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Knotting of algebraic curves in complex surfaces

Sergey Finashin

1. Introduction

Theorem 1.1. *For any $d \geq 5$ there exist infinitely many smooth oriented closed surfaces $F \subset \mathbb{C}P^2$ representing class $d \in H_2(\mathbb{C}P^2) = \mathbb{Z}$, having $\text{genus}(F) = \frac{1}{2}(d-1)(d-2)$ and $\pi_1(\mathbb{C}P^2 \setminus F) \cong \mathbb{Z}/d$, such that the pairs $(\mathbb{C}P^2, F)$ are pairwise smoothly non-equivalent. Moreover, d -fold cyclic coverings over $\mathbb{C}P^2$ branched along F differ by their Seiberg-Witten invariants and thus are non-diffeomorphic.*

This theorem, which answers an old question (cf. [6], Problem 4.110), is proved in [2] for even $d \geq 6$. In this paper the proof for odd d and generalized Theorem 1.1 (see below Theorem 1.6) are added. Sections 2-3 and the Appendix reproduce the content of [2] whereas Section 5 extends the results from there.

Remark 1.1. Note that the surfaces that are constructed are not symplectic. Some speculation referring to Gromov's theorem suggests that any symplectic surface in $\mathbb{C}P^2$ may be isotopic to an algebraic curve. As far as I know, at the moment it is proved only for degrees $d \leq 4$.

The knotting construction used to obtain surfaces F is a relative of the rim-surgery defined in [5]. An alternative way to achieve Theorem 1.1 is to use the tangle-surgery of Viro introduced in [3]. For technical reasons I prefer to use the rim-surgery in this paper, and give below an idea about the other approach just because it inspired this paper.

1.1. The idea that inspired my construction

Any kind of a surgery on a codimension two submanifold, F , in some fixed n -manifold X gives rise to some n -dimensional surgery on the double covering $Y \rightarrow X$ branched along F . Vice versa, considering a surgery on Y , one can try to perform it equivariantly with respect to the covering transformation, which results in some surgery on a pair (X, F) . Sometimes X is preserved, and only F as an embedded submanifold is modified by this surgery. Such an ambient surgery on F in X will be called *the folding* of the corresponding surgery on Y .

For example, if Y is a complex surface defined over \mathbb{R} , and $X = Y/\text{conj}$ is the quotient by the complex conjugation $\text{conj} : Y \rightarrow Y$, then the projection $p : Y \rightarrow X$ is a double covering branched along $F = \text{Fix}(\text{conj})$ (the real locus of Y). Algebraic transformations (say, a blow-up, or a logarithmic transform) can be applied to Y in the real category. It

turns out (at least in the examples known to the author) that the quotient $X = Y/\text{conj}$ is not changed if a transformation is irreducible over \mathbb{C} , i.e., if it does not contain a pair of conj-symmetric transformations localized outside the real part F .

Say, the folding of a blow-up at a real point of Y is a real blow-up of F , that is an ambient connected sum $(X, F)\#(S^4, \mathbb{R}P^2)$, because $\mathbb{C}P^2/\text{conj} \cong S^4$. Viro observed [3] that the folding of a logarithmic transform is a certain tangle-surgery on F . This yields “exotic knottings” of $F = \#_{10}\mathbb{R}P^2$ in $S^4 = Y/\text{conj}$, where $Y = E(1) = \mathbb{C}P^2\#_9\overline{\mathbb{C}P^2}$ is a rational elliptic surface, being modified by logarithmic transforms (which produce Dolgachev surfaces defined over \mathbb{R}).

The same construction applied to a K3 surface, $Y = E(2)$, instead of $E(1)$, gives “exotic knottings” of $F = \text{Fix}(\text{conj})$ in $X = Y/\text{conj}$. For a suitable choice of the real structure in Y , the quotient X is diffeomorphic to $\mathbb{C}P^2$ and F becomes a sextic in X , so the surgery gives examples for $d = 6$ in Theorem 1.1. Viro’s tangle surgery can be applied, in general, along any null-framed annulus membrane on a surface in a four-manifold, which gives in the covering space a logarithmic transform. Suitable membranes on algebraic curves in $\mathbb{C}P^2$ are described in what follows.

It turned out that the Fintushel-Stern’s surgery on Y admits also a folding, i.e., can be made equivariantly, with the quotient X being preserved, provided the knot that we use is a double knot, i.e., $K\#K$. This folding is just what I call below “an annulus rim surgery”.

1.2. An annulus rim-surgery

Our surgery, like the Viro tangle surgery, requires a suitable annulus membrane and produces a new surface via knotting an old one along such a membrane. By an annulus membrane for a smooth surface F in 4-manifold X we mean a smoothly embedded surface $M \subset X$, $M \cong S^1 \times I$, with $M \cap F = \partial M$ and such that M comes to F normally along ∂M . Assume that such a membrane has framing 0, or equivalently, admits a diffeomorphism of its regular neighborhood $\phi : U \rightarrow S^1 \times D^3$ mapping $U \cap F$ onto $S^1 \times f$, where $f = I \perp I \subset D^3$ is a disjoint union of two segments, which are unknotted and unlinked in D^3 , that is to say that a union of f with a pair of arcs on a sphere ∂D^3 bounds a trivially embedded band, $b \subset D^3$, $b \cong I \times I$, so that $f = I \times (\partial I) \subset b$ (see Figure 1). The annulus M can be viewed as $S^1 \times \{\frac{1}{2}\} \times I$ in $S^1 \times b \subset S^1 \times D^3 \cong U$.

