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The k-Derivation of a Gamma-Ring

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Abstract

In this paper, the k-derivation is defined on a Γ-ring M (that is, if M is a Γ-ring, $d : M \to M$ and $k : \Gamma \to \Gamma$ are to additive maps such that $d(a\beta b) = d(a)\beta b + ak(\beta)b + a\beta d(b)$ for all $a, b \in M$, $\beta \in \Gamma$, then d is called a k-derivation of M) and the following results are proved. (1) Let R be a ring of characteristic not equal to 2 such that if $xry = 0$ for all $x, y \in R$ then $r = 0$. If d is a k-derivation of the $(R =)\Gamma$-ring R with $k = d$, then d is the ordinary derivation of R. (2) Let M be a nonzero prime Γ-ring of characteristic not equal to 2, γ be an element of *Gamma* and a is an element in M such that $[[x, a]_\gamma, a]_\gamma = 0$ for all $x \in M$. Then $a\gamma a = 0$ or $a \in C_\gamma$. (3) Let M be a prime Γ-ring with Char$M \neq 2$, d be a nonzero k-derivation of M, γ be a nonzero element of Γ and $k(\gamma) \neq 0$. If $d(M) \subseteq C_\gamma$, then M is a commutative Γ-ring.

Key Words: k-derivation, derivation, commutativity, gamma-ring

1. Preliminaries

Let M be additive abelian groups. If there exits a mapping of $M \times \Gamma \times M$ to M (the image of (a, γ, b), $a, b \in M$, $\gamma \in \Gamma$, being denoted by $(a\gamma b)$), satisfying for all $a, b, c \in M$, $\alpha, \beta \in \Gamma$:

B1. $(a + b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha + \beta)b = a\alpha b + a\beta b$, $a\alpha(b + c) = a\alpha b + a\alpha c$

B2. $(a\alpha b)\beta c = a\alpha(b\beta c)$,

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then M is called a Γ-ring in the sense of Barnes [1]. This definition is due to Barnes, and is slightly weaker than the original one due to Nobusava [7].

If, in addition, there exits a mapping of $\Gamma \times M \times \Gamma$ to Γ (the image of (γ, a, β), being denoted by $\gamma a \beta$) such that the following axioms are satisfied for all $a, b, c \in M, \quad \alpha, \beta \in \Gamma$:

N1. Same as B1

N2. $(a\alpha b)\beta c = a(\alpha b \beta)c = a\alpha(b\beta c)$

N3. $a\alpha b = 0$ for all $a, b \in M$ implies $\alpha = 0$,

then M is called a Γ-ring in the sense of Nobusawa.

Let M be a Γ-ring in the sense a of Barnes. A subgroup A of the additive group M is said to be a right (resp. left) ideal of Γ-ring M if $a\gamma y$ (resp. $y\gamma a$) for all $a \in A$, $\gamma \in \Gamma$, $y \in M$. If A is both a left and a right ideal, then A is said to be a two-sided ideal or simply an ideal of M. When S and T are subsets of M, and Ω is a subset of Γ, we denote by $S\Omega T$ the set of all finite sums of the form $\sum s_i \gamma_i t_i$ where $s_i \in S, \quad \gamma_i \in \Omega \quad and \quad t_i \in T$. If $\Omega = \{\gamma\}$, then $S\Omega T$ is denoted by $S\gamma T$ end so on [4]. If I and J are a left ideal and a right ideal of M, respectively, then $I\Omega J$ is an ideal of M. Similar properties hold depending on ideal properties of I and J. If $a\Gamma M \Gamma b = 0$ with $a, b \in M$ implies either a=0 or b=0, then M is called a prime Γ-ring [5]. Moreover, a Γ-ring M is said to be completely prime $a\Gamma b = 0$ with $a, b \in M$ implies $a = 0$ or $b = 0$ [6]. We also note that, for a Γ-ring in the sense Nobusawa, primeness and completely primeness are equivalent. $C_\Gamma = \{c \in M : c\alpha m = m\alpha c \quad \forall \alpha \in \Gamma \quad and \quad \forall m \in M\}$ and $C_\alpha = \{c \in M : c\alpha m = m\alpha c \quad \forall m \in M\}$ with $\alpha \in \Gamma$ are called the center and the α-center of a Γ-ring M, respectively. If $C_\Gamma = M$ then M is called a commutative Γ-ring. If M is a Γ-ring in the sense of Nobusawa, the center C_M and the the a-center C_a of a M-ring Γ are similarly defined.

