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## A Borsuk-Ulam Theorem for Heisenberg Group Actions

*Necdet Güner*

### Abstract

Let  $G = H_{2n+1}$  be a  $(2n + 1)$ -dimensional Heisenberg Lie group acts on  $M = C^m - \{0\}$  and  $M' = C^{m'} - \{0\}$  exponentially. By using Cohomological Index we proved the following theorem.

If  $f : M \rightarrow M'$  is a  $G$ -equivariant map, then  $m \leq m'$ .

**Key Words:** Borsuk-Ulam Type Theorem, Cohomological Index, Group Action.

### 1. Introduction

By using cohomological index and relative index theories Fadell, Husseini and Rabinowitz proved Borsuk-Ulam type theorems for compact Lie groups. The ideal-valued index  $Index_G(X)$  of a  $G$ -space  $X$  for a compact Lie group, is the kernel of the map  $H_G^*(pt) \rightarrow H_G^*(X)$ , where  $H_G^*(pt)$  is the Borel cohomology of a point, which is isomorphic to  $H^*(BG)$ , the cohomology of the classifying space of  $G$ , [2,5]. If  $G$  is a non-compact Lie group, where  $BG$  may be acyclic, then the preceding method fails.

Fadell and Husseini introduced infinitesimal ideal-valued index theory to overcome difficulties of this type. Infinitesimal index is the kernel of the map from  $B\mathcal{G}$ , the basic subcomplex of  $G$  to  $H_{\mathcal{G}}^*(X)$ , the infinitesimal  $G$ -deRham cohomology of a  $G$ -space  $X$ . They proved a Borsuk-Ulam type theorem for the non-compact abelian Lie group  $G = C$  [3,4].

In this work we would like to extend their results to the Heisenberg groups. The

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main theorem of this work is a Borsuk-Ulam type theorem about a Heisenberg Lie group action.

## 2. Preliminaries

First recall the definitions of the Lie derivative  $\Theta(X)$ , the substitution operation  $i(X)$ , and the differential operator  $\delta$ .

The Lie derivative of a  $p$ -form  $\alpha$  with respect to  $X \in X(M)$  is the linear map  $\Theta(X)$  homogeneous of degree zero, given by

$$\Theta(X)\alpha = X(\alpha(X_1, X_2, \dots, X_p)) - \left( \sum_{j=1}^p \alpha(X_1, \dots, [X, X_j], \dots, X_p) \right).$$

The substitution operator  $i(X)$ , induced by  $X$  define a  $(p-1)$ -form  $i(X)\alpha$  by

$$i(X)\alpha(X_1, X_2, \dots, X_{p-1}) = \alpha(X, X_1, X_2, \dots, X_{p-1})$$

The map  $i(X) : \Omega(M) \rightarrow \Omega(M)$  is a homogeneous operator of degree  $(-1)$ .

The exterior derivative is the real linear map  $\delta$ , homogeneous of degree 1, defined by

$$\begin{aligned} \delta\alpha(X_0, X_1, \dots, X_p) &= \sum_{j=0}^p (-1)^j X_j(\alpha(X_0, \dots, \hat{X}_j, \dots, X_p)) \\ &+ \sum_{0 \leq i < j \leq p} (-1)^{i+j} \alpha([X_i, X_j], \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_p) \end{aligned}$$

where  $\alpha$  is a  $p$ -form.

Let  $G$  be a connected Lie group with its Lie algebra  $\mathcal{G}$  and dual  $\mathcal{G}^*$ . The Weil algebra  $W(\mathcal{G}) = \Lambda(\mathcal{G}^*) \otimes S(\mathcal{G}^*)$  where  $\Lambda(\mathcal{G}^*)$  is the exterior algebra of the dual  $\mathcal{G}^*$  and  $S(\mathcal{G}^*)$  is the symmetric algebra generated by elements of degree 2. Let  $s_k$  be a basis of  $S(\mathcal{G}^*)$ ,  $h : \mathcal{G}^* \rightarrow S^2(\mathcal{G}^*)$  defined by  $h(\alpha_k) = s_k$ , where  $\alpha_k$  is a basis for  $\mathcal{G}^*$ . Also we define  $\Theta_S(X)s_k = h(\Theta_E(X)\alpha_k)$ , on  $S(\mathcal{G}^*)$ , where  $\Theta_E(X)$  is the usual Lie derivative defined on  $\mathcal{G}^*$ . The substitution operation  $i(X)$  is as defined above on  $\Lambda(\mathcal{G}^*)$  and 0 on  $S(\mathcal{G}^*)$ .

