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Lifts of Derivations to the Semitangent Bundle

A.A. Salimov and Ekrem Kadıoğlu

Abstract

The main purpose of this paper is to investigate the complete lifts of derivations for semitangent bundle and to discuss relations between these and lifts already known.

Key Words: Vector, Field, Derivation, Complete lift, Semitangent bundle.

1. Semitangent bundle

Let M_n be an n -dimensional differentiable manifold of class C^∞ and $\pi : M_n \rightarrow B_m$ the differentiable bundle determined by a submersion π . Suppose that (x^a, x^α) , $a, b, \dots = 1, \dots, n-m; \alpha, \beta, \dots = n-m+1, \dots, n; i = 1, 2, \dots, n$ is a system of local coordinates adapted to the bundle $\pi : M_n \rightarrow B_m$ where x^α are coordinates in B_m , x^a are fibre coordinates of the bundle (see [1, p. 190]). If $(x^{a'}, x^{\alpha'})$ is another system of local coordinates in the bundle, then we have

$$\begin{cases} x^{a'} = x^{a'}(x^a, x^\alpha), \\ x^{\alpha'} = x^{\alpha'}(x^\alpha). \end{cases} \quad (1)$$

The Jacobian of (1) is given by the matrix

$$(A_i^{i'}) = \left(\frac{\partial x^{i'}}{\partial x^i} \right) = \begin{pmatrix} A_a^{a'} & A_\alpha^{a'} \\ 0 & A_\alpha^{\alpha'} \end{pmatrix}.$$

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Let $T_p(B_m)$ ($p = \pi(\tilde{p}), \tilde{p} = (x^a, x^\alpha) \in M_n$) be the tangent space at a point p of B_m . If $X^\alpha = dx^\alpha(X)$ are components of X in tangent space $T_p(B_m)$ with respect to the natural base $\{\partial_\alpha\}$ ($\partial_\alpha = \frac{\partial}{\partial x^\alpha}$), then we have the set of all points $(x^a, x^\alpha, x^{\bar{\alpha}})$, $x^{\bar{\alpha}} = X^\alpha, \bar{\alpha} = \alpha + m$ is by definition, the semitangent bundle $t(M_n)$ over the manifold M_n (see [1], [2]), $\dim t(M_n) = n + m$. In the special case $n = m, t(M_n)$ is a tangent bundle $T(M_n)$.

To a transformation of local coordinates of M_n (see (1)), there corresponds in $t(M_n)$ the coordinate transformation

$$\begin{cases} x^{a'} = x^{a'}(x^a, x^\alpha), \\ x^{\alpha'} = x^{\alpha'}(x^\alpha), \\ x^{\bar{\alpha}'} = \frac{\partial x^{\alpha'}}{\partial x^\alpha} x^{\bar{\alpha}}. \end{cases} \quad (2)$$

The Jacobian of (2) is given by

$$\bar{A} = \begin{pmatrix} A_a^{a'} & A_\alpha^{a'} & 0 \\ 0 & A_\alpha^{\alpha'} & 0 \\ 0 & A_{\alpha\sigma}^{\alpha'} x^{\bar{\sigma}} & A_\alpha^{\alpha'} \end{pmatrix}, \quad (3)$$

where

$$A_{\alpha\sigma}^{\alpha'} = \frac{\partial^2 x^{\alpha'}}{\partial x^\alpha \partial x^\sigma}.$$

We denote by $\mathcal{T}_q^p(M_n)$ the module over $F(M_n)$ of all tensor fields of class C^∞ and of type (p, q) in M_n , where $F(M_n)$ denotes the ring of real-valued C^∞ functions on M_n . Let $v \in \mathcal{T}_0^1(M_n)$ be a projectable vector field, that is,

$$v = (v^i) = \begin{pmatrix} v^a(x^b, x^\beta) \\ v^\alpha(x^\beta) \end{pmatrix}.$$

Taking account of (3), we easily see that ${}^c v' = \bar{A}^c v$, where

$${}^c v' = \begin{pmatrix} v^{a'} \\ v^{\alpha'} \\ x^{\bar{\sigma}'} \frac{\partial v^{\alpha'}}{\partial x^{\sigma'}} \end{pmatrix}, {}^c v = \begin{pmatrix} v^a \\ v^\alpha \\ x^{\bar{\sigma}} \frac{\partial v^\alpha}{\partial x^\sigma} \end{pmatrix}, \quad (4)$$

that is ${}^c v \in \mathcal{T}_0^1(t(M_n))$. The vector field ${}^c v$ is called the complete lift of v to the semitangent bundle $t(M_n)$ [1, p. 194].

