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Remarks on the Paper “on the Commutant of the Ideal Centre”

Şafak Alpay, Bahri Turan

In memory of Yunus Aran (1976 - 2000)

Abstract

We continue with the work started in [4] and give a new sufficient condition on Riesz spaces having topologically full centres for $Z^\sim(E)_C = Orth(E^\sim)$ to hold.

If E is a Riesz space E^\sim , the order dual of E will be the Riesz space of all order bounded linear functionals on E . Riesz spaces considered in this note are assumed to have separating order duals. $Z(E)$ will denote the ideal centre, $Orth(E)$, will denote the orthomorphisms of E . If E is a topological Riesz space E' will denote continuous dual of E . When $T : E \rightarrow F$ is an order bounded operator between two Riesz spaces, the adjoint of T carries F^\sim into E^\sim and it will be denoted by T^\sim . In all undefined terminology concerning Riesz spaces we will adhere to the definitions in [1], [5] and [8].

When the order dual E^\sim separates the points of the Riesz space E , an order bounded operator $T : E \rightarrow E$ is an orthomorphism if and only if its adjoint $T^\sim : E^\sim \rightarrow E^\sim$ is an orthomorphism. Moreover, the operator $\psi : Orth(E) \rightarrow Orth(E^\sim); \psi(T) = T^\sim$ is a one to one Riesz homomorphism [1]. The image under ψ of the centre $Z(E)$ will be denoted by $Z^\sim(E)$. $Z^\sim(E)$ is a Riesz subspace of $Z(E^\sim)$.

Definition A Riesz space E , is said to have topologically full centre if, for each pair x, y in E with $0 \leq y \leq x$, there exists a net (π_α) in $Z(E)$ with $0 \leq \pi_\alpha \leq I$ for each α , such that $\pi_\alpha x \rightarrow y$ in $\sigma(E, E^\sim)$.

Banach lattices with topologically full centre were initiated in [7]. These spaces were also studied in [2],[3], [4] and [6]. The class of Riesz spaces and the class of Banach spaces have topologically full centres are quite large. σ -Dedekind complete Riesz spaces have topologically full centres. However, not all Riesz spaces have topologically full centres.

Order bounded maps on the Riesz space E will be denoted by $L_b(E)$. $Z(E)_C$ will denote the commutant of $Z(E)$ in $L_b(E)$. That is, $Z(E)_C = \{T \in L_b(E) : T\pi = \pi T \text{ for each } \pi \in Z(E)\}$. The Riesz space $Orth(E)$ under composition is an Archimedean f -algebra and therefore it is commutative. Hence $Orth(E) \subset Z(E)_C$.

We have studied the commutant $Z(E)_C$ of the ideal centre $Z(E)$ in the order bounded operators $L_b(E)$ [4]. If E is a Riesz space with topologically full centre, we have identified $Z(E)_C$ with $\text{Orth}(E)$.

If E has topologically full centre, it was claimed that $Z^\sim(E)_C = \text{Orth}(E^\sim)$. However, as Arenson has pointed out, part of the proof of this claim contains an error. If $E = C(K)$, then we can embed E' into $E''' = C(K)'''$ in two different ways. One of these embeddings is the usual embedding of a Banach space into its bidual as: $\mu \in E' \rightarrow \hat{\mu} \in E'''$. Let \hat{E}' denote the image of E' in E''' . For $\psi \in E'''$, we consider $\mu = \psi|_{E \in \hat{E}'}$. For each $\psi \in E'''$, $\psi - \hat{\mu} \in E^\circ \subset E'''$ with $\mu = \psi|_E$. Thus, $\psi = (\psi - \hat{\mu}) + \hat{\mu}$ implies that $E''' = \hat{E}' \oplus E^\circ$. The correspondence $\psi \rightarrow \hat{\mu}$ is a positive operator which fails to be a lattice homomorphism.