The differential operator  $\delta$  on  $W(\mathcal{G})$  defined as follows:

$$\delta = \delta_E + \delta_S + h$$

$$h = \sum_k i(X_k)\mu(h(\alpha_k))$$

$$\delta_E = (1/2)\sum_k \mu(\alpha_k)\Theta_E(X_k)$$

$$\delta_S = \sum_k \mu(\alpha_k)\Theta_S(X_k)$$

$X_k \in \mathcal{G}$ ,  $\alpha_k \in \mathcal{G}^*$  and  $\mu$  is a multiplication operator defined as  $\mu(\alpha)\beta = \alpha \wedge \beta$ ,  $\alpha, \beta \in \Lambda(\mathcal{G}^*)$ .

Let  $L$  be a finite dimensional Lie algebra and  $i$ ,  $\delta$  and  $\Theta$  are defined as above. Let  $R = \sum_{p \geq 0} R^p$  be a graded commutative algebra with differential  $\delta$ .

The horizontal subalgebra of  $R$ :

$$R_{i=0} = \bigcap_{X \in L} \ker i(X).$$

The invariant subalgebra of  $R$ :

$$R_{\Theta=0} = \bigcap_{X \in L} \ker \Theta(X).$$

The basic subalgebra of  $R$ :

$$R_{i=0, \Theta=0} = (R_{i=0}) \cap (R_{\Theta=0}).$$

The basic subalgebra of the Weil algebra of a Lie group  $G$ :

$$W(\mathcal{G})_{i=0, \Theta=0} = (\Lambda(\mathcal{G}^*) \otimes S(\mathcal{G}^*))_{i=0, \Theta=0} \cong S(\mathcal{G}^*)_{\Theta=0} = B\mathcal{G}.$$

The Basic Weil subalgebra serves as the algebraic analogue of the classifying space  $BG$  and we will denote it by  $B\mathcal{G}$ , [3].

### Infinitesimal Index:

The infinitesimal deRham complex of a differentiable  $G$ -manifold  $M$  is  $\Omega(M) \otimes W(\mathcal{G})$ , with  $\Theta$ ,  $i$  and  $\delta$  the differential operator, where  $\Omega(M)$  denotes the differential forms on  $M$ . The basic subcomplex  $\Omega_{\mathcal{G}}(M)$  of  $\Omega(M) \otimes W(\mathcal{G})$  is defined as  $\Omega_{\mathcal{G}}(M) = (\Omega(M) \otimes W(\mathcal{G}))_{i=0, \Theta=0}$  and the cohomology of  $\Omega_{\mathcal{G}}(M)$  is called the infinitesimal deRham cohomology of  $M$  and

denoted by  $H_{\mathcal{G}}^*(M)$ . The inclusion map

$$j_M : W(\mathcal{G}) \rightarrow \Omega(M) \otimes W(\mathcal{G})$$

$$x \rightarrow 1 \otimes x$$

$j_M$  induces morphisms

$$\tilde{j}_M : W(\mathcal{G})_{i=0, \Theta=0} \rightarrow (\Omega(M) \otimes W(\mathcal{G}))_{i=0, \Theta=0}$$

$$\tilde{j}_M : B\mathcal{G} \rightarrow \Omega_{\mathcal{G}}(M)$$

a morphism  $\tilde{j}_M$  of differential graded algebras is called the classifying map for the  $G$ -space  $M$ . The classifying map  $\tilde{j}_M$  induces

$$j_M^* : B\mathcal{G} \rightarrow H_{\mathcal{G}}^*(M)$$

since  $\delta = 0$  on  $S(\mathcal{G}^*)$ .

The infinitesimal  $\mathcal{G}$ -index of  $M$ ,  $Index_{\mathcal{G}}M$ , is the kernel of the map

$$j_M^* : B\mathcal{G} \rightarrow H_{\mathcal{G}}^*(M)$$

where  $j_M^*$  is induced by  $\tilde{j}_M : B\mathcal{G} \rightarrow \Omega_{\mathcal{G}}(M)$ .

