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## On Non-Homogeneous Riemann Boundary Value Problem

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### Abstract

In this paper we consider non-homogeneous Riemann boundary value problem with unbounded oscillating coefficients on a class of open rectifiable Jordan curve.

**Key Words:** Curve, Cauchy type integral, singular integral, Riemann problem.

In [1], homogeneous Riemann boundary value problem was studied when curve  $\gamma$  satisfies the condition  $\theta(\delta) \approx \delta$  and  $G$  is an oscillating function at the end points of the curve. In this work we investigate the non-homogeneous Riemann problem in the same case and we will use terminology and notations introduced in [1].

We need the following class of functions for our future references:

$$\begin{aligned} H^{a_1}(\mu_1, \nu_1) + H^{a_2}(\mu_2, \nu_2) &= \{g \in C_\gamma : g = g_1 + g_2, g_k \in C_\gamma, \Omega_{g_k}^{a_k}(\xi) \\ &= O(\xi^{-\nu_k}), \omega_{g_k}^{a_k}(\delta, \xi) = O(\delta^{\mu_k} \xi^{-\mu_k - \nu_k})\} \end{aligned}$$

where  $k = 1, 2, \mu_k \in (0, 1), \nu_k \in [0, 1), \delta, \xi \in (0, d], \delta \leq \xi, d = \text{diam } \gamma$ .

**Lemma 1.** [3] Suppose that  $\gamma$  satisfies  $\theta(\delta) \approx \delta, G(t) = \exp(2\pi i f(t)), \Omega_f^{a_k}(\xi) = O(\ln \frac{1}{\xi}), \omega_f^{a_k}(\delta, \xi) = O(\frac{\delta}{\xi}), \delta \leq \xi, k = 1, 2, g \in H^{a_1}(\mu_1, \nu_1) + H^{a_2}(\mu_2, \nu_2)$  and suppose

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$h$  is holomorphic in  $\mathbb{C} \setminus \gamma$ , continuously extendable to  $\hat{\gamma}$  from both sides,  $h(z) \neq 0$  for all  $z \in \mathbb{C} \setminus \gamma$ ,  $h^\pm(t) \neq 0$  for all  $t \in \hat{\gamma}$ ,  $h^+(t) = G(t)h^-(t)$ ,  $g/h^+ \in L(\gamma)$ . Then the function

$$\Phi_0(z) = \frac{h(t)}{2\pi i} \int_{\gamma} \frac{g(\tau)}{h^+(\tau)(\tau - z)} d\tau, z \notin \gamma \tag{1}$$

is holomorphic in  $\mathbb{C} \setminus \gamma$ , continuously extendable to  $\hat{\gamma}$  from both sides and satisfies the homogeneous boundary conditions

We introduce the quantity

$$\overline{\Delta}_G^k = \overline{\lim}_{z \rightarrow a_k} \frac{1}{\ln |z - a_k|} \operatorname{Re} \int_{\gamma} \frac{f(\tau)}{\tau - z} d\tau, z \notin \gamma \tag{2}$$

and if  $\overline{\Delta}_G^k$  is finite introduce

$$\alpha'_k = \begin{cases} \overline{\Delta}_G^k, & \text{if } \overline{\Delta}_G^k \in \mathbb{Z} \\ \lceil \overline{\Delta}_G^k \rceil + 1, & \text{if } \overline{\Delta}_G^k \notin \mathbb{Z} \end{cases} \tag{3}$$

$k=1,2$ .

**Lemma 2.** Suppose that  $\gamma$  satisfies  $\theta(\delta) \approx \delta$ ,  $G$  and  $g$  are as in lemma1 and

$$\chi_0(z) = (z - a_1)^{-\alpha'_1} (z - a_2)^{-\alpha'_2} \exp \int_{\gamma} \frac{f(\tau)}{\tau - z} d\tau, z \notin \gamma.$$

Then  $g/\chi_0^+$  is integrable on  $\gamma$ .