As it is well known, if R is a semiprime 2-torsion-free ring and $t \in R$ commutes with all $tx - xt$ for $x \in R$ then $t \in Z$ (the center of R) [2]. This corollary is used in the proofs of many theorems on commutativity of rings. In this paper, we shall consieder a similar problem on the Γ-ring M. That is, let M be a nonzero prime Γ-ring of characteristic not equal to 2, γ be a nonzero element of Γ, a is an element in M such that $a\gamma a \neq 0$ and a commutes with all $x\gamma a - a\gamma x$ for $x \in M$, then a must be in C_γ.

2. k-Derivation of Γ-Ring

Let M be a Γ-ring (in the sense of Barnes), d and k two additive maps from M to M and from Γ to Γ, respectively. If for all $a, b \in M$ and $\beta \in \Gamma$, $d(a\beta b) = d(a)\beta b + ak(\beta)b + a\beta d(b)$ is satisfied, then d is called a k-derivation of M.

Every associative ring R is a Γ-ring where $R = \Gamma$ in the sense of Barnes. Let be a derivation of a ring R, that is, d is an additive map from R to R and $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. It is clear that d is a k-derivation of the $(R =)\Gamma$-ring R with $d = k$.

Remark : If M is a Γ-ring in the sense of Barnes and d is a k-derivation of the Γ-ring M, k need not be determined uniquely. But if M is a Γ-ring in the sense of Nobusawa and d is a k-derivation of the Γ-ring M, then k is uniquely determined. Particularly, if a ring R satisfies N3 (or R is semiprime or R has unity or R has no nonzero zero divisor), then R is a $(R =)\Gamma$-ring in the sense of Nobusawa. In this case, if d is a k- derivation of the $(R =)\Gamma$-ring R with characteristic not equal to 2, then d is the ordinary derivation of this ring R if and only if $d = k$ (This proves in Theorem 1).

Lemma 1: Let M be a Γ-ring in the sense of Nobusawa. If d is a k-derivation of the Γ-ring M, then $k(\alpha a\beta) = k(\alpha)a\beta + \alpha d(a)\beta + \alpha ak(\beta)$ for all $a \in M$ and $\alpha, \beta \in \Gamma$.

Proof: It is clear by using N3.

Lemma 2: Let M be a Γ-ring in the sense of Nobusawa. If d is both a k_1- and k_2-derivation of the Γ-ring M, then $k_1 = k_2$.

Proof: Using the definition, $k_1 = k_2$ is obtained by N3.

Theorem 1: Let R be a ring of characteristic not equal to 2 satisfying N3, d be a k-derivation of the $(R =)\Gamma$-ring R. d is the ordinary derivation of the ring R if and only if $d = k$.

Proof: Let R be a ring of characteristic not equal to 2 satisfying N3 and d be a k-derivation of the $(R =)\Gamma$-ring R. If d is the ordinary derivation of R, then it is clear that $d = k$. Now we prove the converse. Let d be a k- derivation of the $(R =)\Gamma$-ring R with $k = d$. Since d is an additive map from R to R, we need only to show that $d(xy) = d(x)y + xd(y)$

for all $x, y \in R$. By hypothesis, we have $d(xyt) = d(x)yt + xd(y)t + xyd(t)$ for all $x, y, t \in R$. Replace y by yz and t by tn in the equation where $z, n \in R$, and using $d(x(yz)(tn)) = d(xy(ztn))$, we get $xd(yz)tn + xyzd(tn) = xd(y)ztn + xyd(z)tn + xyzd(t)n + xyztd(n)$. This gives $x(d(yz)tn + yzd(tn) - d(y)ztn - yd(z)tn - yzd(t)n - yztd(n))m = 0$ for all $m \in R$. Using N3, we have

$$d(yz)tn + yzd(tn) - d(y)ztn - yd(z)tn - yzd(t)n - yztd(n) = 0 \quad \forall y, z, t, n \in R.$$