The infinitesimal  $\mathcal{G}$ -index possesses the following properties:

**Continuity.** [3] *If  $B\mathcal{G}$  is Noetherian, there is an open  $G$ -set  $V_0$  such that  $X \subset V_0$  and for every open  $G$ -set  $U$ ,  $X \subset U \subset V_0$ ,*

$$Index_{\mathcal{G}}X = Index_{\mathcal{G}}U.$$

**Monotonicity.** [3] *Let  $B\mathcal{G}$  be Noetherian, and  $f : M \rightarrow N$  is a differentiable  $G$ -map,  $X \subset M$  and  $Y \subset N$  are  $G$ -subsets with  $f(X) \subset Y$ , then*

$$Index_{\mathcal{G}}Y \subset Index_{\mathcal{G}}X.$$

**Additivity.** [3] Let  $XUY \subset M$ ,  $X$  and  $Y$  be  $G$ -sets and  $B\mathcal{G}$  is Noetherian, then

$$(Index_{\mathcal{G}}X)(Index_{\mathcal{G}}Y) \subset Index_{\mathcal{G}}(XUY).$$

Recall that, by definition, the  $(2n + 1)$ -dimensional Heisenberg Lie group  $H_{2n+1}$  is the Lie group of real matrices of the form:

$$\begin{pmatrix} 1 & x_1 & x_2 & \cdots & x_n & z \\ 0 & 1 & 0 & \cdots & 0 & y_1 \\ 0 & 0 & 1 & \cdots & 0 & y_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & y_n \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

where  $x_i, y_i$ , and  $z \in \mathbb{R}$ .

$H_{2n+1}$  is a two-step, nilpotent Lie group. Let  $\mathcal{H}_{2n+1}$  denote the Lie algebra of  $H_{2n+1}$ .  $\mathcal{H}_{2n+1}$  is generated by  $\{X_1, \dots, X_n, Y_1, \dots, Y_n, Z\}$  with all its commutators equal to zero except  $[X_i, Y_i] = Z, i = 1, 2, \dots, n$ . The dual  $\mathcal{H}_{2n+1}^*$  is generated by  $\{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n, \gamma\}$ , where

$$X_i = \frac{\partial}{\partial x_i}, \quad Y_i = \frac{\partial}{\partial y_i} + x_i \frac{\partial}{\partial z}, \quad Z = \frac{\partial}{\partial z}$$

and  $\alpha_i = dx_i, \beta_i = dy_i, \gamma = dz - \sum_{i=1}^n x_i dy_i, i = 1, \dots, n$ .

**Proposition 1.** Let  $G$  be a  $(2n + 1)$ -dimensional Heisenberg Lie group and  $A = G/[G, G]$  be its abelinization, then  $B\mathcal{G} \cong BA$ . Where  $BA$  is the polynomial algebra in  $s_1, \dots, s_n, t_1, \dots, t_n$ , and  $s_i = h(\alpha_i)$ , and  $t_i = h(\beta_i)$ .

**Proof.** The basic subcomplex  $B\mathcal{G} \cong (S\mathcal{G}^*)_{\Theta=0}$ , where  $\Theta = 0$  means that the Lie derivatives is zero with respect to all  $X \in \mathcal{G}$ . Let  $\{X_1, \dots, X_n, Y_1, \dots, Y_n, Z\}$ , and  $\{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n, \gamma\}$  denote the generators of  $\mathcal{G}$  and  $\mathcal{G}^*$  respectively, then  $\mathcal{A}$  and  $\mathcal{A}^*$  are generated by  $\{X_1, \dots, X_n, Y_1, \dots, Y_n\}$ , and  $\{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n\}$  respectively. It is known that, if  $A$  is abelian, all the generators of  $\mathcal{A}^*$  are invariant differential forms, then  $\Theta(X)\alpha_i = \Theta(X)\beta_j = 0$  for all  $X \in \mathcal{G}$  and  $1 \leq i, j \leq n$ , [3]. Therefore

$$BA = (SA^*)_{\Theta=0} = SA^* \subset (S\mathcal{G}^*)_{\Theta=0} = B\mathcal{G}$$