Proof. It is obvious that  $g/\chi_0^+$  is bounded on a compact subset of  $\gamma \setminus \{a_1, a_2\}$  and measurable. Therefore it is integrable on compact subset of  $\gamma$ . Now we estimate  $g/\chi_0^+$  on  $\gamma_\delta(a_1)$  for small enough  $\delta$ . Since  $g \in H^{a_1}(\mu_1, \nu_1) + H^{a_2}(\mu_2, \nu_2)$  we have

$$|g(t)| \leq \Omega_{g_2}^{a_1}(|t - a_1|) + \Omega_{g_2}^{a_2}(|t - a_2|) \leq$$

$$\Omega_{g_1}^{a_1}(|t - a_1|) + \Omega_{g_2}^{a_2}(|a_2 - a_1| - \delta) \leq C |t - a_1|^{-\nu_1} + C \leq C |t - a_1|^{-\nu_1}.$$

From (2) we have

$$-\operatorname{Re} \int_{\gamma} \frac{f(\tau)}{\tau - z} d\tau \leq -(\overline{\Delta}_G^1 + \varepsilon) \ln |z - a_1|,$$

where  $z \notin \gamma$  is close enough to  $a_1$ . Hence

$$\begin{aligned} \left| \frac{1}{\chi_0^+(t)} \right| &= |(t - a_1)^{\alpha_1'} (t - a_2)^{\alpha_2'}| \exp\left(-\operatorname{Re} \int_{\gamma} \frac{f(\tau)}{\tau - z} d\tau\right) \leq \\ &\leq C |t - a_1|^{\alpha_1'} \exp\left(-(\overline{\Delta}_G^1 + \varepsilon) \ln |z - a_1|\right) = C |t - a_1|^{a_1 e_1' - \overline{\Delta}_G^1 - \varepsilon}. \end{aligned}$$

Therefore

$$\left| \frac{g(t)}{\chi_0^+(t)} \right| \leq C |t - a_1|^{-\nu_1 + \alpha_1' - \overline{\Delta}_G^1 \varepsilon}.$$

For small enough  $\varepsilon$ , we have  $q = \nu_1 - \alpha_1' + \overline{\Delta}_G^1 + \varepsilon > -1$ , that is, in the vicinity of  $a_1$

$$\left| \frac{g(t)}{\chi_0^+(t)} \right| \leq C |t - a_1|^q, q > -1.$$

Analogously, similar estimation exists in the vicinity of  $a_2$ . This yields  $\frac{g(t)}{\chi_0^+(t)}$  as integrable.

**Lemma 3.** Suppose  $\gamma$  satisfies  $\theta(\delta) \approx \delta$ ,  $G$  satisfies the conditions of lemma1,  $g \in H^{a_1}(\mu_1, \nu_1) + H^{a_2}(\mu_2, \nu_2)$  and

$$\Delta_G^k = \lim_{z \rightarrow a_k} \frac{1}{\ln |z - a_k|} \operatorname{Re} \int_{\gamma} \frac{f(\tau)}{\tau - z} d\tau, z \notin \gamma \tag{4}$$

exists. Then the function in (1) is piecewise holomorphic while  $h = \chi$ .

Proof. It is obvious that according to condition (4) we may take  $\alpha_1 = \alpha_1', \alpha_2 = \alpha_2'$  and  $\chi_0 = \chi$ . Then for function (1) we only need to estimate in endpoints.

We shall investigate function (1) in the vicinity of  $a_1$  (in the other end we may show the proof analogously). Take  $2\eta = |z - a_1|, q = -\alpha_1 + \overline{\Delta}_G^1$ . Since  $q \in (-1, 0], \nu_1 \in [0, 1)$ .

Choose  $\varepsilon$  small enough such that  $q + \nu_1 + \varepsilon < 1$ . Let  $\lambda > 0$  such that for  $z \in \{\xi \in \mathbb{C} : |\xi - a_1| < \lambda\} \setminus \gamma$ ,

$$(\Delta_G^1 + \varepsilon) \ln |z - a_1| \leq \operatorname{Re} \int \frac{f(\tau)}{\tau - z} d\tau \leq (\Delta_G^1 - \varepsilon) \ln |z - a_1|. \quad (5)$$

