_2' - 4a_2 a_1' + 2k_{2g} a_1 a_3 \\ & + k_{1g} (-k_{1g}^2(l) - a_1^2(l) - k_{1g}^2 - a_1^2 - 2k_{2g}^2 - 2a_2^2) \} ds \\ & + 2\mu'(l)k_{1g}(l) - 2\mu(l)k_{1g}'(l) + 4\mu(l)a_1(l)a_2(l). \end{aligned}$$

In order that $K'(0) = 0$ for all choices of the function $\mu(s)$ satisfying (2.2), with arbitrary values of $\mu(l)$ and $\mu'(l)$, the given timelike arc α must satisfy two boundary conditions and differential equation:

$$\begin{aligned} (1) \quad & k_{1g}(l) = 0 \\ (2) \quad & k_{1g}'(l) = 2a_1(l)a_2(l) \\ (3) \quad & 2k_{1g}'' - 2a_1 a_2' - 4a_2 a_1' + 2k_{2g} a_1 a_3 \\ & + k_{1g} (-a_1^2(l) - k_{1g}^2 - a_1^2 - 2k_{2g}^2 - 2a_2^2) = 0. \end{aligned} \tag{2.18}$$

2.2. Intrinsic equations for a relaxed elastic line on an spacelike hypersurface

If V_1, V_2, \dots, V_{n-1} is spacelike and N is timelike,

i) In the case of $k_{1g}^2 < a_1^2$

$$|\varepsilon_2 k_{1g}^2 + \varepsilon_n a_1^2| = -k_{1g}^2 + a_1^2 \tag{2.19}$$

Substituting (2.8), (2.9), (2.12), (2.13), (2.15), (2.16) and (2.19) in (2.14), $K'(0)$ can be written as

$$\begin{aligned}
 K'(0) = & \int_0^l \mu \{ -2k''_{1g} - 2a_1a'_2 - 4a_2a'_1 + 2k_{2g}a_1a_3 \\
 & + k_{1g} (-k^2_{1g}(l) + a^2_1(l) - k^2_{1g} + a^2_1 + 2k^2_{2g} - 2a^2_2) \} ds \\
 & - 2\mu'(l)k_{1g}(l) + 2\mu(l)k'_{1g}(l) + 4\mu(l)a_1(l)a_2(l).
 \end{aligned}$$

In order that $K'(0) = 0$ for all choices of the function $\mu(s)$ satisfying (2.2), with arbitrary values of $\mu(l)$ and $\mu'(l)$, the given timelike arc α must satisfy two boundary conditions and differential equation

$$\begin{aligned}
 (1) \quad & k_{1g}(l) = 0 \\
 (2) \quad & k'_{1g}(l) = -2a_1(l)a_2(l) \\
 (3) \quad & -2k''_{1g} - 2a_1a'_2 - 4a_2a'_1 + 2k_{2g}a_1a_3 \\
 & + k_{1g} (a^2_1(l) - k^2_{1g} + a^2_1 + 2k^2_{2g} - 2a^2_2) = 0.
 \end{aligned} \tag{2.20}$$

ii) In the case of $k^2_{1g} > a^2_1$

$$|\varepsilon_2 k^2_{1g} + \varepsilon_n a^2_1| = k^2_{1g} - a^2_1. \tag{2.21}$$

Substituting (2.8), (2.9), (2.12), (2.13), (2.15), (2.16) and (2.21) in (2.14), $K'(0)$ can be written as

$$\begin{aligned}
 K'(0) = & \int_0^l \mu \{ 2k''_{1g} + 2a_1a'_2 + 4a_2a'_1 - 2k_{2g}a_1a_3 \\
 & + k_{1g} (k^2_{1g}(l) - a^2_1(l) + k^2_{1g} - a^2_1 - 2k^2_{2g} + 2a^2_2) \} ds \\
 & + 2\mu'(l)k_{1g}(l) - 2\mu(l)k'_{1g}(l) - 4\mu(l)a_1(l)a_2(l).
 \end{aligned}$$