If $f = f(x^a, x^\alpha)$ is a function in M_n , we write ${}^c f$ for the function in $t(M_n)$ defined by

$${}^c f = {}_1(df) = x^{\bar{\beta}} \partial_\beta f$$

and call of the complete lift of f to $t(M_n)$.

Lemma: Let \tilde{v} and \tilde{w} be vector fields in $t(M_n)$ such that $\tilde{v}^c f = \tilde{w}^c f$ for any $f \in \mathcal{T}_0^0(M_n)$. Then $\tilde{v} = \tilde{w}$.

Proof: It is sufficient to show that if $'\tilde{v}^c f = (\tilde{v} - \tilde{w})^c f = 0$ for any $f \in \mathcal{T}_0^0(M_n)$, then

$'\tilde{v} = 0$. If $\begin{pmatrix} '\tilde{v}^a \\ '\tilde{v}^\alpha \\ '\tilde{v}^{\bar{\alpha}} \end{pmatrix}$ are components of $'\tilde{v}$ with respect to the coordinates $(x^a, x^\alpha, x^{\bar{\alpha}})$ in

$t(M_n)$, then we have from $'\tilde{v}^c f = 0$

$$' \tilde{v}^a x^{\bar{\beta}} \partial_a \partial_\beta f + ' \tilde{v}^\alpha x^{\bar{\beta}} \partial_\alpha \partial_\beta f + ' \tilde{v}^{\bar{\alpha}} \partial_\alpha f = 0.$$

If this holds for any $f \in \mathcal{T}_0^0$, $\partial_\alpha f$, $\partial_\alpha \partial_\beta f$ and $\partial_a \partial_\beta f$ taking any preassigned values at a fixed point, we have

$$' \tilde{v}^\alpha x^{\bar{\beta}} + ' \tilde{v}^\beta x^{\bar{\alpha}} = 0, \quad ' \tilde{v}^a x^{\bar{\beta}} = 0, \quad ' \tilde{v}^{\bar{\alpha}} = 0. \tag{5}$$

Suppose $x^{\bar{\alpha}} \neq 0$ and assume that $x^{\bar{1}} \neq 0$. Then from $' \tilde{v}^a x^{\bar{\beta}} = 0$ we have $' \tilde{v}^a = 0$. Putting $\alpha = 1$ in the first equation of (5), we have $' \tilde{v}^\beta x^{\bar{1}} + ' \tilde{v}^1 x^{\bar{\beta}} = 0$, from which $' \tilde{v}^\beta = \lambda x^{\bar{\beta}}$ for a certain function $\lambda = -\frac{' \tilde{v}^1}{x^{\bar{1}}}$. Substituting this into the first equation of (5), we find $2\lambda x^{\bar{\alpha}} x^{\bar{\beta}} = 0$, from which, putting $\alpha = \beta = 1$, we have $\lambda = 0$, i.e. $' \tilde{v}^\alpha = 0$. Thus we see that the vector field $'\tilde{v}$ is zero at a point such that $y^i \neq 0$, that is, in $t(M_n) - M_n$. But the vector field $'\tilde{v}$ is continuous at every point of $t(M_n)$. So, we have $'\tilde{v} = 0$ in $t(M_n)$.

Thus we have the following.

Remark: Any element \tilde{v} of $\mathcal{T}_0^1(t(M_n))$ is completely determined by its action on functions of the type ${}^c f \in \mathcal{T}_0^0(t(M_n))$.

2. Lifts of Derivations to $t(M_n)$

Let $\mathcal{T}(M_n)$ be the direct sum $\sum_{p,q} \mathcal{T}_q^p(M_n)$. A map $\mathcal{D} : \mathcal{T}(M_n) \rightarrow \mathcal{T}(M_n)$ is a derivation on M_n if [3]

- a) \mathcal{D} is linear with respect to constant coefficients,
- b) for all p, q , $\mathcal{D}\mathcal{T}_q^p(M_n) \subset \mathcal{T}_q^p(M_n)$,
- c) for all C^∞ tensor fields T_1 and T_2 on M_n

$$\mathcal{D}(T_1 \otimes T_2) = (\mathcal{D}T_1) \otimes T_2 + T_1 \otimes \mathcal{D}T_2,$$

- d) \mathcal{D} commutes with contraction.