Let  $t_z \in \{y \in \gamma : |z - y| = \operatorname{dist}(z, \gamma_\lambda(a_1) \setminus \gamma_\eta(z))\}$ . We decompose (1) as follows:

$$\begin{aligned} \frac{\Phi_0(z)}{\chi(z)} &= \frac{1}{2\pi i} \int_{\gamma} \frac{g(\tau)}{\chi^+(\tau)(\tau - z)} d\tau = \frac{1}{2\pi i} \int_{\gamma \setminus \gamma_\lambda(a_1)} \frac{g(\tau)}{\chi^+(\tau)(\tau - z)} d\tau + \\ &\frac{1}{2\pi i} \int_{\gamma_\lambda(a_1) \setminus \gamma_\eta(z)} \frac{g(\tau)}{\chi^+(\tau)(\tau - z)} d\tau + \frac{1}{2\pi i} \int_{\gamma_\eta(z)} \frac{g(\tau) - g(t_z)}{\chi^+(\tau)(\tau - z)} d\tau + \frac{g(t_z)}{2\pi i} \int_{\gamma_\eta(z)} \frac{d\tau}{\chi^+(\tau)(\tau - z)} \\ &= A_1 + A_2 + A_3 + A_4. \end{aligned}$$

It is obvious that since  $A_1$  does not depend on  $\eta$  it is bounded in the vicinity of  $a_1$ .

Let  $\tau \in \gamma_\lambda(a_1) \setminus \gamma_\eta(z)$  therefore  $|\tau - a_1| + \eta \leq |\tau - z| + 3\eta \leq |\tau - z|$ . From lemma 2 and [2] we have

$$\begin{aligned} |A_2| &\leq \frac{1}{2\pi} \int_{\gamma_\lambda(a_1) \setminus \gamma_\eta(z)} \frac{\Omega_{g_1}^{a_1}(|\tau - a_1|) + \Omega_{g_1}^{a_2}(|a_2 - a_1| - \lambda)}{|\chi^+(\tau)| (|\tau - a_1| + \eta)} |d\tau| \\ &\leq C \int_{\gamma_\lambda(a_1)} \frac{|\tau - a_1|^{-\nu_1} d\tau}{|\tau - a_1|^{q+\varepsilon} (|\tau - a_1| + \eta)} d\tau \leq C \int_0^\lambda \frac{x^{-\nu_1 - q - \varepsilon}}{x + \eta} d\theta(x) \\ &\leq C \int_0^\lambda \frac{x^{-\nu_1 - q - \varepsilon}}{x + \eta} dx \leq C \int_0^\lambda \frac{x^{-\nu_1 - q - \varepsilon}}{\eta + \varepsilon} dx \\ &\leq C \left( \frac{1}{\eta} \int_0^\lambda x^{-\nu_1 - q - \varepsilon} dx + \int_\eta^\lambda x^{-\nu_1 - q - \varepsilon - 1} dx \right) \leq C \eta^{-\nu_1 - q - \varepsilon}. \end{aligned}$$

If  $\gamma_\eta(z) = \emptyset$ , then  $A_3 = 0$ . Otherwise, since  $|z - t_z| \leq \eta$  and  $\gamma_\eta(z) \subset \gamma_{2\eta}(t_z)$  we get