In order that $K'(0) = 0$ for all choices of the function $\mu(s)$ satisfying (2.2), with arbitrary values of $\mu(l)$ and $\mu'(l)$, the given timelike arc α must satisfy two boundary conditions and differential equation

$$\begin{aligned}
 (1) \quad & k_{1g}(l) = 0 \\
 (2) \quad & k'_{1g}(l) = -2a_1(l)a_2(l) \\
 (3) \quad & 2k''_{1g} + 2a_1a'_2 + 4a_2a'_1 - 2k_{2g}a_1a_3 \\
 & + k_{1g}(k^2_{1g}(l) - a^2_1(l) + k^2_{1g} - a^2_1 - 2k^2_{2g} + 2a^2_2) = 0.
 \end{aligned} \tag{2.22}$$

3. Applications

Theorem 3.1. An arc of a geodesic on hyperbolic n-space $H^n(r)$ is a relaxed elastic line .

Proof. For a geodesic arc on hyperbolic n-space $H^n(r)$, $k_{1g} = 0$ (so $k_{2g} = 0$), $a^2_1 = c^2 = \frac{1}{r^2}$ and $a_2 = a_3 = 0$. Therefore (2.20) and (2.22) are satisfied.

Theorem 3.2. In the spacelike hyperplane in \mathbb{R}^n_1 , an arc is a relaxed elastic line if and only if it lies on a geodesic.

Proof. In the spacelike hyperplane in \mathbb{R}^n_1 , k_{2g} , a_2 , a_3 vanishes for all curves and $a_1 = 0$. Then the third equation in (2.20) and (2.22) reduces to

$$2k''_{1g} + k^3_{1g} = 0. \tag{3.1}$$

With integrating factor k'_{1g} , the first integral is

$$(k'_{1g})^2 + \frac{1}{4}k^4_{1g} = \text{const.}$$

The boundary conditions in(2.20) and (2.22), which reduces to $k'_{1g}(l) = 0$, require that the constant be zero. But then we must have $k_{1g} \equiv 0$.

Conversely, any arc of a geodesic in the spacelike hyperplane satisfies (3.1), (2.20) and (2.22), trivially.

Theorem 3.3. On the spacelike hypersurface in \mathbb{R}^n_1 , an arc of a geodesic is a relaxed elastic line if and only if it satisfies

$$a^2_1a_2 = 0.$$

Proof. If $k_{1g} \equiv 0$ (so $k_{2g} = 0$), then the third equation in (2.20) and (2.22) reduces to

$$a_1 a'_2 + 2a'_1 a_2 = 0.$$

The first integral is

$$a_1^2 a_2 = \text{const}$$

and the constant must vanish because of the second equation in (2.20), (2.22). The boundary conditions in (2.20) and (2.22) are trivial.

Theorem 3.4. An arc of a geodesic on a pseudo-hypersphere $S_1^n(r)$ is a relaxed elastic line.

Proof. For a geodesic arc on hyperbolic n-space $S_1^n(r)$, $k_{1g} = 0$ (so $k_{2g} = 0$), $a_1^2 = c^2 = \frac{1}{r^2}$ and $a_2 = a_3 = 0$. Therefore (2.18) is satisfied.

Theorem 3.5. In the timelike hyperplane in \mathbb{R}_1^n , an arc is a relaxed elastic line if and only if it lies on a geodesic.

Proof. In the timelike hyperplane, k_{2g}, a_2, a_3 vanishes for all curves and $a_1^2 = c^2 = 0$. The third equation in (2.18) reduces to

$$2k''_{1g} - k_{1g}^3 = 0. \tag{3.2}$$

With integrating factor k'_{1g} , the first integral is

$$(k'_{1g})^2 - \frac{1}{4}k_{1g}^4 = \text{const}.$$

The boundary conditions in (2.18), which reduces to $k'_{1g}(l) = 0$, require that the constant be zero. But then we must have $k_{1g} \equiv 0$.

Conversely, any arc of a geodesic in the timelike hyperplane satisfies (26) and (20), trivially.

Theorem 3.6. On the timelike hypersurface in \mathbb{R}_1^n , an arc of a geodesic is a relaxed elastic line if and only if it satisfies

$$a_1^2 a_2 = 0.$$

Proof. If $k_{1g} \equiv 0$, then the third equation in (2.18) reduces to

$$a_1 a_2' + 2a_1' a_2 = 0.$$

The first integral is

$$a_1^2 a_2 = \text{const}$$

and the constant must vanish because of the second equation in (2.18). The boundary conditions in (2.18) are trivial.

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