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AYŞE ALTIN

AYSEL TURGUT VANLI

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The Pitch and the Angle of Pitch of a Closed Nonnull Ruled Hypersurface Whose Generator is Spacelike in

$$R_1^{k+2}$$

Ayşe Altın, Aysel Turgut Vanlı

Abstract

In this paper, the pitch and the angle of pitch of a closed nonnull ruled hypersurface whose generators are spacelike are calculated in R_1^{k+2} .

1. Introduction

A hypersurface in $k+2$ - dimensional Minkowski space $R_1^{k+2} = (R^{k+2}, \sum_{i=1}^{k+1} dx_i - dx_{k+2})$ is called a spacelike (timelike) hypersurface if the induced metric tensor on the hypersurface is a positive definite Riemannian metric (Lorentz metric) [2]. If the tangent vector at every point of a given curve in R_1^{k+2} is a spacelike (timelike) vector, then the given curve is called a spacelike (timelike) curve [6].

Let $\eta: I \rightarrow R_1^{k+2}$ be a spacelike curve in R_1^{k+2} where $I \subset R$, and let $\{e_1(t), e_2(t), \dots, e_k(t)\}$ be a given orthonormal vector defined at each point $\eta(t)$. The set $\{e_1(t), e_2(t), \dots, e_k(t)\}$ spans a space of the tangent space $T_{\eta(t)}R_1^{k+2}$ at the point $\eta(t)$ of R_1^{k+2} . Let us denote this space by $E_k(t)$. The set $M = \cup_{t \in I} E_k(t)$ is a hypersurface of R_1^{k+2} . A parametrization for this hypersurface is

$$\varphi: I \times R^k \rightarrow R_1^{k+2}, \varphi(t, v_1, \dots, v_k) = \eta(t) + \sum_{i=1}^k v_i e_i(t) \quad (1)$$

if

$$\text{rank}(\varphi_t, \varphi_{v_1}, \dots, \varphi_{v_k}) = \text{rank}(\eta'(t) + \sum_{i=1}^k v_i e'_i(t), e_1(t), \dots, e_k(t)) = k + 1$$

then the hypersurface M is said to be a ruled hypersurface, the space $E_k(t)$ is called the generator space of the ruled hypersurface at $\eta(t)$ and the curve η is called base curve of the ruled hypersurface. The line, whose director vector is $e_i(t)$, that passes through $\eta(t)$ is said to be i^{th} generator line of the hypersurface.

For real numbers $p > 0$, the condition $\eta(t + p) = \eta(t)$ is satisfied, then the surface is called a closed ruled hypersurface [4,p.518]. The smallest $p > 0$ satisfying $\eta(t + p) = \eta(t)$ is called the period of the closed ruled hypersurface. A curve which intersects each space $E_k(t)$ orthogonally is said to be an orthogonal trajectory of M .

Adapting the algorithm of [5] to R_1^{k+2} , if the initial basis $\{e_1(t_0), e_2(t_0), \dots, e_k(t_0)\}$ is given, then the basis $\{e_1(t), e_2(t), \dots, e_k(t)\}$ satisfying

$$\langle e_i(t), e_j(t) \rangle = \epsilon_i \delta_{ij} \quad \text{and} \quad \langle e'_i(t), e_j(t) \rangle = 0, \quad 1 \leq i, j \leq k \quad (2)$$

is uniquely determined, where we denote the derivative of vector field e_ν along the curve η by e'_ν , and \langle, \rangle denotes the scalar product in R_1^{k+2} .

In [3], H. Frank and O.Giering computed the k - number of the pitches and the angles of pitch for a simple closed $C^2 - (k + 1)$ dimensional ruled surface. That is not the generalization of the pitch and the angle of pitch. Desired generalization has been done in A.Altin [1].

In this paper, the results obtained by [1] is investigated in R_1^{k+2} .