$$\begin{aligned}
 |A_3| &\leq \frac{1}{2\pi} \int_{\gamma_\eta(z)} \frac{\omega_{g_1}^{\alpha_1}(|\tau - t_z|, \eta/2) + \omega_{g_2}^{\alpha_2}(|\tau - t_z|, |a_2 - a_1| - \lambda)}{|\chi^+(\tau)| |\tau - t_z|} |d\tau| \\
 &\leq \frac{1}{2\pi} \int_{\gamma_\eta(z)} \frac{|\tau - t_z|^{\mu_1} \eta^{-\mu_1 - \nu_1} + |\tau - t_z|^{\mu_2}}{\eta^{q+\varepsilon} |\tau - t_z|} |d\tau| \\
 &\leq C \eta^{-\eta-\varepsilon} (\eta^{-\mu_1 - \nu_1} \int_{\gamma_{2\eta}(t_z)} |\tau - t_z|^{\mu_1 - 1} |d\tau| \\
 &\quad + \int_{\gamma_{2\eta}(t_z)} |\tau - t_z|^{\mu_2 - 1} |d\tau|) \leq C \eta^{-\eta-\varepsilon} (\eta^{-\mu_1 - \nu_1} \int_0^{2\eta} \tau^{\mu_1 - 1} d\theta(\tau) \\
 &\quad + \int_0^{2\eta} \tau^{\mu_2 - 1} d\theta(\tau)) \leq C \eta^{-\eta-\varepsilon} (\eta^{-\mu_1 - \nu_1} \int_0^{2\eta} \tau^{\mu_1 - 1} d\tau \\
 &\quad + \int_0^{2\eta} \tau^{\mu_2 - 1} d\tau) \leq C \eta^{-q-\varepsilon} (C \eta^{-\nu_1} + C \eta^{\mu_2}) \leq C \eta^{-q-\varepsilon-\nu_1}.
 \end{aligned}$$

Now we investigate  $A_4$ . If  $\gamma_\eta(z) = \emptyset$ , then  $A_4 = 0$ . Otherwise since  $|z - t_z| \leq \eta$  and  $|\tau_z - a_1| \geq \eta$  then

$$\begin{aligned}
 |g(t_z)| &\leq \Omega_{g_1}^{\alpha_1}(|t_z - a_1|) + \Omega_{g_2}^{\alpha_2}(|t_z - a_2|) \leq \Omega_{g_1}^{\alpha_1}(|t_z - a_1|) \\
 &\quad + \Omega_{g_2}^{\alpha_2}(|a_2 - a_1| - \delta) \leq C |t_z - a_1|^{-\nu_1} + C \leq C \eta^{-\nu_1}.
 \end{aligned}$$

Suppose that  $a_2 \notin \gamma_\eta(z)$  and  $\gamma_\eta(z) = \Lambda \cup (\bigcup_{k=1}^p \widetilde{c_k d_k})$ ,  $1 \leq p \leq \infty$ ,  $c_k, d_k \in \sum_\eta(z) = \{\xi \in \mathbb{C} : |\xi - z| = \eta\}$ . Arcs  $\widetilde{c_k d_k}$  are connected components of  $\gamma_\eta(z)$  and  $\Lambda \subset \sum_\eta(z)$ . The number of  $\widetilde{c_k d_k}$ 's may not be more than countable since arbitrary partition of interval  $[0, d]$ ,  $d = \text{diam } \gamma$ , is countable.

The points  $c_k, d_k, c_k \neq d_k$  divide  $\sum_\eta(z)$  into two arcs with endpoints  $c_k, d_k$ . Denote one of them by  $\Lambda_k$  oriented from  $c_k$  to  $d_k$ . Let  $D$  be the domain bounded by  $\Lambda_k \cup \widetilde{c_k d_k}$

and  $z \notin D$ . If  $\text{meas } \Lambda_k \leq \pi\eta$  then  $\text{meas } \Lambda_k \leq |c_k - d_k| \pi/2 \leq (\text{meas } \widetilde{c_k d_k})\pi/2$ . If  $\text{meas } \Lambda_k > \pi\eta$  then  $\text{meas } \widetilde{c_k d_k} \geq 2\eta \geq \text{meas } \Lambda_k/\pi$ . Therefore  $\text{meas } \Lambda_k \leq C(\text{meas } \widetilde{c_k d_k})$ ,  $C = \max\{\pi, 2/\pi\} = \pi$ . Meanwhile if  $\tau \in \sum_{\eta}(z)$  we have  $|\tau - z| = \eta$ . By means of Cauchy theorem we get