Now assume that  $B\mathcal{G}$  has some elements  $\omega$  which is not an element of  $BA$ . Then  $\omega$  contains a polynomial of  $r = h(\gamma)$ . Let

$$\wp(r) = \sum_{l=0}^m a_l r^l \neq 0$$

where  $a_l$ 's are linearly independent in  $B\mathcal{G}$ .

$$\omega = \sum_{j=1}^k c_j s_{j_1} \wedge \dots \wedge s_{j_p} \wedge t_{j_1} \wedge \dots \wedge t_{j_q} \wedge \wp(r)$$

and  $\Theta(X)\omega = 0$  for all  $X \in \mathcal{G}$ . Since  $\Theta(X)s_i = h(\Theta(X)\alpha_i) = 0$  and  $\Theta(X)t_j = h(\Theta(X)\beta_j) = 0$ ,

$$\Theta(X)\omega = \left( \sum_{j=1}^k c_j s_{j_1} \wedge \dots \wedge s_{j_p} \wedge t_{j_1} \wedge \dots \wedge t_{j_q} \right) \wedge \Theta(X)\wp(r) = 0$$

$$\Theta(X)\wp(r) = 0$$

$$\Theta(X)\wp(r) = \Theta(X)(a_0 + a_1 r + a_2 r^2 + \dots + a_m r^m) = 0$$

$$a_1(\Theta(X)r) + a_2(\Theta(X)r^2) + \dots + a_m(\Theta(X)r^m) = 0$$

$$a_1(\Theta(X)r) + 2a_2 r(\Theta(X)r) + \dots + m a_m r^{m-1}(\Theta(X)r) = 0$$

$$(a_1 + 2a_2 r + \dots + m a_m r^{m-1})(\Theta(X)r) = 0$$

since  $\Theta(X)r = h(\Theta(X)\gamma) \neq 0$ , then

$$a_1 + 2a_2 r + \dots + m a_m r^{m-1} = 0$$

by linearly independence,  $a_i = 0$  for  $i = 1, \dots, m$ , thus  $\wp(r) = a_0$ . Therefore,  $B\mathcal{G} \cong BA$ .  $\square$

### Exponential $G$ Action:

Let  $C^n$  denote the complex  $n$ -space and  $M = C^n$ . Fadell and Husseini defined the right  $C$ -action

$$M \times G \rightarrow M$$

$$(z_1, \dots, z_n)(x + iy) = e^x(z_1(e^{iy})^{\lambda_1}, \dots, z_n(e^{iy})^{\lambda_n})$$

where  $(\lambda_1, \dots, \lambda_n)$  is an  $n$ -tuple of non-zero real numbers, such that  $\lambda_i/\lambda_j$  are irrational for  $i \neq j$ . This action takes  $C^n - \{0\}$  onto itself. Fadell and Husseini called this action an exponential action with parameters  $\lambda_1, \dots, \lambda_n$ , and then proved the following Borsuk-Ulam type theorem,[3]:

*If  $G = C$  acts on  $C^n$  and  $C^m$  with exponential actions with parameters  $\lambda_1, \dots, \lambda_n$  and  $\mu_1, \dots, \mu_m$ , respectively, with  $m < n$ . Then every  $G$ -map  $f : C^n \rightarrow C^m$  has a non-trivial zero. Alternatively, there does not exist a  $G$ -map  $f : C^n - \{0\} \rightarrow C^m - \{0\}$ .*

Here we want to use the same action for  $G = G_1 \oplus \dots \oplus G_n$  and  $M = C^m - \{0\} = (C^{m_1} \oplus \dots \oplus C^{m_n}) - \{0\}$  where  $G_k = R + iR$  and  $M_k = C^{m_k} - \{0\}$ .

The  $G_k$  action on  $M_k$  is defined as follows:

$$\varphi_k : (\vec{z}, \xi_k, \vec{\lambda}_k) \rightarrow e^{x_k}(z_1(e^{iy_k})^{\lambda_1}, \dots, z_{m_k}(e^{iy_k})^{\lambda_{m_k}}).$$

where  $\xi_k = x_k + iy_k \in G_k$ ,  $\vec{z} = (z_1, \dots, z_{m_k}) \in M_k$ , and  $\vec{\lambda}_k = (\lambda_1, \dots, \lambda_{m_k})$  is an  $m_k$ -tuple of non-zero real numbers such that  $\lambda_i/\lambda_j$  are irrational for  $i \neq j$ .