2. The Pitch of a Closed Nonnull Ruled Hypersurface in R_1^{k+2}

Theorem 2.1. *Let M be a nonnull ruled hypersurface in R_1^{k+2} . There exists a unique orthogonal trajectory at each point of M .*

Proof. An orthogonal trajectory, if exists, is of the form:

$$\beta : I \rightarrow M, \quad \beta(s) = \eta(s) + \sum_{i=1}^k f_i(s) e_i(s) \quad (3)$$

where f is a function from I into R . □

Differentiating (3), we obtain

$$\beta'(s) = \eta'(s) + \sum_{i=1}^k f'_i(s)e_i(s) + \sum_{i=1}^k f_i(s)e'_i(s). \quad (4)$$

Since the curve β is orthogonal to generator space $E_k(s)$, we have

$$\langle \beta'(s), e_j(s) \rangle = 0, \quad \text{for all } j = 1, \dots, k. \quad (5)$$

Replacing (4) in (5), we obtain

$$\langle \eta'(s), e_j(s) \rangle + \sum_{i=1}^k f'_i(s) \langle e_i(s), e_j(s) \rangle + \sum_{i=1}^k f_i(s) \langle e'_i(s), e_j(s) \rangle = 0.$$

Using (2) leads to

$$\langle \eta'(s), e_j(s) \rangle + \epsilon_j f'_j(s) = 0.$$

Thus we have

$$f'_j(s) = - \int \epsilon_j \langle \eta'(s), e_j(s) \rangle ds + c_j.$$

If we denote

$$- \int \epsilon_j \langle \eta'(s), e_j(s) \rangle ds = F_j(s) \quad (6)$$

then we have

$$f'_j(s) = F'_j(s) + c_j \quad 1 \leq j \leq k. \quad (7)$$

Since c_j is chosen arbitrarily, there are many curves satisfying (5). However, there exists a unique orthogonal trajectory passing through each $p_0 \in M$ which is of the form

$$p_0 = \varphi(s_0, v_{1_0}, \dots, v_{k_0}) = \eta(s_0) + \sum_{i=1}^k v_{i_0} e_i(s_0).$$

Definition 2.1. Let M be a closed nonnull ruled hypersurface in R_1^{k+2} and let $\{e_1(t), e_2(t), \dots, e_k(t)\}$ be the orthonormal frame of M at the point $\eta(t)$. Let β denote the orthogonal trajectory at $\eta(t)$. The distance between $\beta(t)$ and $\beta(t+p)$ is called the pitch of M .

Theorem 2.2. Assume that L_t is the pitch of a closed nonnull ruled hypersurface M whose generator η is a spacelike curve in R_1^{k+2} .

If M is a spacelike then

$$L_t = \sqrt{L_{1_t}^2 + L_{2_t}^2 + \dots + L_{k_t}^2}.$$

If M is a timelike then

$$L_t = \sqrt{|L_{1_t}^2 + L_{2_t}^2 + \dots + L_{(k-1)_t}^2 - L_{k_t}^2|}.$$

Proof. Let M be a closed spacelike ruled hypersurface in R_1^{k+2} , that is, e_i ($1 \leq i \leq k$) is a spacelike vector field in R_1^{k+2} .

If we choose p_0 of Theorem 2.1 as $\eta(t)$ then we have $\nu_{i_t} = 0$, subsequently $f_i(t) = 0$ and thus $c_i = -F_i(t)$. From (7), for $s = t + p$, we have

$$f_i(t + p) = F_i(t + p) - F_i(t)$$

and

$$f_i(t + p) - f_i(t) = F_i(t + p) - F_i(t).$$

This, together with (6), implies that

$$f_i(t + p) - f_i(t) = - \int_t^{t+p} \epsilon_i \langle \eta'(s), e_i(s) \rangle ds. \tag{8}$$

Now, let

$$f_i(t + p) - f_i(t) = L_{i_t} \quad 1 \leq i \leq k.$$

L_{i_t} is called the pitch on the i^{th} generator at the point $q(t)$. Clearly,

$$\beta(t)\vec{\beta}(t + p) = L_{1_t}e_1 + L_{2_t}e_2 + \dots + L_{k_t}e_k. \tag{9}$$

The length of this vector renders the pitch of the closed spacelike ruled hypersurface M at $\eta(t)$. We have

$$L_t = \sqrt{L_{1_t}^2 + L_{2_t}^2 + \cdots + L_{k_t}^2}.$$

Let M be a closed timelike ruled hypersurface in R_1^{k+2} , that is the normal vector field of M is spacelike. Then one of the e_i ($1 \leq i \leq k$) must be a timelike vector field. Assume that e_k is a timelike vector field.