If X and F are oriented, then f inherits an orientation as a transverse intersection, $f = F \pitchfork D^3$, and we may choose a band b so that the orientation of f is induced from some orientation of b . It is convenient to view $f = I \perp I$ as is shown on Figure 1, so that the segments of f are parallel and oppositely oriented, with b being a thin band between them. Such a presentation is always possible if we allow a modification of ϕ , since one of the segments of f may be turned around by a diffeomorphism of $D^3 \rightarrow D^3$ leaving the other segment fixed.

Given a knot $K \subset S^3$, we construct a new smooth surface, $F_{K,\phi}$, obtained from F by tying a pair of segments $I \perp I$ along K inside D^3 , as is shown on Figure 1. More precisely, we consider a band $b_K \subset D^3$ obtained from b by knotting along K and let f_K denote

the pair of arcs bounding b_K inside D^3 . We assume that the framing of b_K is chosen the same as the framing of b , or equivalently, that the inclusion homomorphisms from $H_1(\partial D^3 \setminus (\partial f)) = H_1(\partial D^3 \setminus (\partial f_K))$ to $H_1(B^3 \setminus f)$ and to $H_1(B^3 \setminus f_K)$ have the same kernel. Then $F_{K,\phi}$ is obtained from F by replacing $S^1 \times f \subset S^1 \times D^3 \cong U$ with $S^1 \times f_K$. It is obvious that $F_{K,\phi}$ is homeomorphic to F and realizes the same homology class in $H_2(X)$.

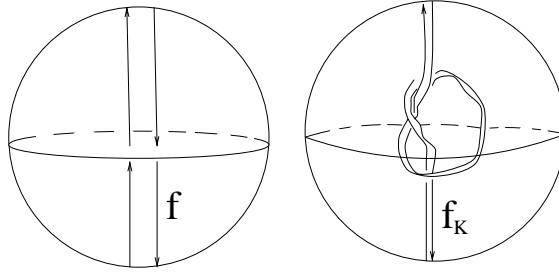


FIGURE 1. Knotting of a band b_K

The above construction is called in what follows *an annulus rim-surgery*, since it looks like the rim-surgery of Fintushel and Stern [5], except that we tie two strands simultaneously, rather than one. Recall that the usual rim-surgery is applied in [5] to surfaces $F \subset X$ which are *primitively embedded*, that is $\pi_1(X \setminus F) = 0$, which is not the case for the algebraic curves in $\mathbb{C}P^2$ of degree > 1 . The primitivity condition is required to preserve the fundamental group of $X \setminus F$ throughout the knotting. An annulus rim-surgery may preserve a *non-trivial* group $\pi_1(X \setminus F)$, if we require commutativity of $\pi_1(X \setminus (F \cup M))$, instead of primitivity of the embedding.

Proposition 1.2. *Assume that X is a simply connected closed 4-manifold, $F \subset X$ is an oriented closed surface with an annulus-membrane M of index 0, $\phi : U \rightarrow S^1 \times D^3$ is a trivialization like described above and $K \subset S^3$ is any knot. Assume furthermore that $F \setminus \partial M$ is connected and the group $\pi_1(X \setminus (F \cup M))$ is abelian. Then the group $\pi_1(X \setminus F_{K,\phi})$ is cyclic and isomorphic to $\pi_1(X \setminus F)$.*

1.3. Maximal nest curves

To prove Theorem 1.1, we apply an annulus rim-surgery inside $X = \mathbb{C}P^2$ letting $F = \mathbb{C}A$ be the complex point set of a suitable non-singular real algebraic curve, containing an annulus, M , among the connected components of $\mathbb{R}P^2 \setminus \mathbb{R}A$, where $\mathbb{R}A = \mathbb{C}A \cap \mathbb{R}P^2$ is the real locus of the curve.

One may take, for instance, a real algebraic curve $\mathbb{C}A$ of degree d , with a *maximal nest real scheme*. Such a curve for $d = 2k$ is constructed by a small real perturbation of a union of k real conics, whose real parts (ellipses) are ordered by inclusion in $\mathbb{R}P^2$. For

$d = 2k + 1$, we add to such conics a real line not intersecting the conics in $\mathbb{R}P^2$ and then perturb the unions. The real part, $\mathbb{R}A$, of our non-singular curve contains k components, O_1, \dots, O_k , called ovals (just deformed ellipses). We order the ovals so that O_i lies inside O_{i+1} and denote by R_i the annulus-component of $\mathbb{R}P^2 \setminus \mathbb{R}A$ between O_i and O_{i+1} for $i = 1, \dots, k - 1$. R_0 is a topological disk bounded from outside by O_1 , and R_k is the component bounded from inside by O_k .

The closures, $Cl(R_i)$, for $i = 1, \dots, k - 1$ are obviously 0-framed annulus-membranes on $\mathbb{C}A$. For simplicity, let us choose $M = Cl(R_1)$.