Moreover, since $d((yz)tn) = d(yz(tn))$, using the definition of k-derivation we have

$$d(yz)tn - yzd(tn) - d(y)ztn - yd(z)tn + yzd(t)n + yztd(n) = 0 \quad \forall y, z, t, n \in R.$$

Adding up the last two equations, using Char$R \neq 2$ we have

$$d(yz)tn - d(y)ztn - yd(z)tn = 0.$$

This implies $s(d(yz)t - d(y)zt - yd(z)t)n = 0$, for all $s \in R$. Using N3, we have $d(yz)t - d(y)zt - yd(z)t = 0$. In the same way, we get,

$$d(yz) - d(y)z - yd(z) = 0, \quad \forall y, z \in R.$$

Hence, the theorem is proved.

From now on, (except where stated otherwise) M will be a Γ-ring in the sense of Nobusawa. For $a, b \in M$ and $\alpha, \beta \in \Gamma$, $[a, b]_\alpha$ and $[\alpha, \beta]_b$ will be denoted $a\alpha b - b\alpha a$ and $\alpha b\beta - \beta b\alpha$ respectively.

Lemma 3: Let M be a Γ-ring and d be a k-derivation of M. Then the following equalities are satisfied for $a, b, c, x \in M$ and $\alpha, \beta, \gamma \in \Gamma$:

i. $[a, b]_\beta = -[b, a]_\beta$, $[\alpha, \beta]_a = -[\beta, \alpha]_a$

ii. $[a + b, c]_\beta = [a, c]_\beta + [b, c]_\beta$, $[\alpha + \beta, \gamma]_a = [\alpha, \gamma]_a + [\beta, \gamma]_a$

iii. $[a\alpha b, x]_\beta = [a, x]_\beta \alpha b + a[\alpha, \beta]_x b + a\alpha[b, x]_\beta$

iv. $[\alpha b\beta, \gamma]_a = [\alpha, \gamma]_a b\beta + \alpha[b, a]_\gamma \beta + \alpha b[\beta, \gamma]_a$

v. $[[\alpha,\beta]_a,\gamma]_a+[[\gamma,\alpha]_a,\beta]_a+[[\beta,\gamma]_a,\alpha]_a=0$

vi. $[[a,b]_\beta,c]_\beta+[[c,a]_\beta,b]_\beta+[[b,c]_\beta,a]_\beta=0$

vii. $d([a,b]_\beta)=[d(a),b]_\beta+[a,b]_{k(\beta)}+[a,d(b)]_\beta$

viii. $k([\alpha,\beta]_a)=[k(\alpha),\beta]_a+[\alpha,\beta]_{d(a)}+[\alpha,k(\beta)]_a$.

Proof: Obvious.

Lemma 4: Let M be a prime Γ-ring, U,Ω be nonzero ideals of M and Γ, respectively. Then the following statements are satisfied for $a,b\in M$ and $\alpha,\beta\in\Gamma$:

i. $a\Omega b=0\Rightarrow a=0 \quad or \quad b=0$

ii. $\alpha U\beta=0\Rightarrow\alpha=0 \quad or \quad \beta=0$

iii. $a\Gamma U\Gamma b=0\Rightarrow a=0 \quad or \quad b=0$

iv. $\alpha M\Omega M\beta=0\Rightarrow\alpha=0 \quad or \quad \beta=0$

v. If $u\alpha v=0$ for all $u,v\in U$ then $\alpha=0$

vi. $C_\Gamma=0\Leftrightarrow C_M=0$

vii. Either $C_\Gamma\neq 0$ or $C_M\neq 0\Rightarrow M$ is a commutative Γ-ring.

viii. $U\subseteq C_\gamma$, for $0\neq\gamma\in\Gamma\Rightarrow M$ is a commutative Γ-ring.

ix. $0\neq\gamma\in\Gamma$ and for all $u,v\in U \quad [u,v]_\gamma=0\Rightarrow M$ is a commutative Γ-ring [3].