From definition of Derivations we obtain $\mathcal{D}I = 0$, where I denotes the identity tensor field of type (1,1) in M_n .

For a derivation \mathcal{D} in M_n , there exists a vector field P in M_n such that

$$Pf = \mathcal{D}f \tag{6}$$

for any $f \in \mathcal{T}_0^0(M_n)$. If we put

$$\mathcal{D}\frac{\partial}{\partial x^i} = Q_i^h \frac{\partial}{\partial x^h} \tag{7}$$

in each coordinate neighborhood U of M_n , we have in U

$$\mathcal{D}(dx^k) = -Q_i^k dx^i. \tag{8}$$

Let $X \in \mathcal{T}_0^1(M_n)$ and $w \in \mathcal{T}_1^0(M_n)$. From (7) and (8), we see that $\mathcal{D}X$ and $\mathcal{D}w$ have, respectively, components of the form

$$\mathcal{D}X : (P^i \partial_i X^h + Q_i^h X^i); \quad \mathcal{D}w : (P^j \partial_j w_i - Q_i^h w_h), \tag{9}$$

where P^i are components of P given by (6). The pair (P^h, Q_i^h) is called components of the Derivation \mathcal{D} in U [4, p. 26].

We define vector fields ${}^c\mathcal{D}$ in $t(M_n)$ by (see Remark in Section 1)

$${}^c\mathcal{D}^c f = \iota(\mathcal{D}df), \quad f \in \mathcal{T}_0^0(M_n) \tag{10}$$

and call ${}^c\mathcal{D}$ the complete lift of \mathcal{D} to $t(M_n)$, where ι is the operator defined by

$$\iota S = x^{\bar{\beta}} S_{\beta} \tag{11}$$

$S = S_b dx^b + S_\beta dx^{\bar{\beta}}$ being an arbitrary covector field in M_n .

If $\begin{pmatrix} {}^c\mathcal{D}^a \\ {}^c\mathcal{D}^\alpha \\ {}^c\mathcal{D}^{\bar{\alpha}} \end{pmatrix}$ are components of ${}^c\mathcal{D}$ with respect to the coordinates $(x^a, x^\alpha, x^{\bar{\alpha}})$, then

taking account of the argument similar to that used in the proof of Lemma in Section 1 we have, from (9)-(11)

$$\begin{aligned} {}^c\mathcal{D}^a x^{\bar{\beta}} \partial_a \partial_\beta f + {}^c\mathcal{D}^\alpha x^{\bar{\beta}} \partial_\alpha \partial_\beta f + {}^c\mathcal{D}^{\bar{\alpha}} \partial_\alpha f &= x^{\bar{\beta}} (P^i \partial_i \partial_\beta f - Q_\beta^i \partial_i f) \\ &= x^{\bar{\beta}} P^a \partial_a \partial_\beta f + x^{\bar{\beta}} P^\alpha \partial_\alpha \partial_\beta f - \\ &\quad - x^{\bar{\beta}} Q_\beta^a \partial_a f - x^{\bar{\beta}} Q_\beta^\alpha \partial_\alpha f \end{aligned}$$

and ${}^c\mathcal{D}^a = P^a$, ${}^c\mathcal{D}^\alpha = P^\alpha$, ${}^c\mathcal{D}^{\bar{\alpha}} = -x^{\bar{\beta}} Q_\beta^\alpha$. Thus ${}^c\mathcal{D}$ has components

$${}^c\mathcal{D} = \begin{pmatrix} P^a \\ P^\alpha \\ -x^{\bar{\beta}} Q_\beta^\alpha \end{pmatrix} \quad (12)$$

with respect to the coordinates $(x^a, x^\alpha, x^{\bar{\alpha}})$.