Since  $z_j(e^{iy_k})^{\lambda_j} = z_j$  for all  $j$ , if and only if  $\lambda_j y_k = 2q\pi$ ,  $q \in Z$ , the discrete subgroup  $\Gamma_j = \{(2q\pi/\lambda_j)\}$ ,  $q \in Z$  appears as non-trivial isotropy. If we require that  $\lambda_i/\lambda_j$  be irrational for  $i \neq j$ , then these would be the only non-trivial isotropy subgroups. We also note that the representation  $iy_k \rightarrow \text{diag}((e^{iy_k})^{\lambda_1}, \dots, (e^{iy_k})^{\lambda_{m_k}})$  has compact image  $R/D = S^1$ ,  $D$  discrete, in  $U(m_k)$  if all ratios  $\lambda_i/\lambda_j$  are rational. Otherwise, this imbedding of  $R$  in  $U(m_k)$  has closure which is a torus of dimension  $\geq 2$ .

The  $G$ -action on  $M$  is defined as follows:

$$M \times G \rightarrow M$$

$$(\oplus_{k=1}^n M_k) \times (\oplus_{k=1}^n G_k) \rightarrow (\oplus_{k=1}^n M_k)$$

$$((\vec{z}_1, \dots, \vec{z}_n), (\xi_1, \dots, \xi_n), (\vec{\lambda}_1, \dots, \vec{\lambda}_n)) \rightarrow e^{(\sum_{j=1}^n x_j)}(\varphi_1(\vec{z}_1, iy_1, \vec{\lambda}_1), \dots, \varphi_n(\vec{z}_n, iy_n, \vec{\lambda}_n)).$$

Here we have

$$G = (G_1 \oplus iG_1) \oplus \dots \oplus (G_n \oplus iG_n).$$

This can be written as,

$$G = (G_1 \oplus \dots \oplus G_n) \oplus iG_1 \oplus \dots \oplus iG_n$$



$$G = R \oplus iR_1 \oplus \cdots \oplus iR_n.$$

Let  $G' = R$  and  $G'' = R_1 \oplus \cdots \oplus R_n$ . Here  $G'$  acts freely on  $C^m - \{0\}$  and  $C^m - \{0\} \cong R^+ \times S^{2m-1}$  with orbit space  $(C^m - \{0\})/G' \cong S^{2m-1}$ . If  $\Gamma$  is a discrete subgroup of  $G'$ , note that  $G'/\Gamma = S^1$  is a compact group and

$$M/\Gamma = (C^m - \{0\})/\Gamma \cong S^1 \times S^{2m-1}.$$

By lemmas 4.2, 4.3, and 4.4 of Fadell and Husseini [3], there is a natural chain equivalence

$$\nu : \Omega(M)_{\Theta=0} \rightarrow \Omega(S^1 \times S^{2m-1}).$$

Now, consider  $M = C^m - \{0\}$  as a  $G'$ -space by restricting the  $G$  action. Then the natural projection  $M \rightarrow M/G' = S^{2m-1}$  is a locally trivial principle  $G$ -bundle by Palais' theorems [7].

**Proposition 2.** [3] *There is a chain equivalence  $\gamma : \Omega_{G'}(M) \rightarrow \Omega(S^{2m-1})$ .*

Atiyah and Bott showed that, since torus  $T$  is compact, the Borel cohomology  $H_T^*(S^{2m-1}; R)$  is naturally isomorphic to the infinitesimal cohomology  $H_{\mathcal{T}}^*(S^{2m-1})$  [1]. Furthermore, the ideal-valued index,  $Index_{\mathcal{T}}(S^{2m-1})$  and the infinitesimal ideal-valued index,  $Index_{\mathcal{T}}(S^{2m-1})$  coincide when  $H^*(BT; R)$  and  $B\mathcal{T}$  are naturally identified. The inclusion map  $T \subset T^m \subset U(m)$  induces homomorphisms  $\lambda_j : T \rightarrow S^1$ ,  $j = 1, 2, \dots, m$ , and if  $\mathcal{S}$  is the Lie algebra of  $S^1$ ,  $\lambda_j$  induces  $\lambda_j^* : B\mathcal{S} \rightarrow BT$ . If  $\sigma$  is the generator of  $B\mathcal{S}$  set  $\lambda'_j = \lambda_j^*(\sigma)$ . Then, the natural inclusion

