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## Formula for the Highly Regularized Trace of the Sturm-Liouville Operator with Unbounded Operator Coefficients Having Singularity

*İ. Albayrak, O. Baykal, E. Gül*

### Abstract

In this work, a formula for the  $n^{th}$  regularized trace of the Sturm-Liouville operator with unbounded operator coefficients having singularity is obtained

### 1. Introduction

The regularized trace of the scalar differential operators has been calculated by I. M. Gelfand and B. M. Levitan [1], L. A. Dikiy [2], C. J. Halberg and V. A. Kramer [3] among other works. The list of published work on this subject can be found in B. M. Levitan and I. S. Sargsyan [4], I. C. Fulton and S. A. Pruess [5]. The regularized trace of differential operators with unbounded operator coefficients has been computed in [6, 8, 9, 10].

In this study, the  $n$  ( $n \in \mathbb{N}$ ) regularized trace of Sturm-Liouville operator with unbounded operator coefficients having singularity has been calculated in the interval  $[0,1]$ .

Let  $H$  be a separable Hilbert Space. Denote by  $H_1 = L_2(H : [0, 1])$  the Hilbert space of all vector-valued measurable functions  $y(t)$  ( $t \in [0, 1]$ ) with values in  $H$  and such that

$$\int_0^1 \|y(t)\|^2 dt < \infty.$$

The scalar product vectors  $y(t), z(t)$  in  $H_1$  is defined by

$$(y, z)_1 = \int_0^1 (y(t), z(t)) dt.$$

Let us write

$$l_0(y) = -y''(x) + \frac{v^2 - \frac{1}{4}}{x^2} y(x) + Ay(x)$$

$$l(y) = -y''(x) + \frac{v^2 - \frac{1}{4}}{x^2} y(x) + Ay(x) + Q(x)y(x), \quad v \geq \frac{1}{2},$$

where  $A$  is an operator from  $D(A)$  to  $H$  which satisfies the following conditions such that  $\overline{D(A)} = H$ :

$$A = A^* \geq I, \quad A^{-1} \in \sigma_\infty(H).$$

Expressions  $l_0(y)$  and  $l(y)$  form two operators  $L$  and  $L_0$  in space  $H_1$ :

$D(L'_0) = \{y(x) \in H_1 : y(x) \in D(A), y''(x), Ay(x)$  continuous in  $[0,1]$  according to norm in  $H$   $y(0) = y(1) = 0$  and  $l_0(y) \in H_1\}$

$$L'_0 y = l_0(y), \quad y \in D(L'_0).$$

Let us denote  $\overline{L_0} = L_0$ , where the overbar symbol shows closure of the operator. It can be shown that  $L_0$  is a self-adjoint operator in  $H_1$ . Operator  $L = L_0 + Q(x)$  ( $Q(x) = Q^*(x)$  and is bounded in  $H_1$ : see condition 2) ) with domain  $D(L) = D(L_0)$  is a self-adjoint operator in  $H_1$ .

Let us accept that the operator function  $Q(x)$  in the expression  $l(y)$  satisfies the following conditions:

- 1)  $Q(x)$  has a weak derivative of the second order in  $[0,1]$ . For  $\forall x \in [0,1]$ ,  $Q^{(i)}(x)$  ( $i=0,1,2$ ) are self-adjoint operator from  $H$  to  $H$  and  $A^{n-1}Q^{(i)}(x) \in \sigma_1(H)$ ;
- 2) The functions  $\|A^{n-1}Q^{(i)}(x)\|_{\sigma_1(H)}$  ( $i=0,1,2$ ) are bounded and measurable in  $[0,1]$ ;

3) For  $\forall f \in H$ ,  $\int_0^1 (Q(x)f, f)dx = 0$ ;

4) For  $\forall f \in H$  and  $x \in [0, \delta]$ , there is an operator  $B \in \sigma_1(H)$  such that

$$|(Q^{(i)}(x)f, f)| \leq |(Bf, f)|.$$

Here,  $\sigma_1(H)$  is the space of kernel operators from  $H$  to  $H$  as in [11], and  $(., .)$  denotes the inner product in  $H$ . We denote the inner product by  $(., .)_1$  and the norm of operator by  $\|.\|_1$  in  $H_1$ .

In this study, we will use the following inequalities proved in [11]:

$$|tr B_1| \leq \|B_1\|_{\sigma_1(H)}$$

$$\|B_1 B_2\|_{\sigma_1(H)} \leq \|B_1\|_{\sigma_1(H)} \cdot \|B_2\|$$

$$\|B_2 B_1\|_{\sigma_1(H)} \leq \|B_1\|_{\sigma_1(H)} \cdot \|B_2\|,$$

where  $B_1 \in \sigma_1(H)$  and  $B_2 \in L(H, H)$ .