Proof: The clarity of **ii, iii, iv, v, viii** is evident. Now we prove **i, vi** and **vii**.

i: Let $a\Omega b=0$. So $a\Gamma M\Omega M\Gamma b\subseteq a\Omega b=0$. By primeness of M $a=0$ or $b=0$, since $M\Omega M\neq 0$:

iv: Let $C_M=0$. Suppose that $C_\Gamma\neq 0$. Then, there exists a nonzero element a of C_Γ. So, $a\gamma x-x\gamma a=0$ for all $\gamma\in\Gamma$ and $x\in M$. By this equation, replace γ by $\gamma y\delta$ where $y\in M$ and $\delta\in\Gamma$, using $a\in C_\Gamma$ we obtain

$0=a\gamma y\delta x-x\gamma y\delta a= a\gamma y\delta x-x\gamma a\delta y =a\gamma y\delta x-a\gamma x\delta y=a\gamma(y\delta x-x\delta y)$

That is, $a\gamma(y\delta x - x\delta y) = 0$ for all $\gamma, \delta \in \Gamma$ and $x, y \in M$. So $a\Gamma(y\delta x - x\delta y) = 0$. By primeness of M, we get $x\delta y - y\delta x = 0$ for all $x, y \in M, \quad \delta \in \Gamma$. This implies that $\delta \in C_M$ for all $\delta \in \Gamma$. This contradicts by $C_M = 0$.

vii: Suppose that $C_\Gamma \neq 0$. There should be a nonzero element a of C_Γ. That is, $x\gamma a = a\gamma x$ for all $\gamma \in \Gamma, \quad x \in M$. We obtain

$a\delta(x\gamma y - y\gamma x) = a\delta x\gamma y - a\delta y\gamma x = y(\delta x\gamma)a - a\delta(y\gamma x) = y\delta(x\gamma a) - a\delta(y\gamma x) = y\delta(a\gamma x) - a\delta(y\gamma x) = (y\delta a)\gamma x - a\delta(y\gamma x) = (a\delta y)\gamma x - (a\delta y)\gamma x = 0$. Hence $a\Gamma(x\gamma y - y\gamma x) = 0$ for all $x, y \in M, \quad \gamma \in \Gamma$. By primeness of M, we have $x\gamma y - y\gamma x = 0$ for all $x, y \in M, \quad \gamma \in \Gamma$. So, M is a commutative Γ-ring.

Theorem 2: Let M be a nonzero prime Γ-ring of characteristic not equal to 2 and γ be an element of Γ. If there exists $a \in M$ such that $[[x, a]_\gamma, a]_\gamma = 0$ for all $x \in M$, then $a\gamma a = 0$ or $a \in C_\gamma$.

Proof : We suppose $\gamma \neq 0$ (otherwise $a\gamma a = 0$). By the hypothesis, we have $[[x\beta y, a]_\gamma, a]_\gamma = 0$ for all $x, y \in M$ and $\beta \in \Gamma$. Using Lemma 3 (iii) and hypothesis, we get

$$2[x, a]_\gamma[\beta, \gamma]_a y + 2[x, a]_\gamma \beta[y, a]_\gamma + 2x[\beta, \gamma]_a[y, a]_\gamma + x[[\beta, \gamma]_a, \gamma]_a y = 0. \tag{2.1}$$

Replace x and y by $[x, a]_\gamma$ and $[y, a]_\gamma$, respectively, then we have

$$[x, a]_\gamma[[\beta, \gamma]_a, \gamma]_a[y, a]_\gamma = 0, \qquad \forall x, y \in M \quad \forall \beta \in \Gamma. \tag{2.2}$$