$$BT \rightarrow \Omega(M)_{\Theta_T=0} \otimes BT$$

induces a surjection

$$BT \rightarrow H_{\mathcal{T}}^*(S^{2m-1})$$

with kernel  $P_T$  the principal ideal generated by  $\varepsilon = \lambda'_1 \lambda'_2 \dots \lambda'_m$ . Here each  $\lambda_j : T \rightarrow S^1$  is nontrivial because  $(S^{2m-1})^{G''} = (S^{2m-1})^T = \emptyset$ . This implies that if  $g : BT \rightarrow B\mathcal{G}''$  is induced by inclusion  $g(\lambda'_j) \neq 0$  for  $j = 1, 2, \dots, m$ .

**Lemma.** [6]  $g(\varepsilon)$ , is a polynomial of  $t_1, \dots, t_n$ , of degree  $m$ .

**3. Results**

Now consider the exponential  $G = G_1 \oplus \dots \oplus G_n$  action on

$$M = (C^{m_1} \oplus \dots \oplus C^{m_n}) - \{0\} = C^m - \{0\}$$

where  $m = m_1 + \dots + m_n$ , and  $M = C^m - \{0\} = R^+ \times S^{2m-1}$ .

**Theorem 1.** Let  $G = G_1 \oplus \dots \oplus G_n$  acts on  $M = (C^{m_1} \oplus \dots \oplus C^{m_n}) - \{0\}$  with an exponential action with parameters  $(\lambda_1, \dots, \lambda_m)$ . Then, the following inclusion map

$$j_M : W(G) \rightarrow \Omega_{\mathcal{G}}(M)$$

induces a surjection

$$j_M^* : B\mathcal{G} \rightarrow H_{\mathcal{G}}^*(M)$$

The kernel of this map is an ideal generated by  $s_1 + \dots + s_n$  and  $\lambda'_1 \lambda'_2 \dots \lambda'_m$ .

$$Index_{\mathcal{G}}(M) = \langle s_1 + \dots + s_n, \lambda'_1 \lambda'_2 \dots \lambda'_m \rangle .$$

**Proof.** I. First compare the spectral sequences  $\Omega(M)_{\Theta=0} \otimes S(\mathcal{G}^*)$  and  $\Omega(M)_{\Theta'=0} \otimes S(\mathcal{G}'^*)$  via the filtration preserving map

$$\Omega(M)_{\Theta=0} \otimes S(\mathcal{G}^*) \rightarrow \Omega(M)_{\Theta'=0} \otimes S(\mathcal{G}'^*)$$

induced by  $G' \subset G$ . Induced map on fibers  $\Omega(M)_{\Theta=0} \rightarrow \Omega(M)_{\Theta'=0}$  is a chain equivalence and on the base  $S(\mathcal{G}^*) = R[s_1, \dots, s_n, t_1, \dots, t_n] \rightarrow R[s_1 + \dots + s_n] = S(\mathcal{G}'^*)$ . At  $E_2$ -level we have the following diagram:

$$\begin{array}{ccc} H^*(\Omega(M)_{\Theta=0}) \otimes S(\mathcal{G}^*) & \rightarrow & H^*(\Omega(M)_{\Theta'=0}) \otimes S(\mathcal{G}'^*) \\ \downarrow d_2 & & \downarrow d'_2 \\ H^*(\Omega(M)_{\Theta=0}) \otimes S(\mathcal{G}^*) & \rightarrow & H^*(\Omega(M)_{\Theta'=0}) \otimes S(\mathcal{G}'^*) \end{array}$$

Let  $u' \in H^1(\Omega(M)_{\Theta'=0}) = H^1(S^1 \times S^{2m-1})$  denotes a generator corresponding to the  $S^1$ -factor. Since  $H_{\mathcal{G}'}^*(M) = H^*(S^{2m-1})$  then  $d'_2 u' \neq 0$ . We may assume without loss

that, if  $u \in H^1(\Omega(M)_{\Theta=0}) = H^1(S^1 \times S^{2m-1})$  is the generator corresponding to  $u'$ , then  $d_2 u = s_1 + \dots + s_n$ .