Proposition 1.3. *The assumptions of Proposition 1.2 hold if we put $X = \mathbb{C}P^2$, let $F = \mathbb{C}A$ be a maximal nest real algebraic curve of degree $d \geq 5$ and choose $M = Cl(R_1)$.*

1.4. Proof of Theorem 1.1 for even d

Assuming that the class $[F] \in H_2(X; \mathbb{Z}/2)$ vanishes, one can consider a double covering $p : Y \rightarrow X$ branched along F ; such a covering is unique if we require in addition that $H_1(X; \mathbb{Z}/2) = 0$. Similarly, we consider the double coverings $Y(K, \phi) \rightarrow X$ branched along $F_{K, \phi}$. To prove non-equivalence of pairs $(\mathbb{C}P^2, F_{K, \phi})$ for some family of knots K , it is enough to show that $Y(K, \phi)$ are not pairwise diffeomorphic. It follows from that $Y(K, \phi)$ is diffeomorphic to the 4-manifolds $Y_{K\#K}$ obtained from Y by a surgery introduced in [4] (Let us call it *FS-surgery*).

Proposition 1.4. *The above $Y(K, \phi)$ is diffeomorphic to a 4-manifold obtained from Y by the FS-surgery along the torus $T = p^{-1}(M)$ via the knot $K\#K \subset S^3$.*

To distinguish the diffeomorphism types of $Y_{K\#K}$ one can use the formula of Fintushel and Stern [4] for SW-invariants of a 4-manifold Y after FS-surgery along a torus $T \subset Y$. Recall that this formula can be applied if the SW-invariants of Y are well-defined and a torus T , realizing a non-trivial class $[T] \in H_2(Y)$, is *c-embedded* (the latter means that T lies as a non-singular fiber in a cusp-neighborhood in Y , cf. [4]). Being an algebraic surface of genus ≥ 1 , the double plane Y has well-defined SW-invariants. The conditions on T are also satisfied.

Proposition 1.5. *Assume that X , F and M are like in Proposition 1.2, $[F] \in H_2(X; \mathbb{Z}/2)$ vanishes and $p : Y \rightarrow X$ is like above. Then the torus $T = p^{-1}(M)$ is primitively embedded in Y and therefore $[T] \in H_2(Y)$ is an infinite order class. If, moreover, X , F and M are chosen like in Proposition 1.3, then $T \subset Y$ is c-embedded.*

Recall that the product formula [4]

$$SW_{Y_K} = SW_Y \cdot \Delta_K(t), \quad \text{where } t = \exp(2[T])$$

expresses the Seiberg-Witten invariants (combined in a single polynomial) of the manifold Y_K , obtained by an FS-surgery, in terms of the Seiberg-Witten invariants of Y and the Alexander polynomial, $\Delta_K(t)$, of K .

This formula implies that the basic classes of Y_K can be expressed as $\pm\beta + 2n[T]$, where $\pm\beta \in H_2(Y)$ are the basic classes of Y and $|n| \leq \deg(\Delta_K(t))$, are the degrees of

the non-vanishing monomials in $\Delta_K(t)$. So, if $[T]$ has infinite order, then the manifolds $Y(K, \phi) \cong Y_{K\#K}$ differ from each other by their SW-invariants, and moreover, by the numbers of their basic classes, for an infinite family of knots K , since the number of the basic classes is determined by the number of the terms in $\Delta_{K\#K} = (\Delta_K)^2$ (one can take any family of knots with Alexander polynomials of distinct degrees). \square

1.5. A generalization

More generally, one can produce “fake algebraic curves” under the following conditions.

Theorem 1.6. *Assume that F is a non-singular connected curve in a simply connected complex surface X , which admits a deformation degenerating F into an irreducible curve $F_0 \subset X$, with a singularity of the type X_9 , such that the fundamental group $\pi_1(X \setminus F_0)$ is abelian. Then there exists an infinite family of surfaces $F_{K,\phi} \subset X$ homeomorphic to F and realizing the same homology class as F , having the same fundamental group of the complement, but with the smoothly non-equivalent pairs $(X, F_{K,\phi})$.*

Recall that X_9 -singularity is a point where 4 non-singular branches meet pairwise transversally. Nori’s theorem [7] gives conditions under which $\pi_1(X \setminus F_0)$ must be abelian. For instance, it is so if A_0 has no other singularities except X_9 and $A \circ A > 16$.

Remark 1.2. The claim of Theorem 1.6 holds also if F_0 has a more complicated than X_9 singularity, provided the group $\pi_1(X \setminus F_0)$ is abelian.

2. Commutativity of the fundamental group throughout the knotting

Lemma 2.1. *The assumptions of Proposition 1.2 imply that $\pi_1(X \setminus (F \cup M)) = \pi_1(X \setminus F)$ is cyclic with a generator presented by a loop around F .*

Proof. The Alexander duality in X combined with the exact cohomology sequence of a pair $(X, F \cup M)$ gives

$$H_1(X \setminus (F \cup M)) \cong H^3(X, F \cup M) = H^2(F \cup M)/i^*H^2(X)$$

where $i : F \cup M \rightarrow X$ is the inclusion map. If F is oriented and $F \setminus \partial M$ is connected, then the Mayer-Vietoris Theorem yields $H^2(F \cup M) \cong H^2(F) \cong \mathbb{Z}$, and thus $H_1(X \setminus (F \cup M)) \cong H_1(X \setminus F)$ is cyclic with a generator presented by a loop around F . The same property holds for the fundamental groups of $X \setminus (F \cup M)$ and $X \setminus F$, since they are abelian by the assumption of Proposition 1.2. \square

Proof of Proposition 1.2. Put $X_0 = \text{Cl}(X \setminus U)$. Then $\partial X_0 = \partial U \cong S^1 \times S^2$ and $\partial U \setminus F$ is a deformational retract of $U \setminus (F \cup M)$, so

$$\pi_1(X_0 \setminus F) = \pi_1(X \setminus (F \cup M))$$

Since this group is cyclic and is generated by a loop around F , the inclusion homomorphism $h : \pi_1(\partial U \setminus F) \rightarrow \pi_1(X_0 \setminus F)$ is epimorphic and thus $\pi_1(X_0 \setminus F) = \pi_1(\partial U \setminus F)/k$, where k is the kernel of h .