On the other hand, Lemma 3 (iv) implies

$$[\beta[z, a]_\gamma\delta, \gamma]_a = [\beta, \gamma]_a[z, a]_\gamma\delta + \beta[z, a]_\gamma[\delta, \gamma]_a, \quad \forall z \in M \quad \forall \beta, \delta \in \Gamma. \tag{2.3}$$

In (2.2) replacing β by $\beta[z, a]_\gamma\delta$ where $z \in M$, $\delta \in \Gamma$, using (2.2) (2.3) and considering $Char M \neq 2$, we obtain

$$[x, a]_\gamma[\beta, \gamma]_a[z, a]_\gamma[\delta, \gamma]_a[y, a]_\gamma = 0 \quad \forall x, y, z \in M \quad \forall \beta, \delta \in \Gamma. \tag{2.4}$$

In (2.4), replace β by $\beta[m, a]_\gamma\sigma$ where $m \in M$, $\sigma \in \Gamma$ and use (2.3) and (2.4), we have

$$[x, a]_\gamma[\beta, \gamma]_a[m, a]_\gamma\sigma[z, a]_\gamma[\delta, \gamma]_a[y, a]_\gamma = 0.$$

That is,

$$[x,a]_\gamma[\beta,\gamma]_a[m,a]_\gamma\Gamma[z,a]_\gamma[\delta,\gamma]_a[y,a]_\gamma = 0.$$

Since M is prime Γ-ring, we get

$$[x,a]_\gamma[\beta,\gamma]_a[m,a]_\gamma = 0 \quad \forall x,m \in M \quad \forall \beta \in \Gamma. \tag{2.5}$$

Replacing x by $[x,a]_\gamma$ in (2.1), and using (2.5) and the hypothesis, we have

$$[x,a]_\gamma[[\beta,\gamma]_a,\gamma]_a y = 0 \quad \forall x,y \in M \quad \forall \beta \in \Gamma. \tag{2.6}$$

In the same way, if we replace y by $[y,a]_\gamma$ in (2.1) we obtain

$$x[[\beta,\gamma]_a,\gamma]_a[y,a]_\gamma = 0 \quad \forall x,y \in M \quad \forall \beta \in \Gamma. \tag{2.7}$$

In (2.6), replace y and β by $[y,a]_\gamma$ and $\beta z\delta$ where $z \in M, \quad \delta \in \Gamma$, respectively, and use (2.5), (2.6), (2.7) and $CharM \neq 2$, we get

$$[x,a]_\gamma[\beta,\gamma]_a z[\delta,\gamma]_a[y,a]_\gamma = 0 \quad \forall x,y,z \in M \quad \forall \beta,\delta \in \Gamma.$$

In the last statement, replace z by $z\sigma n$ with $n \in M$, $\sigma \in \Gamma$, we have

$$[x,a]_\gamma[\beta,\gamma]_a z = 0 \quad \mathit{or} \quad n[\delta,\gamma]_a[y,a]_\gamma = 0 \quad \forall x,y,z,n \in M \quad \forall \beta,\delta \in \Gamma.$$

Suppose that $[x,a]_\gamma[\beta,\gamma]_a z = 0$ for all $x,z \in M, \quad \beta \in \Gamma$. Replace β by $\beta y[\delta,\gamma]_a$ where $y \in M, \quad \delta \in \Gamma$ and using $[x,a]_\gamma[\beta,\gamma]_a z = 0$ we get

$$[x,a]_\gamma\beta y[[\delta,\gamma]_a,\gamma]_a z = 0 \quad \forall \beta \in \Gamma.$$

That is, $[x,a]_\gamma\Gamma y[[\delta,\gamma]_a,\gamma]_a z = 0$. So $[x,a]_\gamma = 0$ or $y[[\delta,\gamma]_a,\gamma]_a z = 0$, $\forall x,y,z \in M$, $\forall \delta \in \Gamma$. If $[x,a]_\gamma = 0$ for all $x \in M$ then $a \in C_\gamma$. Now, suppose that $y[[\delta,\gamma]_a,\gamma]_a z = 0$ $\forall y,z \in M, \quad \forall \delta \in \Gamma$. By (N3), we have

$$[[\delta,\gamma]_a,\gamma]_a = 0 \quad \forall \delta \in \Gamma. \tag{2.8}$$

Since $CharM \neq 2$, by the assumption and (2.8), equation (2.1) implies

$$[x,a]_\gamma\beta[y,a]_\gamma + x[\beta,\gamma]_a[y,a]_\gamma = 0 \quad \forall x,y \in M, \quad \forall \beta \in \Gamma. \tag{2.9}$$