On the other hand, \hat{E}' can be identified with the space of order continuous linear functionals on $E'' = C(K)''$. Consequently, \hat{E}' is a band in E''' and there exists an order projection $P : E''' \rightarrow \hat{E}'$. P is an orthomorphism and $E''' = \hat{E}' \oplus (I - P)E'''$. However, $E^\circ \neq (I - P)E'''$ and $P\psi \neq \psi|_E$ as it was erroneously claimed in [4].

The next example of Arenson's (private communication) explains the situation even better.

Example: (Arenson) Let K be a compact Hausdorff space with no isolated points and E be $C(K)$. Then $Z(E) = E$ and $E^\sim = Z(E)'$ is the space of measures on K . If Q is the Stone compact space of the Banach lattice $Z(E)'$, we identify $Z(E^\sim)$ with $C(Q)$. Since $Z(E)$ and $Z^\sim(E)$ are isometrically isomorphic, we are able to identify $Z^\sim(E)$ with $C(K)$.

Let us note that $Z(E^\sim) = C(Q)$ and $Z^\sim(E)'' = C(K)''$. Therefore we have $Z(E^\sim)' = C(Q)' = C(K)''' = Z^\sim(E)'''$. Let j be the natural embedding of $Z^\sim(E)' = C(K)'$ into $C(K)''' = Z(E^\sim)'$ and let $H_1 = j(Z^\sim(E)')$, $H_2 = H_1^d$. H_1 is a band of $Z(E^\sim)'$ as $Z^\sim(E)'$ is an AL-space. Therefore $Z(E^\sim)' = C(Q)' = H_1 \oplus H_2$. It is well known that H_1 is the class of order continuous functionals on $C(Q)$ and therefore:

- (1) If $\mu \in H_1$ then the support of μ is a closed and open subset of Q ;
- (2) If the support of $\mu \in C(Q)'$ is nowhere dense then $\mu \in H_2$.

Under this circumstances $\{Z^\sim(E)^0\}^d = \{0\}$ and $P = 0$. To see this, let $S(\mu) = j(\mu|_{Z^\sim(E)}) : H_2 \rightarrow H_1$ be the restriction map. If ϑ is a nonzero measure in H_2 then the measure $\mu = \vartheta - S(\vartheta)$ is in $Z^\sim(E)^0$ and $|\mu| \wedge |\vartheta| = |\vartheta| \neq 0$. Therefore $P(\vartheta) = 0$. If μ is a non-zero measure in H_1 , then by the following lemma, there exists a measure $\vartheta \in H_2$ with $S(\vartheta) = \mu$. The measure $\eta = \vartheta - \mu$ is an element of $Z^\sim(E)^0$ and $|\eta| \wedge |\mu| = |\mu| \neq 0$.

Therefore $P(\mu) = 0$.

Let us note that if Q_1 is a nowhere dense closed subset of Q then $C(Q_1)'$ (considered as the space of measures on Q whose supports are contained in Q_1) is contained in H_2 . To complete the proof of $(Z^\sim(E)^\circ)^d = 0$ we only need to prove the following lemma.

Lemma 1. *There exists a nowhere dense closed subset Q_1 of Q such that $S(C(Q_1)') = H_1$.*

Proof. Let $\varphi : Q \rightarrow K$ be the continuous surjection which gives rise the natural embedding $\pi \rightarrow \pi \cdot \varphi$ of $C(K)$ into $C(Q)$.

For each $t \in K$, let δ_t be the point evaluation at t . i.e, δ_t is : $\pi \rightarrow \pi(t)$ on $C(K)$. Similarly, for each $q \in Q$, let Δ_q be the functional $\pi \rightarrow \pi(q)$ on $C(Q)$. If $t = \varphi(q)$, then $\Delta_q |_{C(K)} = \delta_t$.