Applying the Van Kampen theorem to the triad $(X_0 \setminus F, U \setminus F_{K,\phi}, \partial U \setminus F)$, we conclude that $\pi_1(X \setminus F_{K,\phi}) \cong \pi_1(U \setminus F_{K,\phi})/j(k)$, where $j : \pi_1(\partial U \setminus F) \rightarrow \pi_1(U \setminus F_{K,\phi})$ is the inclusion homomorphism. Furthermore, in the splitting

$$\pi_1(U \setminus F_{K,\phi}) \cong \pi_1(S^1 \times (D^3 \setminus f_K)) \cong \mathbb{Z} \times \pi_1(D^3 \setminus f_K)$$

factorization by $j(k)$ kills the first factor \mathbb{Z} and adds some relations to $\pi_1(D^3 \setminus f_K)$, one of which effects to $\pi_1(D^3 \setminus f_K)$ as if we attach a 2-cell along a loop, m_b , going once around the band b_K (to see it, note that factorization by k leaves only one generator of $\pi_1(\partial D^3 \setminus f_K) = \pi_1(S^2 \setminus \{4\text{pts}\})$). Attaching such a 2-cell effects to π_1 as connecting together a pair of the endpoints of f_K , which transforms f_K into an arc (see Figure 2). This arc is unknotted and thus factorization by $j(k)$ makes $\pi_1(D^3 \setminus f_K)$ cyclic and leaves $\pi_1(X \setminus F_{K,\phi})$ isomorphic to $\pi_1(X_0 \setminus F) \cong \pi_1(X \setminus (F \cup M)) \cong \pi_1(X \setminus F)$. \square

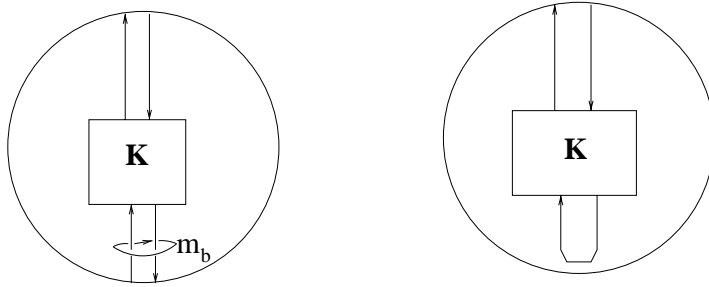


FIGURE 2. Gluing a 2-cell along m_b effects as transforming f_K into an unknotted arc

Proof of Proposition 1.3. All the assumptions of Proposition 1.2 except the last two are obviously satisfied. It is well known that $\mathbb{C}A \setminus \mathbb{R}A$ splits for a maximal nest curve $\mathbb{C}A$ into a pair of connected components permuted by the complex conjugation, and thus, $\mathbb{C}A \setminus \partial M$ is connected, provided $\partial M \not\subseteq \mathbb{R}A$, which is the case for $d \geq 5$. So, it is only left to check that the group $\pi_1(\mathbb{C}P^2 \setminus (\mathbb{C}A \cup M))$ is abelian.

There are several ways to check it. For instance, one can refer to my old work [1] containing computation of the homotopy type of $\mathbb{C}P^2 \setminus (\mathbb{C}A \cup \mathbb{R}P^2)$ and, in particular, of its fundamental group (see also §4 in [3]). This computation concerns a real curve $\mathbb{C}A \subset \mathbb{C}P^2$ if it is an *L-curve*, i.e., $\mathbb{C}A$ can be obtained by a non-singular perturbation from a curve $\mathbb{C}A_0 = \mathbb{C}L_1 \cup \dots \cup \mathbb{C}L_d$ splitting into d real lines, $\mathbb{C}L_i$, in a generic position. The maximal nest curves, $\mathbb{C}A \subset \mathbb{C}P^2$, can be easily constructed as *L-curves*, and the result of [1] gives a presentation $\pi = \pi_1(\mathbb{C}P^2 \setminus (\mathbb{C}A \cup \mathbb{R}P^2)) = \langle a, b \mid a^d b^d = 1 \rangle$, where a, b are represented by loops around the two connected components of $\mathbb{C}A \setminus \mathbb{R}A$. More specifically, a basis point and these loops can be taken on the conic $C = \{x^2 + y^2 + z^2 = 0\} \subset \mathbb{C}P^2$, which have the real point set empty. The group $\pi_1(\mathbb{C}P^2 \setminus (\mathbb{C}A \cup M))$ is obtained from π by adding the relations corresponding to puncturing the components R_i , $0 \leq i \leq k$,

$i \neq 1$, of $\mathbb{R}P^2 \setminus \mathbb{R}A$ (here $d = 2k$ or $d = 2k + 1$). Such a relation (as we puncture R_i) is $a^{d-i}b^i = b^{d-i}a^i = 1$, see [1], or §4 in [3]. A pair of the relations for $i = 2$ and $i = 3$ implies that $a = b$.