Replace x by $x\delta z$ with $z \in M, \quad \delta \in \Gamma$ and use (2.9), then

$$([x,a]_\gamma \delta z + x[\delta,\gamma]_a z)\beta[y,a]_\gamma = 0 \quad \forall \beta \in \Gamma.$$

This implies either $a \in C_\gamma$ or $([x,a]_\gamma \delta z + x[\delta,\gamma]_a z) = 0$ $\forall x,y,z \in M, \quad \forall \delta \in \Gamma$. Hence, we have $([x,a]_\gamma \delta z + x[\delta,\gamma]_a z) = 0$ $\forall x,y,z \in M, \quad \forall \delta \in \Gamma$. In view of (2.9) the equation in Lemma 3 (iii) reduces to

$$[x\delta z, a]_\gamma = x\delta[z,a]_\gamma \quad \forall x,z \in M, \quad \forall \delta \in \Gamma. \tag{2.10}$$

Now, by Lemma 3 (ii) and (2.10) we get

$$0 = [[x,a]_\gamma, a]_\gamma] = [x\gamma a - a\gamma x, a]_\gamma = -a\gamma[x,a]_\gamma.$$

From the last equality we obtain

$$a\gamma x\gamma a = a\gamma a\gamma x \quad \forall x \in M. \tag{2.11}$$

Moreover, by hypothesis we have $a\gamma[x,a]_\gamma = [x,a]_\gamma\gamma a$. By (2.11), the left side of this equation is zero. Hence

$$a\gamma x\gamma a = x\gamma a\gamma a \quad \forall x \in M \tag{2.12}$$

is obtained. By (2.11) and (2.12), we get $x\gamma a\gamma a = a\gamma a\gamma x \quad \forall x \in M$. That is, $a\gamma a \in C_\gamma$. On the other hand, using (2.10), we get $a\beta a\gamma a - a\gamma a\beta a = [a\beta a, a]_\gamma = a\beta[a,a]_\gamma = 0$ for all $\beta \in \Gamma$, and so $a\beta a\gamma a = a\gamma a\beta a \quad \forall \beta \in \Gamma$. Finally, using this equality and $a\gamma a \in C_\gamma$, we obtain $a\gamma a\beta[x,a]_\gamma = 0$ for all $x \in M \quad \beta \in \Gamma$, that is, $a\gamma a\Gamma[x,a]_\gamma = 0$ for all $x \in M$. Consequently, either $a\gamma a = 0$ or $a \in C_\gamma$.

One can prove the case of $n[\delta,\gamma]_a[y,a]_\gamma = 0$ for all $y,n \in M, \quad \delta \in \Gamma$ similarly.

Remark: Let a and γ be nonzero elements of M and Γ, respectively. Then $d: M \to M$ defined by $d(x) = [a,x]_\gamma$ and $k: \Gamma \to \Gamma$ defined by $k(\beta) = [\gamma,\beta]_a$ are two additive maps. Moreover d is a k-derivation of M. We call d an inner k-derivation of M as an inner derivation of an associative ring.

Lemma 5: Let M be a prime Γ-ring, γ and a be nonzero elements of Γ and C_γ, respectively. For each $x,y \in M$ and $\beta \in \Gamma$, the following conditions are satisfied.

i. $[\gamma, \beta]_a = 0$

ii. $[a\gamma x, y]_\beta = a\gamma[x, y]_\beta$ and $[x\gamma a, y]_\beta = [x, y]_\beta\gamma a$

iii. $[a\beta x, y]_\gamma = [a, y]_\beta\gamma x + a\beta[x, y]_\gamma$

iv. If $b \in C_\gamma$ then $[a\gamma b, x]_\beta = [a\beta b, x]_\gamma = a\gamma[b, x]_\beta = a[\beta, \gamma]_x b$

v. If $b \in C_\gamma$ and if $a\Gamma b \subseteq C_\gamma$ then $b = 0$ or M is commutative Γ-ring.