For each $t \in K$, there is a unique point in Q , say $\psi(t)$, such that $j(\delta_k) = \Delta_{\psi(t)} \cdot \psi(t)$ is an isolated point of Q and $\psi : K \rightarrow Q$ is discontinuous and maps K onto an open subset $V = \psi(K)$ of Q . Let $Q_1 = \overline{V} \setminus V$. Q_1 is nowhere dense and closed. To prove the lemma, it suffices to show that $\varphi(Q_1) = K$. Let $t \in K$. As there are no isolated points in K , there exists a net $\{t_\alpha\}, t_\alpha \neq t$ for each α in K with $t = \lim_\alpha t_\alpha$. Let q be a cluster point of the net $\{\psi(t_\alpha)\}$. Then $q \in Q_1$ and $\varphi(q) = t$ as $t_\alpha = \varphi\{\psi(t_\alpha)\}$ for each α . \square

Let us note that the conclusion $Z^\sim(E)_C = Orth(E^\sim)$ remains valid for Arenson's example. The details are below.

We now give a sufficient condition for $Z^\sim(E)_C = Orth(E^\sim)$. We first give a lemma that will be needed.

Lemma 2. *Let E be a Riesz space with topologically full centre and satisfying $(E^\sim)^\sim = (E^\sim)_n^\sim$. Then the bilinear map*

$$(f, F) \rightarrow \psi_{f,F} \text{ of } E^\sim \times (E^\sim)^\sim \rightarrow Z^\sim(E)^\sim \text{ defined by } \psi_{f,F}(\tilde{\pi}) = F(\tilde{\pi}f)$$

is a bi-lattice homomorphism.

Proof. For each $f \in E_+^\sim$, the map $\psi_f : (E^\sim)^\sim \rightarrow Z^\sim(E)^\sim$ defined by $F \rightarrow \psi_{f,F}$ is positive. Hence we have $\psi_f(F)^+ \leq \psi_f(F^+)$ for each $F \in (E^\sim)^\sim$. Let $\tilde{\pi} \in Z^\sim(E)_+$ be arbitrary, then

$$\psi_f(F^+)(\tilde{\pi}) = \psi_{f,F^+}(\tilde{\pi}) = F^+(\tilde{\pi}f) = \sup\{F(g) : 0 \leq g \leq \tilde{\pi}f\}$$

If $0 \leq g \leq \tilde{\pi}f$, we claim there exists $\{\pi_\alpha\}$ in $Z(E)$ satisfying $0 \leq \pi_\alpha \leq I$ for each α and $\tilde{\pi}_\alpha(\tilde{\pi}f) \rightarrow g$ in $\sigma(E^\sim, (E^\sim)^\sim)$. As E^\sim is Dedekind complete, we can find $S \in Z(E^\sim)$

with $0 \leq S \leq I$ and $S(\tilde{\pi}f) = g$. The Arens homomorphism $m : Z(E)'' \rightarrow Z(E^\sim)$ is surjective and continuous when the domain is equipped with $\sigma(Z(E)'', Z(E)')$ and the range has the $\sigma(E^\sim, (E^\sim)_n^\sim)$ operator topology [6]. Therefore there exists F in $Z(E)''$ with $0 \leq F \leq I$ satisfying $m(F) = S$. Using the fact that $Z(E)$ is $\sigma(Z(E)'', Z(E)')$ dense in $Z(E)''$, we can find a net $\{\pi_\alpha\}$ in $Z(E)$ satisfying $0 \leq \pi_\alpha \leq I$ for each α and $\pi_\alpha \rightarrow F$ in $\sigma(Z(E)'', Z(E)')$. Continuity of the map $m : Z(E)'' \rightarrow Z(E^\sim)$ imply that $m(\pi_\alpha) = \tilde{\pi}_\alpha \rightarrow m(F) = S$ in $\sigma(E^\sim, (E^\sim)_n^\sim)$ operator topology. This is to say $G(\tilde{\pi}_\alpha h) \rightarrow G(S h)$ for each $h \in E^\sim$ and $G \in (E^\sim)_n^\sim$. Thus $\tilde{\pi}_\alpha(\pi f) \rightarrow g$ in $\sigma(E^\sim, (E^\sim)_n^\sim)$.