The arguments from [1] and [3] relevant to the above calculation are briefly summarized in the Appendix. \square

Remark 2.1. It follows from the proof above that $\pi_1(\mathbb{C}P^2 \setminus (\mathbb{C}A \cup M))$ is not abelian and $\mathbb{C}A \setminus \partial M$ is not connected for a maximal nest quartic, $\mathbb{C}A$.

3. The double surgery in the double covering

Proof of Proposition 1.4. The proof is based on the following two observations. First, we notice that $Y(K, \phi)$ is obtained from Y by a pair of FS-surgeries along the tori parallel to T , then we notice that such pair of surgeries is equivalent to a single FS-surgery along T . The both observations are corollaries of Lemma 2.1 in [5], so, I have to recall first the construction from [4], [5].

An FS-surgery [4] on a 4-manifold X along a torus $T \subset X$, with the self-intersection $T \circ T = 0$, via a knot $K \subset S^3$ is defined as a *fiber sum* $X \#_{T=S^1 \times m_K} S^1 \times M_K$, that is an amalgamated connected sum of X and $S^1 \times M_K$ along the tori T and $S^1 \times m_K \subset S^1 \times M_K$. Here M_K is a 3-manifold obtained by the 0-surgery along K in S^3 , and m_K denotes a meridian of K (which may be seen both in S^3 and in M_K). Such a fiber sum operation can be viewed as a direct product of S^1 and the corresponding 3-dimensional operation, which I call S^1 -*fiber sum*.

More precisely, S^1 -fiber sum $X \#_{K=L} Y$ of oriented 3-manifolds X and Y along oriented framed knots $K \subset X$ and $L \subset Y$ is the manifold obtained by gluing the complements $\text{Cl}(X \setminus N(K))$ and $\text{Cl}(Y \setminus N(L))$ of tubular neighborhoods, $N(K)$, $N(L)$, of K and L via a diffeomorphism $f : \partial N(K) \rightarrow \partial N(L)$ which identifies the longitudes of K with the longitudes of L preserving their orientations, and the meridians of K with the meridians of L reversing the orientations. As it is shown in Lemma 2.1 of [5], tying a knot K in an arc in D^3 can be interpreted as a fiber sum $D^3 \#_{m=m_K} M_K$, where m is a meridian around this arc. The meridians m and m_K are endowed here with the 0-framings (0-framing of a meridian makes sense as a meridian lies in a small 3-disc). To understand this observation, it is useful to view an S^1 -fiber sum with M_K as surgering a tubular neighborhood, $N(m)$, of m and replacing it by the complement, $S^3 \setminus N(K)$ of a tubular neighborhood, $N(K)$, of K , so that the longitudes of m are glued to the meridians of K and the meridians of m to the longitudes of K . The framing of an arc in D^3 is preserved under such a fiber sum, so tying a knot in the band $b \subset D^3$ is equivalent to taking an S^1 -fiber sum with M_K along a meridian m_b around b .

The double covering over D^3 branched along f is a solid torus, $N \cong S^1 \times D^2$, and the pull back of m_b splits into a pair of circles, $m_1, m_2 \subset N$, parallel to $m = S^1 \times \{0\}$. Therefore, $Y(K, \phi)$ is obtained from Y by performing FS-surgery twice, along the tori

$$T_i = S^1 \times m_i \subset p^{-1}(U) \cong S^1 \times N, \quad i = 1, 2$$

The following Lemma implies that this gives the same result as a single FS-surgery along $T = p^{-1}(M)$ via the knot $K\#K$. \square

Lemma 3.1. *For any pair of knots, K_1, K_2 , the manifold*

$$M_{K_1} \#_{m_{K_1}=m_1} N \#_{m_2=m_{K_2}} M_{K_2}$$

obtained by taking an S^1 -fiber sum twice, is diffeomorphic to $N \#_{m=m_K} M_K$, for $K = K_1 \# K_2$, via a diffeomorphism identical on ∂N .

Proof. A solid torus N can be viewed as the complement $N = S^3 - N'$ of an open tubular neighborhood N' of an unknot, so that m, m_1, m_2 represent meridians of this unknot. Taking a fiber sum of S^3 with M_{K_i} along $m_i = m_{K_i}$ is equivalent to knotting N' in S^3 via K_i . So, performing S^1 -fiber sum twice, along m_1 and m_2 , we obtain the same result as after taking fiber sum along m once, via $K = K_1 \# K_2$. \square

Remark 3.1. The above additivity property can be equivalently stated as

$$M_{K_1} \#_{m_{K_1}=m_{K_2}} M_{K_2} \cong M_{K_1 \# K_2}$$

Proof of Proposition 1.5. Lemma 2.1 implies that, in the assumptions of Proposition 1.2, $\pi_1(Y \setminus (F \cup T))$ is a cyclic group with a generator represented by a loop around F . Thus, $\pi_1(Y \setminus T) = 0$ and, by the Alexander duality, $H_3(Y, T) = H^1(Y \setminus T) = 0$, which implies that $[T] \in H_2(Y)$ has infinite order.