II. Now compare the spectral sequences for  $\Omega(M)_{\Theta=0} \otimes S(\mathcal{G}^*)$  and  $\Omega(S^{2m-1})_{\Theta''=0} \otimes S(\mathcal{G}''^*)$  via natural map

$$\Omega(M)_{\Theta=0} \otimes S(\mathcal{G}^*) \rightarrow \Omega(S^{2m-1})_{\Theta''=0} \otimes S(\mathcal{G}''^*)$$

induced by  $M = C^m - \{0\} \rightarrow S^{2m-1}$  and  $G'' \subset G$ .

If we take a generator  $v'' \in H^*(\Omega(S^{2m-1})_{\Theta''=0}) \cong H^*(S^{2m-1})$  and apply Hussein's Lemma, we will see that  $d_m'' v'' = C \lambda'_1 \lambda'_2 \dots \lambda'_m$ ,  $C \neq 0$ . Since  $v''$  may be chosen as the image of  $v$ , where  $v \in H^{2m-1}(\Omega(M)_{\Theta=0})$ , which denotes a generator corresponding to  $S^{2m-1}$  factor, then we have  $d_m v = C \lambda'_1 \lambda'_2 \dots \lambda'_m$ .  $\square$

Now assume  $G \cong R^{2n}$ , and  $M = C^m - \{0\} = (C^{m_1} \oplus \dots \oplus C^{m_n}) - \{0\}$ .

Let  $\varphi : G \rightarrow Gl_m(C)$ , where  $m = m_1 + \dots + m_n$  be a homomorphism. Also assume that  $im \varphi \subset$  diagonal matrices and  $\text{closure}(\varphi(G)) = R_1 \oplus iR_1 \oplus \dots \oplus R_n \oplus iR_n$ .

**Corollary 1.** *Let  $G \cong R^{2n}$  and  $M = C^m - \{0\}$ , with exponential  $G$  action given as above. Then  $Index_G(M) = \langle s_1 + \dots + s_n, \lambda'_1 \lambda'_2 \dots \lambda'_m \rangle$ .*

Now consider  $G = H_{2n+1}$ ,  $(2n + 1)$  dimensional nilpotent Heisenberg Lie group of real matrices and let

$$\psi : G \rightarrow G/[G, G] \cong R^{2n}$$

and  $M = C^m - \{0\} = (C^{m_1} \oplus \dots \oplus C^{m_n}) - \{0\}$ , and  $G$  acts on  $M$  via  $\psi$ .

$$M \times G \rightarrow M$$

$$((\vec{z}_1, \dots, \vec{z}_n), g) \rightarrow e^{(\sum_{j=1}^n x_j)} (\varphi_1(\vec{z}_1, iy_1, \vec{\lambda}_1), \dots, \varphi_n(\vec{z}_n, iy_n, \vec{\lambda}_n))$$

**Proposition 3.** *Let  $G$  be a  $(2n + 1)$ -dimensional nilpotent Heisenberg Lie group and  $A = G/[G, G]$  be its abelianization and  $M = C^m - \{0\}$ , the complex  $m$ -space. If the  $G$ -action  $M \times G \rightarrow M$  is defined as above, then*

$$H_G^*(M) \cong H_A^*(M).$$

**Proof.** Let  $\{X_1, \dots, X_n, Y_1, \dots, Y_n, Z\}$ , and  $\{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n, \gamma\}$  denote the generators of  $\mathcal{G}$ , and  $\mathcal{G}^*$  respectively, then  $\mathcal{A}$  and  $\mathcal{A}^*$  are generated by  $\{X_1, \dots, X_n, Y_1, \dots, Y_n\}$  and  $\{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n\}$  respectively. Here  $S\mathcal{A}^*$  is generated by

$$\{h(\alpha_1), \dots, h(\alpha_n), h(\beta_1), \dots, h(\beta_n)\} = \{s_1, \dots, s_n, t_1, \dots, t_n\}$$

also  $S\mathcal{G}^*$  is generated by  $\{s_1, \dots, s_n, t_1, \dots, t_n, r\}$  where  $r = h(\gamma)$ .