$$0 \leq \tilde{\pi}_\alpha(\tilde{\pi}f) \leq \tilde{\pi}(f) \text{ for each } \alpha, \text{ so that } F(\tilde{\pi}_\alpha(\tilde{\pi}f)) \leq \psi_f(F)^+(\tilde{\pi})$$

which yields

$$F(g) \leq \psi_f(F)^+ \text{ for each } g \text{ with } 0 \leq g \leq \tilde{\pi}f. \text{ Hence } \psi_f(F^+) \leq \psi_f(F)^+.$$

We now show that $\psi_F : E^\sim \rightarrow Z^\sim(E)^\sim$ is a lattice homomorphism for an arbitrary F in $(E^\sim)_+^\sim$. Let $f \wedge g = 0$ in E^\sim . As I is a strong order unit in $Z^\sim(E)$, it suffices to show $[\psi_F(f) \wedge \psi_F(g)](I) = 0$.

$$\begin{aligned} [\psi_F(f) \wedge \psi_F(g)](I) &= (\psi_{f,F} \wedge \psi_{g,F})(I) \\ &= \text{inf}\{\psi_{f,F}(\pi_1) + \psi_{g,F}(\pi_2) : \pi_1, \pi_2 \in Z^\sim(E)_+; \pi_1 + \pi_2 = I\} \\ &= \text{inf}\{F(\pi_1 f) + F(\pi_2 g) : \pi_1, \pi_2 \in Z^\sim(E)_+; \pi_1 + \pi_2 = I\} \end{aligned}$$

As E^\sim is Dedekind complete, the principal band generated by f, B_f is a projection band and let $P_f : E^\sim \rightarrow B_f$ be this projection. $P_f \in Z(E^\sim)$, $P_f(g) = 0$, $(I - P_f)(f) = 0$ and $(I - P_f) + P_f = I$. Arguing as above, we can find a net (π_α) in $Z(E)$, $0 \leq \pi_\alpha \leq I$ and $\tilde{\pi}_\alpha \rightarrow P_f$ in $\sigma(E^\sim, (E^\sim)_n^\sim)$ operator topology.

Thus,

$$\begin{aligned} [\psi_F(f) \wedge \psi_F(g)](I) &\leq F(I - \tilde{\pi}_\alpha)f + F(\tilde{\pi}_\alpha g) \text{ for each } \alpha \\ &\leq F(I - P_f)f + F(P_f g) = 0. \end{aligned}$$

Proposition. *Let E be a Riesz space with $(E^\sim)^\sim = (E^\sim)_n^\sim$ and having topologically full centre. Then $Z^\sim(E)_C = \text{Orth}(E^\sim)$.*

Proof. Let $T \in Z^\sim(E)_C$ be arbitrary; let $f, g \in E^\sim$ satisfying $f \perp g$. For each F, G in $(E^\sim)^\sim$, we have $\psi_{f,F} \perp \psi_{g,G}$ [3].

Thus for $f \in E^\sim$ and $F \in (E^\sim)^\sim$,

$$\psi_{Tf,F}(\tilde{\pi}) = F(\tilde{\pi}(Tf)) = F(T(\tilde{\pi}f)) = \tilde{T}(F)(\tilde{\pi}f) = \psi_{f,\tilde{T}F}$$

which yields $|\psi_{Tf,F}| \wedge |\psi_{g,F}| = \psi_{|Tf| \wedge |g|, F} = 0$. Therefore $F(|Tf| \wedge |g|) = 0$ for each $F \in (E^\sim)_+^\sim$ which gives $Tf \perp g$ and T is an orthomorphism. \square

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