To check that T is c-embedded it is enough to observe that there exists a pair of vanishing cycles on T , or more precisely, a pair of D^2 -membranes, $D_1, D_2 \subset Y$, on T , having (-1) -framing and intersecting at a unique point $x \in T$, so that $[\partial D_1], [\partial D_2]$ form a basis of $H_1(T)$. In the setting of Proposition 1.3, $Y \rightarrow \mathbb{CP}^2$ is a double covering branched along a maximal nest curve \mathbb{CA} and T is a connected component of the real part of Y (with respect to a certain real structure on Y lifted from \mathbb{CP}^2). Two nodal degenerations of \mathbb{CA} shown on the top part of Figure 3 give nodal degenerations of the double covering Y .

In the first of the degenerations of \mathbb{CA} , a node appears as an oval O_1 is collapsed into a point. In the second degeneration a crossing-like node can be seen as the fusion point of the ovals O_1 and O_2 . Existence of such degenerations for our explicitly constructed curve \mathbb{CA} is known and trivial. Another simple observation (which is obvious for quartics and thus follows for any maximal nest curve of a higher degree) is that our pair of nodal degenerations can be united into one cuspidal degeneration. This means in particular that the two vanishing cycles in Y intersect transversally at a single point.

Furthermore, our complex vanishing cycles in Y can be chosen conj-invariant. Being a (-2) -sphere, each of such complex cycles is divided by its real pair into a pair of (-1) -discs. Choosing one disc from each pair, we obtain D_1 and D_2 that we need.

It is easy to view these (-2) -spheres and the (-1) -discs explicitly. First, note that R_0 is a (-1) -membrane on \mathbb{CA} and $p^{-1}(R_0)$ is the first of the conj-symmetric vanishing cycles. The (-1) -disk D_1 is any of its halves. Furthermore, there is another (-1) -disk membrane, Q on \mathbb{CA} corresponding to the second nodal degeneration. It can be chosen

conj-invariant and then is split by $Q \cap \mathbb{R}P^2$ into semi-discs $Q = Q_1 \cup Q_2$ permuted by conj. Q_i is bounded by the arcs $Q \cap \mathbb{R}P^2$ and $Q_i \cap \mathbb{C}A$. The disk D_2 is any of the discs $p^{-1}(Q_i)$. \square

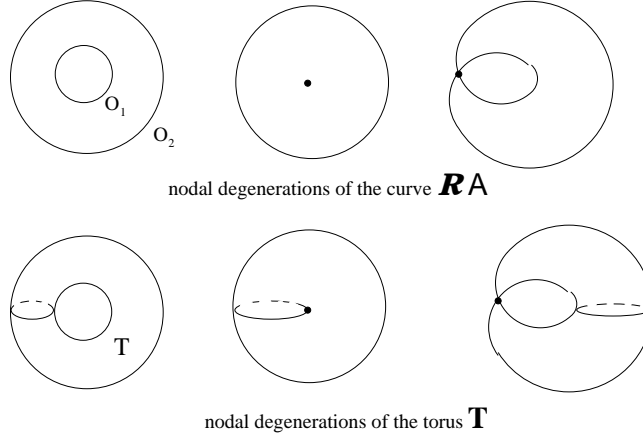


FIGURE 3. Nodal degenerations of $\mathbb{R}A$ providing (-1) -framed D^2 -membranes on T f_K into an unknotted arc

4. The case of d -fold branched covering

Consider as before a maximal nest curve, $\mathbb{C}A \subset \mathbb{C}P^2$, of degree $d \geq 2$, and $\mathbb{C}A_{K,\phi}$ obtained from $\mathbb{C}A$ via an annulus rim-surgery along R_1 , but now let us denote by $p : Y \rightarrow \mathbb{C}P^2$ and $Y(K, \phi) \rightarrow \mathbb{C}P^2$ the d -fold coverings branched along $\mathbb{C}A$ and $\mathbb{C}A_{K,\phi}$ respectively. Consider a d -fold covering $N \rightarrow D^3$ branched along f . The pull-back of m_b consists of d circles, m_1, \dots, m_d , which are cyclically ordered. Using a homeomorphism $(D^3, f) \cong (D^2 \times [0, 1], \{z_1, z_2\} \times [0, 1])$, where $\{z_1, z_2\} \subset \text{Int}(D^2)$, we present N as $F \times [0, 1]$, where F is a sphere with d holes. The circles m_i go around these holes. An annulus rim-surgery in $\mathbb{C}P^2$ along $m_b \times S^1 \subset D^3 \times S^1$, is covered by d copies of FS-surgery along the tori $T_i = m_i \times S^1 \subset N \times S^1$.

The following observation implies that the Fintushel-Stern formula for Seiberg-Witten invariants can be applied in this setting.

Proposition 4.1. *Each of the tori T_i is primitively c -embedded in the complement of the others.*

Proof. A pair of (-1) -disc membranes, D_1^i, D_2^i , on each of T_i is constructed like in the proof of Proposition 1.5. Namely, $p^{-1}(R_0)$ consists of d disks which yield the disks D_1^i , that are glued along $\{\text{pt}\} \times S^1 \subset m_i \times S^1$.

Furthermore, $p^{-1}(Q_1)$ splits also into d disks, Q_1^1, \dots, Q_1^d . Let us choose their orientations induced from a fixed orientation of Q_1 and cyclically order in accord with the ordering of T_i , then the unions $Q_1^i \cup (-Q_1^{i+1})$ provide the required discs D_2^i , which are glued along $m_i \times \{\text{pt}\}$. More precisely, D_1^i are the parts of the components of $p^{-1}(R_0)$ bounded by the intersections of the components with the tori T_i , whereas D_1^i are obtained from $Q_1^i \cup (-Q_1^{i+1})$ by a small shift making them membranes on T_i . \square

Next, we observe that there exists only one linear dependence relation between the classes $[T_i] \in H_2(Y)$.