From (9) we have

$$L_t = \sqrt{|L_{1_t}^2 + L_{2_t}^2 + \cdots + L_{(k-1)_t}^2 - L_{k_t}^2|}.$$

The pitch L_t , by (8), is independent of the choice of orthogonal trajectory.

3. The Angle of Pitch of a Closed Nonnull Ruled Hypersurface in R_1^{k+2}

Definition 3.1. Let M be a closed nonnull ruled hypersurface in R_1^{k+2} and let $e_1(t), e_2(t), \dots, e_k(t)$ be unit director vectors at the point $\eta(t)$ of the generator η . Let $e_{k+1}(t)$ denote the unit tangent vector of orthogonal trajectory at $\eta(t)$. The angle between $e_{k+1}(t)$ and $e_{k+1}(t+p)$ is called the angle of pitch of M where $e_{k+1}(t+p)$ is the tangent vector of the orthogonal trajectory at $\eta(t+p)$ and p is the period of the closed curve η .

Suppose that M is a closed nonnull ruled hypersurface in R_1^{k+2} . Let $\{\bar{e}_1(t), \bar{e}_2(t), \dots, \bar{e}_{k+2}(t)\}$ be an orthonormal frame at the initial point $\beta(t) = \eta(t)$. For all $s \in [t, t+p]$, an orthonormal frame $\{e_1(s), \dots, e_{k+2}(s)\}$ at the point $\beta(s)$ can be written as

$$e_i(s) = \sum_{j=1}^{k+2} a_{ij}(s) \bar{e}_j, \quad 1 \leq i \leq k+2. \quad (10)$$

Since β is an orthogonal trajectory, then, at $s = t+p$, we have

$$e_\mu(s) = \bar{e}_\mu, \quad 1 \leq \mu \leq k$$

$$e_{k+1}(s) = a_{(k+1)(k+1)}(s) \bar{e}_{k+1} + a_{(k+1)(k+2)}(s) \bar{e}_{k+2} \quad (11)$$

$$e_{k+2}(s) = a_{(k+2)(k+1)}(s) \bar{e}_{k+1} + a_{(k+2)(k+2)}(s) \bar{e}_{k+2}$$

Since M is a closed spacelike ruled hypersurface whose generator η is spacelike in R_1^{k+2} , the orthogonal trajectory must be a spacelike curve. Thus, the angle between $\bar{e}_{k+1}(s)$ and $e_{k+1}(s+p)$ is defined.

Assume that M is a closed a timelike ruled hypersurface whose generator η is spacelike in R_1^{k+2} . Then the normal vector field of M is spacelike. Since e_k is a timelike vector field, $\bar{e}_{k+1}(s)$ and $e_{k+1}(s+p)$ must be two spacelike vectors.

Theorem 3.1. *Let M be a closed nonnull ruled hypersurface whose generator η is spacelike in R_1^{k+2} . The angle of pitch of M is*

$$\theta_t = - \int_t^{t+p} \langle e'_{k+1}(s), e_{k+2}(s) \rangle ds.$$

Proof. We can divide the proof in two cases: □

1.case.