Proof: (i) - (iv) obvious. (v) If $a\Gamma b = 0$ then $b = 0$. Otherwise $a\Gamma b\Gamma M$ is a nonzero ideal of M contained in C_γ. By Lemma 4 (viii) the proof is completed.

Lemma 6: Let M be a prime Γ-ring, U be a nonzero left (right) ideal of the Γ-ring M and Ω be a nonzero left (right) of the M-ring Γ. The following statements are satisfied for each $a \in M$ and $\gamma \in \Gamma$:

i. $\gamma U\Gamma = 0 \Rightarrow \gamma = 0 \quad (\Gamma U\gamma = 0 \Rightarrow \gamma = 0)$

ii. $a\Omega M = 0 \Rightarrow a = 0 \quad (M\Omega a = 0 \Rightarrow a = 0)$.

Proof: Obvious.

Lemma 7: Let M be a prime Γ-ring, U be a nonzero left (right) ideal of the Γ-ring M and γ be a nonzero element of Γ. If $U \subseteq C_\gamma$ then M is commutative.

Proof: By hypothesis, $u\gamma x = x\gamma u \in U$ for $x \in M, \quad u \in U$. Hence $M\gamma U \subseteq U$. If $M\gamma U = 0$ then $\Gamma M\gamma U\Gamma = 0$. It is clear that if M is a prime Γ-ring, then Γ is prime M-ring. So, $\Gamma = 0$ or $\gamma U\Gamma = 0$. By Lemma 6 (i), $\gamma = 0$. This is a contradiction. Consequently, $M\gamma U \neq 0$. Moreover, since $u\gamma x \in U \subset C_\gamma$, by Lemma 5 (i) and (ii) we have for every $m, x, y \in M$, $u \in U$, $\beta \in \Gamma$ $m\gamma u\beta[x, y]_\gamma = m\beta u\gamma[x, y]_\gamma = m\beta[u\gamma x, y]_\gamma = 0$. That is, $M\gamma U\Gamma[x, y]_\gamma = 0$. The primeness of M implies that $[x, y]_\gamma = 0$ for all $x, y \in M$. By Lemma 4 (ix), M is a commutative Γ-ring.

The proof is similar if U is a right ideal of M.

Lemma 8: Let M be a prime Γ-ring, d be a nonzero k-derivation of M, γ be a nonzero element of Γ and $d(M)$ is contained in C_γ. If $a \in C_\gamma$, then $a \in C_{k(\gamma)}$.

Proof: It is clear by using Lemma 3 (vii).

Lemma 9: Let M, d and γ be as in Lemma 8. If $d(x)\gamma d(y) = 0$ for all $x, y \in M$, then $d(M)$ is a left or right ideal of M.

Proof: Replace x by $x\beta z$ where $z \in M$, $\beta \in \Gamma$ in the equation $d(x)\gamma d(y) = 0$, we have $d(x)\beta z\gamma d(y) + xk(\beta)z\gamma d(y) = 0$. Replace β by $\beta m\delta$ with $m \in M$, $\delta \in \Gamma$ in the equation, we get $(d(x)\beta m + xk(\beta)m)\delta z\gamma d(y) = 0$. Since M is a prime Γ-ring, this statement implies $(d(x)\beta m + xk(\beta)m) = 0$ or $z\gamma d(y) = 0$. Suppose that for all $x, m \in M$ and $\beta \in \Gamma$ $(d(x)\beta m + xk(\beta)m) = 0$. Then $(d(x\beta m) = x\beta d(y)$ so $d(M)$ is a left ideal of M. Now, let $z\gamma d(y) = 0$ for all $z, y \in M$. Replace y by $y\beta m$ with $m \in M$ and $\beta \in \Gamma$ in the preceding statement to obtain $z\gamma yk(\beta)m + z\gamma y\beta d(m)) = 0$. In the last equation, replace β by $\beta n\delta$, where $n \in M$, $\delta \in \Gamma$, we get $z\gamma y\beta(nk(\delta)m + n\delta d(m)) = 0$. That is $z\gamma y\Gamma(nk(\delta)m + n\delta d(m)) = 0$ for all $n, m, z, y \in M$, $\delta \in \Gamma$. This implies $nk(\delta)m + n\delta d(m) = 0$ for all $n, m \in M$ and $\delta \in \Gamma$. One can then easily show that $d(M)$ is a right ideal of M.