Proposition 4.2. *The inclusion map $H_2(\bigcup_i T_i) \rightarrow H_2(Y)$ has kernel \mathbb{Z} generated by the relation $\sum_{i=1}^d [T_i] = 0$. Here T_i are oriented uniformly in accord with some fixed orientation of $m_b \times S^1$.*

Proof. It is enough to show that $\pi_1(Y \setminus (N \times S^1)) = 0$, since it implies that $H_3(Y, N \times S^1) \cong H^1(Y \setminus (N \times S^1)) = 0$ and thus the inclusion map $H_2(N \times S^1) \rightarrow H_2(Y)$ is monomorphic. The first inclusion map in the composition $H_2(\bigcup_i T_i) \rightarrow H_2(N \times S^1) \rightarrow H_2(Y)$ that we analyze, is just $H_1(\partial F) \otimes H_1(S^1) \rightarrow H_1(F) \otimes H_1(S^1)$, and has kernel $H_2(F, \partial F) \otimes H_1(S^1) \cong \mathbb{Z}$, as stated in the Proposition.

Now note that $p^{-1}(R_1)$ is a deformational retract (spine) of $N \times S^1$, so it is enough to check the triviality of $\pi_1(Y \setminus (p^{-1}(R_1)))$. This triviality follows from that $\pi_1(\mathbb{C}P^2 \setminus (\mathbb{C}A \cup R_1))$ is \mathbb{Z}/d , with a generator represented by a loop around $\mathbb{C}A$ (say, by the computation in [1] reproduced in the Appendix), and thus $\pi_1(Y \setminus p^{-1}(\mathbb{C}A \cup R_1)) = 0$. \square

Proposition 4.2 together with the Fintushel-Stern formula [4] guarantees that the Seiberg-Witten invariants of $Y(K, \phi)$ are distinct for some sequence of knots K with increasing degrees of $\Delta_K(t)$.

Proof of Theorem 1.6 The case of a primitive class $[F] \in H_2(X)$ is considered in [5]. More precisely, the assumptions in Theorem 1.1 in [5] are satisfied because our condition on the fundamental group yields that $\pi_1(X \setminus F)$ is abelian and thus trivial, existence of an irreducible deformation of F implies that $F \circ F \geq 0$, and X_9 -degeneration guarantees that F is not a rational curve.

If $[F]$ is divisible by $d \geq 2$, then we consider a d -fold covering, $p : Y \rightarrow X$, branched along F and perform an annulus rim-surgery on F along a membrane M defined as follows. Consider a local topological model of the singularity X_9 , defined in \mathbb{C}^2 by the equation $(x^2 + y^2)(x^2 + 2y^2) = 0$, and a model of its perturbation, $(x^2 + y^2 - 4\varepsilon)(x^2 + 2y^2 - \varepsilon) = \delta$, where $\varepsilon, \delta \in \mathbb{R}$, $0 \ll \delta \ll \varepsilon \ll 1$. The real locus of a perturbed singularity contains a pair of ovals which bound together in \mathbb{R}^2 an annulus that we take as M .

The assumptions of Theorem 1.6 imply those of Proposition 1.2. Namely, irreducibility of F_0 implies that $F \setminus \partial M$ is connected and commutativity of $\pi_1(X \setminus F_0)$ implies commutativity of $\pi_1(X \setminus (F \cup M))$ via Van Kampen theorem. Moreover, the singularity X_9 provides the topological picture that was used in the above proof of Theorem 1.1, in the case of d -fold covering. Namely, X_9 yields the both (-1) -disk membranes that were used to show that the Fintushel and Stern formula can be applied to Y . \square

Remark 4.1. Note that to apply the formula [5] it is not required that $b_2^+(Y) > 1$. Nevertheless, it is so, because $b_2^+(Y) \geq d$, which can be proved by observing d linearly independent pairwise orthogonal classes in $H_2(Y)$, having non-negative squares. One of these classes is $[F]$, and the other $(d-1)$ come from $p^{-1}(M)$, due to Proposition 4.2 (each of these $(d-1)$ classes has self-intersection 0).

5. Appendix: The topology of $\mathbb{CP}^2 \setminus (\mathbb{RP}^2 \cup \mathbb{CA})$ for L -curves \mathbb{CA}

Let $\mathbb{CA}_0 = \mathbb{CL}_1 \cup \dots \cup \mathbb{CL}_d \subset \mathbb{CP}^2$ denote the complex point set of a real curve of degree d splitting into d lines, \mathbb{CL}_i . Put $\tilde{V} = C \cap \mathbb{CA}_0$, where C is the conic from the proof of Proposition 1.3. Our first observation is that $C \setminus \tilde{V}$ is a deformational retract of $\mathbb{CP}^2 \setminus (\mathbb{RP}^2 \cup \mathbb{CA}_0)$, and moreover, the latter complement is homeomorphic to $(C \setminus \tilde{V}) \times \text{Int}(\mathbb{D}^2)$. To see it, it suffices to note that $\mathbb{CP}^2 \setminus \mathbb{RP}^2$ is fibered over C with a 2-disc fiber, each fiber being a real semi-line, that is a connected component of $\mathbb{CL} \setminus \mathbb{RL}$ for some real line $\mathbb{CL} \subset \mathbb{CP}^2$, where $\mathbb{RL} = \mathbb{CL} \cap \mathbb{RP}^2$. This fibering maps a semi-line into its intersection point with C .