Let M be a closed spacelike ruled hypersurface whose η is spacelike in R_1^{k+2} . Thus, $e_i(s)$ and \bar{e}_i $1 \leq i \leq k+1$ are spacelike vectors, and $e_{k+2}(s), \bar{e}_{k+2}$ are timelike vectors. Hence, $\{\bar{e}_1(t), \bar{e}_2(t), \dots, \bar{e}_{k+2}(t)\}$ is the orthonormal frame at the initial point $\eta(t) = \beta(t)$ and the orthonormal frame is $\{e_1(s), e_2(s), \dots, e_{k+2}(s)\}$ at the point $\beta(s)$ for all $s \in [t, t+p]$, where β is the orthogonal trajectory of M . Let us denote the angle between \bar{e}_{k+1} and $e_{k+1}(s)$ by $\theta(s)$. From (11) we can choose

$$\begin{aligned} a_{(k+1)(k+1)}(s) &= ch\theta(s), & a_{(k+1)(k+2)}(s) &= sh\theta(s), \\ a_{(k+2)(k+1)}(s) &= sh\theta(s) & \text{and} & & a_{(k+2)(k+2)}(s) &= ch\theta(s). \end{aligned}$$

Hence, we get

$$\begin{aligned} e_\mu(s) &= \bar{e}_\mu, & 1 \leq \mu \leq k \\ e_{k+1}(s) &= ch\theta(s)\bar{e}_{k+1} + sh\theta(s)\bar{e}_{k+2} \\ e_{k+2}(s) &= sh\theta(s)\bar{e}_{k+1} + ch\theta(s)\bar{e}_{k+2}. \end{aligned} \tag{12}$$

Using (12) we have

$$e'_{k+1}(s) = e_{k+2}(s)\theta'(s) \quad \text{and} \quad - \langle e'_{k+1}(s), e_{k+2}(s) \rangle = \theta'(s).$$

Thus, we have

$$\theta_t = \int_t^{t+p} d\theta = - \int_t^{t+p} \langle e'_{k+1}(s), e_{k+2}(s) \rangle ds.$$

2.case.

Let M be a closed timelike ruled hypersurface whose η is spacelike in R_1^{k+2} . Thus, $e_k(s)$, and \bar{e}_k are timelike vectors, and $e_i(s)$, and \bar{e}_i $1 \leq i \leq k+2$ ($i \neq k$) are spacelike vectors.

Hence, $\{\bar{e}_1(t), \bar{e}_2(t), \dots, \bar{e}_{k+2}(t)\}$ is the orthonormal frame at the initial point $\alpha(t) = \beta(t)$ and the orthonormal frame is $\{e_1(s), e_2(s), \dots, e_{k+2}(s)\}$ at the point $\beta(s)$ for all $s \in [t, t+p]$, where β is the orthogonal trajectory of M . Let us denote the angle between \bar{e}_{k+1} and $e_{k+1}(s)$ by $\theta(s)$. From (11) we can choose

$$\begin{aligned} a_{(k+1)(k+1)}(s) &= \cos\theta(s), & a_{(k+1)(k+2)}(s) &= -\sin\theta(s), \\ a_{(k+2)(k+1)}(s) &= \sin\theta(s) & \text{and} & \quad a_{(k+2)(k+2)}(s) = \cos\theta(s). \end{aligned}$$

Hence, we get

$$\begin{aligned} e_\mu(s) &= \bar{e}_\mu, & 1 \leq \mu \leq k \\ e_{k+1}(s) &= \cos\theta(s)\bar{e}_{k+1} - \sin\theta(s)\bar{e}_{k+2} \\ e_{k+2}(s) &= \sin\theta(s)\bar{e}_{k+1} + \cos\theta(s)\bar{e}_{k+2}. \end{aligned} \tag{13}$$

Using (13) we have

$$e'_{k+1}(s) = -e_{k+2}(s)\theta'(s) \quad \text{and} \quad - \langle e'_{k+1}(s), e_{k+2}(s) \rangle = \theta'(s).$$

Thus, we have

$$\theta_t = \int_t^{t+p} d\theta = - \int_t^{t+p} \langle e'_{k+1}(s), e_{k+2}(s) \rangle ds.$$

which completes the proof.

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Ayşe ALTIN,
Department of Math. Fac. of Sci. Univ. of
Hacettepe, Ankara-TURKEY
Aysel TURGUT VANLI
Department of Math. Fac. of Sci. and Arts,
Univ. of Gazi, Ankara-TURKEY.

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