Theorem 3: Let M be a prime Γ-ring of characteristic not 2, d be a nonzero k-derivation of M, γ be a nonzero element of Γ and $k(\gamma) \neq 0$. If $d(M) \subseteq C_\gamma$ then M is commutative Γ-ring.

Proof: By hypothesis and Lemma 3 (vii), we have $d([m, n]_\gamma) = [m, n]_{k(\gamma)} \in C_\gamma$ for all $n, m \in M$. In this statement, replace m by $d(x)\beta d(y)$ and n by z where $x, y, z \in M$ and $\beta \in \Gamma$ and use Lemma 5 (iv), we obtain $d(x)[\beta, k(\gamma)]_z d(y) \in C_\gamma$. By Lemma 8, the last statement and hypothesis implies $d(x)[\beta, k(\gamma)]_z d(y) \in C_{k(\gamma)}$ and $d(M) \subseteq C_{k(\gamma)}$, respectively. Hence, we get,

$$[d(x)[\beta, k(\gamma)]_z d(y), z]_{k(\gamma)} = 0 \quad x, y, z \in M, \quad \beta \in \Gamma.$$

Using Lemma 3 (iii) and $d(M) \subseteq C_{k(\gamma)}$, we obtain

$$d(x)[[\beta, k(\gamma)]_z, k(\gamma)]_z d(y) = 0 \quad x, y, z \in M, \quad \beta \in \Gamma.$$

In the last equation, replace β by $\beta d(s)\delta$, where $s \in M, \quad \delta \in \Gamma$. Use Lemma 3 (iv), and Char$M \neq 2$, we get $d(x)[\beta, k(\gamma)]_z d(s)[\delta, k(\gamma)]_z d(y) = 0$. Replacing β by $\beta d(n)\sigma$, where $n \in M, \quad \sigma \in \Gamma$,we have $0 = d(x)[\beta, k(\gamma)]_z d(n)\sigma d(s)[\delta, k(\gamma)]_z d(y)$. That is,

$$d(x)[\beta, k(\gamma)]_z d(n)\Gamma d(s)[\delta, k(\gamma)]_z d(y) = 0 \quad \forall x, y, s, z \in M, \quad \beta, \delta \in \Gamma.$$

The primeness of M gives us

$$d(x)[\beta, k(\gamma)]_z d(n) = 0 \quad \forall x, y, z \in M, \quad \beta \in \Gamma.$$

By Lemma 5 (iv), we obtain

$$0 = d(x)[\beta, k(\gamma)]_z d(n) = [d(x)\beta d(n), z]_{k(\gamma)} = [d(x)k(\gamma)d(n), z]_\beta.$$

This implies $d(x)k(\gamma)d(n) \in C_\beta$, for all $\beta \in \Gamma$, that is, $d(x)k(\gamma)d(n) \in C_\Gamma$. If there are some elements x, n of M such that $d(x)k(\gamma)d(n) \neq 0$, then M is a commutative Γ-ring by Lemma 4 (vii). If $d(x)k(\gamma)d(n) = 0$ for all $x, n \in M$, then $d(M)$ is a right (or left) ideal of M by Lemma 9. Since $0 \neq d(M) \subseteq C_\gamma$ by the hypothesis, Lemma 7 implies M is commutative Γ-ring.

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