Note that the operator  $L_0$  has a pure discrete spectrum. Let  $\mu_1 \leq \mu_2 \leq \dots$  be the eigenvalues of this operator and  $\psi_1(x), \psi_2(x), \dots$  be the orthonormal eigenvectors corresponding to these eigenvalues. Here, every eigenvalue is repeated according to multiplicity number. Since  $Q$  is a self-adjoint bounded operator from  $H_1$  to  $H_1$ ,  $L = L_0 + Q$  is a self-adjoint operator in  $H_1$  which satisfies  $D(L) = D(L_0)$  and it has a pure discrete spectrum. Let  $\lambda_1 \leq \lambda_2 \leq \dots$  be the eigenvalues of this operator. Moreover, let  $\gamma_1 \leq \gamma_2 \leq \dots$  be the eigenvalues of the operator  $A$  from  $D(A)$  to  $H$  and  $\varphi_1, \varphi_2, \dots$  be orthonormal eigenvectors corresponding to these eigenvalues.

If  $\gamma_i \sim ai^\alpha$  while  $i \rightarrow \infty$  ( $a > 0, \alpha > 2$ ), then it is proved in [7], [8] ( and, respectively, for  $\gamma = 1/2; \gamma \geq 1/2$ ) that

$$\mu_m, \lambda_m \sim dm^{\frac{2\alpha}{2+\alpha}} \text{ while } m \rightarrow \infty, (d > 0) \tag{1}$$

is satisfied. By using this formula, we can show that the sequence  $\{\mu_m\}_{m=1}^\infty$  has a subsequence  $\{\mu_{m_p}\}_{p=1}^\infty$  such that

$$\mu_k - \mu_{m_p} \geq d_1 \left( k^{\frac{2\alpha}{2+\alpha}} - m_p^{\frac{2\alpha}{2+\alpha}} \right), \quad (k = m_p, m_p + 1, m_p + 2, \dots), \quad (2)$$

where  $d_1$  is a positive constant. Let  $R_\lambda^0, R_\lambda$  be the resolvents of  $L_0$  and  $L$ , respectively. From (1), if  $\alpha > 2$  and  $\lambda \neq \lambda_k, \mu_k$  ( $k = 1, 2, \dots$ ) then the series  $\sum_{k=1}^{\infty} |\mu_k - \lambda|^{-1}$  and  $\sum_{k=1}^{\infty} |\lambda_k - \lambda|^{-1}$  are convergent. Therefore  $R_\lambda^0$  and  $R_\lambda$  are the kernel operators. In this case, from [11] it is known that

$$tr(R_\lambda - R_\lambda^0) = tr R_\lambda - tr R_\lambda^0 = \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k - \lambda} - \frac{1}{\mu_k - \lambda} \right).$$

If we multiply this equation with  $\frac{\lambda^n}{2\pi i}$  and integrate along the circle

$$|\lambda| = b_p = 2^{-1}(\mu_{m_p} + \mu_{m_p+1})$$

then we obtain

$$\frac{1}{2\pi i} \int_{|\lambda|=b_p} \lambda^n tr(R_\lambda - R_\lambda^0) d\lambda = - \sum_{k=1}^{\infty} \left\{ \frac{1}{2\pi i} \int_{|\lambda|=b_p} \frac{\lambda^n}{\lambda - \lambda_k} d\lambda - \frac{1}{2\pi i} \int_{|\lambda|=b_p} \frac{\lambda^n}{\lambda - \mu_k} d\lambda \right\}. \quad (3)$$

It can be shown that for large values of  $p$ ,

$$\{\lambda_k, \mu_k\}_{k=1}^{m_p} \subset K(0, b_p) = \{\lambda : |\lambda| < b_p\}. \quad (4)$$

Here,  $\lambda_k, \mu_k \notin \overline{K(0, b_p)} = \{\lambda : |\lambda| \leq b_p\}$ , ( $k \geq m_p + 1$ ). From (3) and (4) we find

$$\sum_{k=1}^{m_p} (\lambda_k^n - \mu_k^n) = - \frac{1}{2\pi i} \int_{|\lambda|=b_p} \lambda^n tr(R_\lambda - R_\lambda^0) d\lambda. \quad (5)$$

It is known that  $R_\lambda = R_\lambda^0 - R_\lambda Q R_\lambda^0$ . From here, for any natural number  $N \geq 2$ , we obtain

$$R_\lambda - R_\lambda^0 = \sum_{j=1}^N (-1)^j R_\lambda^0 (QR_\lambda^0)^j + (-1)^{N+1} R_\lambda (QR_\lambda^0)^{N+1}.$$