We want to show that  $\Omega_{\mathcal{G}}(M) \cong \Omega_{\mathcal{A}}(M)$  where

$$\Omega_{\mathcal{G}}(M) = (\Omega(M) \otimes S(\mathcal{G}^*))_{\Theta_{\mathcal{G}}=0}$$

and

$$\Omega_{\mathcal{A}}(M) = (\Omega(M) \otimes S(\mathcal{A}^*))_{\Theta_{\mathcal{A}}=0}.$$

We need to check that  $\Theta_X$ , for  $X \in \{\text{Center of } \mathcal{G}\}$ . Since the center of  $\mathcal{G}$  is generated by  $Z$ ,  $\Theta_Z = 0$  on  $\Omega(M)$  and also  $S(\mathcal{G}^*)_{\Theta_Z=0} = S(\mathcal{A}^*)$ , then

$$(\Omega(M) \otimes S(\mathcal{G}^*))_{\Theta_Z=0} = (\Omega(M) \otimes S(\mathcal{A}^*)).$$

Since

$$(\Omega(M) \otimes S(\mathcal{G}^*))_{\Theta_{\mathcal{G}}=0} = (\Omega(M) \otimes S(\mathcal{G}^*))_{\Theta_Z=0, \Theta_{\mathcal{A}}=0}$$

then

$$(\Omega(M) \otimes S(\mathcal{G}^*))_{\Theta_{\mathcal{G}}=0} = (\Omega(M) \otimes S(\mathcal{A}^*))_{\Theta_{\mathcal{A}}=0}.$$

This gives us

$$\Omega_{\mathcal{G}}(M) \cong \Omega_{\mathcal{A}}(M)$$

and then

$$H_{\mathcal{G}}^*(M) \cong H_{\mathcal{A}}^*(M).$$

□

**Proposition 4.** Let  $G = H_{2n+1}$  acts on  $M = (C^{m_1} \oplus \dots \oplus C^{m_n}) - \{0\}$  with an exponential action with parameters  $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$ . Then,

$$Index_{\mathcal{G}}(M) = \langle s_1 + \cdots + s_n, \lambda'_1 \lambda'_2 \cdots \lambda'_m \rangle .$$

**Proof.** By Proposition.1. and Proposition.3.  $B\mathcal{G} \cong B\mathcal{A}$  and  $H_{\mathcal{G}}(M) \cong H_{\mathcal{A}}(M)$ . Then,

$$Index_{\mathcal{A}}(M) = Index_{\mathcal{G}}(M).$$

The Proposition follows since  $Index_{\mathcal{A}}(M) = \langle s_1 + \cdots + s_n, \lambda'_1 \lambda'_2 \cdots \lambda'_m \rangle$  from Proposition.3. □

We may now give the following Borsuk-Ulam type theorem.

**Theorem 2.** *Let  $G = H_{2n+1}$  acts on  $M = C^p - \{0\} = (C^{p_1} \oplus \cdots \oplus C^{p_n}) - \{0\}$  and  $N = C^q - \{0\} = (C^{q_1} \oplus \cdots \oplus C^{q_n}) - \{0\}$  with an exponential actions with parameters  $\{\lambda_1, \lambda_2, \dots, \lambda_p\}$  and  $\{\mu_1, \mu_2, \dots, \mu_q\}$  respectively. If  $f : M \rightarrow N$  is a  $G$ -equivariant map, then  $p \leq q$ .*

**Proof.** Proposition.4. gives that

$$Index_{\mathcal{G}}(M) = \langle s_1 + \cdots + s_n, \lambda'_1 \lambda'_2 \cdots \lambda'_p \rangle$$

and

$$Index_{\mathcal{G}}(N) = \langle s_1 + \cdots + s_n, \mu'_1 \mu'_2 \cdots \mu'_q \rangle .$$

Where  $\lambda'_1 \lambda'_2 \cdots \lambda'_p$  and  $\mu'_1 \mu'_2 \cdots \mu'_q$  are polynomials of  $t_1, \dots, t_n$  with degrees  $p$  and  $q$  respectively. By monotonicity of  $Index_{\mathcal{G}}$ , if  $f : M \rightarrow N$  is a  $G$ -equivariant map, then

$$Index_{\mathcal{G}}(M) \supset Index_{\mathcal{G}}(N).$$

This implies that the degree of  $\lambda'_1 \lambda'_2 \cdots \lambda'_p$  is smaller than or equal to the degree of  $\mu'_1 \mu'_2 \cdots \mu'_q$ . Thus  $p \leq q$ . □

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