$$\begin{aligned} \left| \int_{\gamma_{\eta}(z)} \frac{d\tau}{\tau - z} \right| &= \left| \left( \int_{\Lambda} + \sum_{k=1}^p \int_{\widetilde{c_k d_k}} \right) \frac{d\tau}{\tau - z} \right| \leq \left| \left( \int_{\Lambda} - \sum_{k=1}^p \int_{\Lambda_k} \right) \frac{d\tau}{\tau - z} \right| \\ &\leq \frac{1}{\eta} (\text{meas } \Lambda + \sum_{k=1}^p \text{meas } \Lambda_k) \\ &\leq \frac{\pi}{\eta} (\text{meas } \Lambda + \sum_{k=1}^p \text{meas } \widetilde{c_k d_k}) = \frac{\pi}{\eta} (\text{meas } \Lambda + \text{meas } \bigcup_{k=1}^p \widetilde{c_k d_k}) = \frac{\pi}{\eta} \text{meas } \gamma_{\eta}(z) \\ &\leq \frac{\pi}{\eta} \text{meas } \gamma_{2\eta}(t_z) = \frac{\pi}{\eta} \theta_{t_z}(2\eta) \leq \frac{\pi}{\eta} \theta(2\eta) \leq \frac{\pi}{\eta} C\eta = \pi C. \end{aligned}$$

Therefore  $|A_4| \leq \frac{C\eta^{-\nu_1}}{2\pi} \eta^{-q-\varepsilon} C\pi = C\eta^{-q-\nu_1-\varepsilon}$ . If we round up the result obtained we have  $|\Phi_0(z)| = \left| \frac{\chi(z)}{2\pi i} \int_{\gamma} \frac{g(\tau)}{\chi^+(\tau)(\tau-z)} d\tau \right| \leq C\eta^{-q-\nu_1-\varepsilon} |\chi(z)|$ . From (5)  $|\chi(z)| \leq C\eta^{q-\varepsilon}$ , therefore  $|\Phi_0(z)| \leq C\eta^{-q-\nu_1-\varepsilon} |\chi(z)| \leq C\eta^{-\nu_1-2\varepsilon}$ .

Since  $\varepsilon$  is small enough we may assume that  $\nu_1 + 2\varepsilon < 1$ . This proves the lemma.

In [1] the solution of the homogeneous boundary value problem was given as  $\chi(z)P_{\varkappa-1}(z)$  where  $P_{\varkappa-1}(z)$  is a polynomial whose degree is not greater than  $\varkappa - 1$ . If  $\varkappa = 0$   $P_{\varkappa-1}(z) \equiv 0$ . For  $\varkappa < 0$  the coefficients of  $z^{-1}, z^{-2}, \dots, z^{-\varkappa}$  in the expansion

$$\frac{1}{2\pi i} \int_{\gamma} \frac{g(\tau)}{\chi^+(\tau)(\tau-z)} d\tau = -\frac{z^{-1}}{2\pi i} \int_{\gamma} \frac{g(\tau)}{\chi^+(\tau)} d\tau - \frac{z^{-2}}{2\pi i} \int_{\gamma} \frac{g(\tau)}{\chi^+(\tau)} \tau d\tau - \frac{z^{-3}}{2\pi i} \int_{\gamma} \frac{g(\tau)}{\chi^+(\tau)} \tau^2 d\tau - \dots$$

must be zero. Thus the following theorem is obtained.

**Theorem.** Suppose the conditions of lemma1 are satisfied and limit in (4) exists.

i) If  $\varkappa \geq 0$ , the Riemann boundary value problem is solvable in  $K(\gamma)$  unconditionally, the solution is given by

$$\Phi(z) = \frac{\chi(z)}{2\pi i} \int_{\gamma} \frac{g(t)}{\chi^+(t)(t-z)} dt + \chi(z) P_{\varkappa-1}(z),$$

where  $P_{\varkappa-1}(z)$  is arbitrary polynomial of degree not greater than  $\varkappa - 1$  ( $P_{\varkappa-1}(z) \equiv 0$  for  $\varkappa = 0$ ). ii) If  $\varkappa < 0$ , then the Riemann boundary value problem is solvable in  $K(\gamma)$  if and only if the conditions

$$\int_{\gamma} \frac{g(t)}{\chi^+(t)} t^j dt = 0, j = 0, 1, \dots, -\varkappa - 1$$

are satisfied. Under these conditions the solution is unique and is given by

$$\Phi(z) = \frac{\chi(z)}{2\pi i} \int_{\gamma} \frac{g(t)}{\chi^+(t)(t-z)} dt.$$

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