If we substitute this expression into (5) then we find

$$\begin{aligned} \sum_{k=1}^{m_p} (\lambda_k^n - \mu_k^n) &= \frac{1}{2\pi i} \int_{|\lambda|=b_p} \lambda^n \operatorname{tr} \left[ \sum_{j=1}^N (-1)^{j+1} R_\lambda^0 (QR_\lambda^0)^j + (-1)^N R_\lambda (QR_\lambda^0)^{N+1} \right] d\lambda \\ &= \sum_{j=1}^N B_{pj} + B_p^{(N)}. \end{aligned} \quad (6)$$

Here,

$$B_{pj} = \frac{(-1)^{j+1}}{2\pi i} \int_{|\lambda|=b_p} \lambda^n \operatorname{tr} [R_\lambda^0 (QR_\lambda^0)^j] d\lambda, \quad (j = 1, 2, \dots, N) \quad (7)$$

$$B_p^{(N)} = \frac{(-1)^N}{2\pi i} \int_{|\lambda|=b_p} \lambda^n \operatorname{tr} [R_\lambda (QR_\lambda^0)^{N+1}] d\lambda. \quad (8)$$

For every natural number  $j$ , it can be shown that the operator function  $(QR_\lambda^0)^j$  is analytic according to the norm in  $\sigma_1(H_1)$  in the resolvent region  $\rho(L_0)$  of the operator  $L_0$ . Moreover,

$$\operatorname{tr} [R_\lambda^0 (QR_\lambda^0)^j] = \operatorname{tr} [(QR_\lambda^0)^{j-1} Q (R_\lambda^0)^2] = \operatorname{tr} \left[ (QR_\lambda^0)^{j-1} \frac{d}{d\lambda} (QR_\lambda^0) \right],$$

$$\operatorname{tr} \left\{ \frac{d}{d\lambda} [(QR_\lambda^0)^j] \right\} = j \operatorname{tr} \left[ (QR_\lambda^0)^{j-1} \frac{d}{d\lambda} (QR_\lambda^0) \right].$$

From the last two relations one obtains

$$\text{tr}[R_\lambda^0(QR_\lambda^0)^j] = \frac{1}{j} \text{tr} \left\{ \frac{d}{d\lambda} [(QR_\lambda^0)^j] \right\}.$$

If this expression is substituted in (7), then

$$\begin{aligned} B_{pj} &= \frac{(-1)^{j+1}}{2\pi i j} \int_{|\lambda|=b_p} \lambda^n \text{tr} \left\{ \frac{d}{d\lambda} [(QR_\lambda^0)^j] \right\} d\lambda \\ &= \frac{(-1)^{j+1}}{2\pi i j} \int_{|\lambda|=b_p} \text{tr} \left\{ \frac{d}{d\lambda} [\lambda^n (QR_\lambda^0)^j] - n\lambda^{n-1} (QR_\lambda^0)^j \right\} d\lambda \end{aligned}$$

is found. So if we note that

$$\text{tr} \left\{ \frac{d}{d\lambda} [\lambda^n (QR_\lambda^0)^j] \right\} = \frac{d}{d\lambda} \{ \text{tr} [\lambda^n (QR_\lambda^0)^j] \}$$

then

$$B_{pj} = \frac{(-1)^j n}{2\pi i j} \int_{|\lambda|=b_p} \lambda^{n-1} \text{tr} [(QR_\lambda^0)^j] d\lambda + \frac{(-1)^{j+1}}{2\pi i j} \int_{|\lambda|=b_p} \frac{d}{d\lambda} \{ \text{tr} [\lambda^n (QR_\lambda^0)^j] \} d\lambda$$

is obtained. On the other hand, by using (4) we obtain

$$\int_{|\lambda|=b_p} \frac{d}{d\lambda} \{ \text{tr} [\lambda^n (QR_\lambda^0)^j] \} d\lambda = 0.$$

Therefore,

$$B_{pj} = \frac{(-1)^j n}{2\pi i j} \int_{|\lambda|=b_p} \lambda^{n-1} \text{tr} [(QR_\lambda^0)^j] d\lambda \quad (9)$$

or

$$B_{pj} = (-1)^j j^{-1} n \sum_{k=1}^{m_p} Res[\lambda^{n-1} tr(QR_\lambda^0)^j]. \quad (10)$$