It is convenient to view the quotient C/conj of the conic C by the complex conjugation as the projective plane, $\widehat{\mathbb{RP}}^2$, dual to $\mathbb{RP}^2 \subset \mathbb{CP}^2$, since each real line, \mathbb{CL} , intersects C in a pair of conjugated points. If we let $V = \{l_1, \dots, l_d\} \subset \widehat{\mathbb{RP}}^2$ denote the set of points l_i dual to the lines $\mathbb{RL}_i \subset \mathbb{RP}^2$, then $\tilde{V} = q^{-1}(V)$, where $q: C \rightarrow C/\text{conj}$ is the quotient map.

The information about a perturbation of \mathbb{CA}_0 is encoded in a *genetic graph of a perturbation*, $\Gamma \subset \widehat{\mathbb{RP}}^2$. The graph Γ is a complete graph with the vertex set V , whose edges are line segments. Note that there exist two topologically distinct perturbations of a real node of \mathbb{RA}_0 at $p_{ij} = \mathbb{RL}_i \cap \mathbb{RL}_j$, as well as there exist two line segments in $\widehat{\mathbb{RP}}^2$ connecting the vertices $l_i, l_j \in V$. Let \mathbb{RA} denotes a real curve obtained from \mathbb{RA}_0 by a sufficiently small perturbation. Then the edge of Γ connecting l_i and l_j contains the points dual to those lines passing through $p_{i,j}$ which do not intersect \mathbb{RA} locally, in a small neighborhood of $p_{i,j}$.

The complement $\mathbb{CP}^2 \setminus (\mathbb{CA} \cup \mathbb{RP}^2)$ turns out to be homotopy equivalent to a 2-complex obtained from $C \setminus \tilde{V}$ by adding 2-cells glued along a figure-eight shaped loops along the edges of $\tilde{\Gamma} = q^{-1}(\Gamma) \subset C$. Such 2-cells identify pairwise certain generators of $\pi_1(C \setminus \tilde{V})$ “along the edges” of $\tilde{\Gamma}$ (cf. [3] for details). This easily implies that the group $\pi_1(\mathbb{CP}^2 \setminus (\mathbb{CA} \cup \mathbb{RP}^2))$ is generated by a pair of elements, a and b , represented by a pair of loops in $C \setminus \tilde{V}$ around a pair of conjugated vertices of \tilde{V} .

For example, for a maximal nest curve, the graph Γ is contained in an affine part of $\widehat{\mathbb{RP}}^2$, i.e., has no common points with some line in $\widehat{\mathbb{RP}}^2$, namely, with a line dual to a point inside the inner oval of the nest. Therefore, the graph $\tilde{\Gamma}$ splits into two connected components separated by a big circle in C . A loop around any vertex of \tilde{V} from one of these components represents a , and a loop around a vertex from the other component represents b . It is trivial to observe also the relation $a^d b^d = 1$ (which is indeed a unique relation in the case of maximal nest curves).

As we puncture \mathbb{RP}^2 at a point $x \in \mathbb{RP}^2 \setminus \mathbb{RA}_0$, we attach a 2-cell to $C \setminus \tilde{V}$ along the big circle $S_x \subset C$ dual to x . If x moves across a line \mathbb{RL}_i , then S_x moves across the pair of points $q^{-1}(l_i)$. Since a small perturbation and puncturing are located at distinct points of \mathbb{CP}^2 and can be done independently, it is not difficult to see that if we choose $x \in R_i$ (in the case of a maximal nest curve \mathbb{CA}), then the big circle S_x cuts C into the hemispheres, one of which

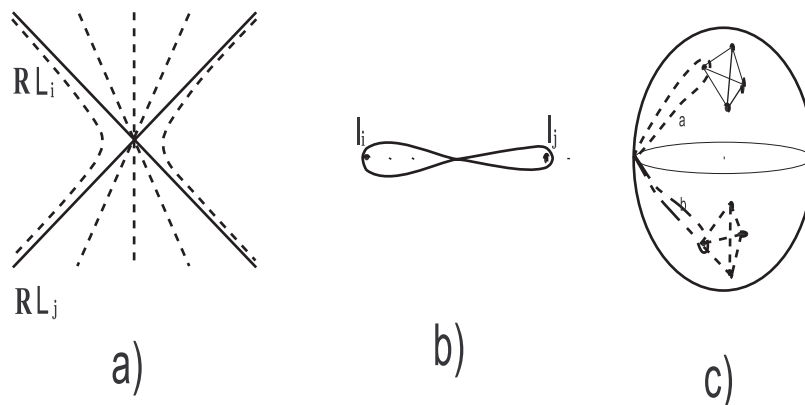


FIGURE 4. a) A perturbation of a real node; the dashed lines are dual to the points of an edge of Γ ; b) A figure-eight loop along an edge of Γ ; c) The loops in $C \setminus \tilde{V}$ representing generators “a” and “b”

contains i vertices from one component of $\tilde{\Gamma}$ and $d - i$ vertices from the other component. This gives relations $a^i b^{d-i} = a^{d-i} b^i = 1$.

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