2.1. The Lifts of Lie Derivations

Let $v \in \mathcal{T}_0^1(M_n)$ is a projectable vector field and L_v denote Lie derivation with respect to v :

$$L_v f = v f, \quad L_v w = [v, w] : (v^i \partial_i w^j - w^i \partial_i v^j).$$

Then L_v is a derivation in M_n having components

$$L_v : (v^h, -\partial_i v^h), \quad Q_i^h = -\partial_i v^h \quad (13)$$

Using (4), (12) and (13), we have

$${}^c(L_v) = \begin{pmatrix} v^a \\ v^\alpha \\ x^{\bar{\beta}} \partial_\beta v^\alpha \end{pmatrix} = {}^c v.$$

2.2. The Lifts of covariant differentiations

Suppose now that ∇ is a projectable linear connection in M_n (see [1, p.198]). Let $v \in \mathcal{T}_0^1(M_n)$ is a projectable vector field and ∇_v denote covariant differentiation with respect to v :

$$\nabla_v f = v f, \quad \nabla_v w : (v^j \partial_j w^h + v^j \Gamma_{jm}^h w^m),$$

where Γ_{ji}^h are local components of ∇ in M_n . Then ∇_v is a derivation in M_n having components

$$\nabla_v : (v^h, v^j \Gamma_{ji}^h), \quad Q_i^h = v^j \Gamma_{ji}^h. \quad (14)$$

Using (12) and (14), we have

$${}^c(\nabla_v) = \begin{pmatrix} v^a \\ v^\alpha \\ -v^\gamma x^\beta \bar{\Gamma}_{\gamma\beta}^\alpha \end{pmatrix}. \quad (15)$$

We introduce now a projectable linear connection $\overset{v}{\nabla}$ in M_n by

$$\begin{aligned} \overset{v}{\nabla}_v w &= \nabla_v w - S(v, w) = \nabla_v w - (\nabla_v w - \nabla_w v - [v, w]) \\ &= \nabla_w v + [v, w], \end{aligned} \quad (16)$$

v and w being arbitrary projectable vector fields in M_n , where S is the torsion tensor of the given connection ∇ . Denoting by Γ_{ij}^h the components of the given connection, we have from (16)

$$\overset{v}{\Gamma}_{ji}^h = \Gamma_{ij}^h, \quad (17)$$

where $\overset{v}{\Gamma}_{ji}^h$ are components of the new connection $\overset{v}{\nabla}$.

We define vector fields $\gamma(\overset{v}{\nabla}v)$ in $t(M_n)$ by

$$\gamma(\overset{v}{\nabla}v) = (x^\beta \overset{v}{\nabla}_\beta v^\alpha) \frac{\partial}{\partial x^\alpha}, \quad (18)$$

where $\overset{v}{\nabla}_\beta v^\alpha = \partial_\beta v^\alpha + \overset{v}{\Gamma}_{\beta\gamma}^\alpha v^\gamma$. Using (3), we can easily verify that the vector field $\gamma(\overset{v}{\nabla}v)$ defined in each coordinate neighborhood in $t(M_n)$ determine global vector field in $t(M_n)$.

From (18), we see that $\gamma(\overset{v}{\nabla}v)$ have components

$$\gamma(\overset{v}{\nabla}v) = \begin{pmatrix} 0 \\ 0 \\ x^{\bar{\beta}} \overset{v}{\nabla}_\beta v^\alpha \end{pmatrix}$$

with respect to the coordinates $(x^a, x^\alpha, x^{\bar{\alpha}})$. Using (4), (15) and (17), we have

$$\begin{aligned} {}^c v - {}^c (\nabla v) &= \begin{pmatrix} v^a \\ v^\alpha \\ x^{\bar{\beta}} \partial_\beta v^\alpha \end{pmatrix} - \begin{pmatrix} v^a \\ v^\alpha \\ -v^\gamma x^{\bar{\beta}} \Gamma_{\gamma\beta}^\alpha \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ x^{\bar{\beta}} (\partial_\beta v^\alpha + \Gamma_{\gamma\beta}^\alpha v^\gamma) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ x^{\bar{\beta}} (\partial_\beta v^\alpha + \overset{v}{\Gamma}_{\beta\gamma}^\alpha v^\gamma) \end{pmatrix} \\ &= \gamma(\overset{v}{\nabla}v). \end{aligned}$$

Thus, we find the formula

$${}^c (\nabla v) = {}^c v - \gamma(\overset{v}{\nabla}v). \tag{19}$$

From (19), we see that

$${}^c (\nabla v) = {}^c v$$

if and only if

$$\overset{v}{\nabla}v = 0.$$

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