From (6) and (10), we obtain

$$\sum_{k=1}^{m_p} \left\{ \lambda_k^n - \mu_k^n - n \sum_{j=2}^N (-1)^j j^{-1} Res_{\lambda=\mu_k} [\lambda^{n-1} tr(QR_\lambda^0)^j] \right\} = B_{p1} + B_{pN}. \quad (11)$$

Let  $\beta_1 < \beta_2 < \dots$ , be the non-negative roots of the Bessel function  $J_\nu(x)$ . The eigenvalues of the operator  $L_0$  and its eigenvectors corresponding to these eigenvalues are of the respective forms

$$\beta_q^2 + \gamma_i, \quad (q, i = 1, 2, \dots)$$

$$\sqrt{2x} \frac{J_\nu(\beta_q x)}{J_{\nu+1}(\beta_q)} \varphi_i, \quad (q, i = 1, 2, \dots).$$

Therefore, the eigenvalues  $\mu_1 \leq \mu_2 \leq \dots$  of the operator  $L_0$

$$\mu_k = \beta_{q_k}^2 + \gamma_{i_k} \quad (k = 1, 2, \dots)$$

and the orthonormal eigenvectors  $\psi_k(x)$  corresponding to these eigenvalues can be in the form

$$\psi_k(x) = \sqrt{2x} \frac{J_\nu(\beta_{q_k} x)}{J_{\nu+1}(\beta_{q_k})} \varphi_{i_k} \quad (k = 1, 2, \dots).$$

From formula (9) we have

$$B_{p1} = \frac{-n}{2\pi i} \int_{|\lambda|=b_p} \lambda^{n-1} tr[(QR_\lambda^0)] d\lambda.$$



Moreover,  $tr(QR_\lambda^0) = \sum_{k=1}^{\infty} (QR_\lambda^0 \psi_k, \psi_k)_1 = \sum_{k=1}^{\infty} (\mu_k - \lambda)^{-1} (Q\psi_k, \psi_k)_1$ .

Hence,

$$\begin{aligned}
 B_{p1} &= n \sum_{k=1}^{\infty} (Q\psi_k, \psi_k)_1 \frac{1}{2\pi i} \int_{|\lambda|=b_p} \frac{\lambda^{n-1}}{\lambda - \mu_k} d\lambda \\
 &= n \sum_{k=1}^{m_p} \mu_k^{n-1} (Q\psi_k, \psi_k)_1 \\
 &= n \sum_{k=1}^{m_p} (\beta_{q_k}^2 + \gamma_{i_k})^{n-1} (Q\psi_k, \psi_k)_1 \\
 &= n \sum_{k=1}^{m_p} \sum_{l=0}^{n-1} C_{n-1}^p \beta_{q_k}^{2(n-1-l)} \gamma_{i_k}^l (Q\psi_k, \psi_k)_1 \\
 &= n \sum_{k=1}^{m_p} \sum_{l=0}^{n-2} C_{n-1}^l \beta_{q_k}^{2(n-1-l)} \gamma_{i_k}^l (Q\psi_k, \psi_k)_1 + n \sum_{k=1}^{m_p} \gamma_{i_k}^{n-1} \cdot (Q\psi_k, \psi_k)_1 \quad (12)
 \end{aligned}$$

From (11) and (12) we find

$$\begin{aligned}
 \sum_{k=1}^{m_p} \left\{ \lambda_k^n - \mu_k^n - n \sum_{j=2}^N (-1)^j j^{-1} \operatorname{Res}_{\lambda=\mu_k} [\lambda^{n-1} tr(QR_\lambda^0)^j] - n \sum_{l=0}^{n-2} C_{n-1}^l \beta_{q_k}^{2(n-1-l)} \gamma_{i_k}^l (Q\psi_k, \psi_k)_1 \right\} \\
 = n \sum_{k=1}^{m_p} \gamma_{i_k}^{n-1} (Q\psi_k, \psi_k)_1 + \beta_p^{(N)}, \quad (13)
 \end{aligned}$$

where the indices  $q_k$  and  $i_k$  are the natural numbers such that  $\mu_k = \beta_{q_k}^2 + \gamma_{i_k}$  ( $\mu_1 \leq \mu_2 \leq \dots$  and  $k = 1, 2, \dots$ ). In the following we will compute the limit

$$\lim_{p \rightarrow \infty} \sum_{k=1}^{m_p} \left\{ \lambda_k^n - \mu_k^n - n \sum_{j=2}^N (-1)^j j^{-1} \operatorname{Res}_{\lambda=\mu_k} [\lambda^{n-1} tr(QR_\lambda^0)^j] - n \sum_{l=0}^{n-2} C_{n-1}^l \beta_{q_k}^{2(n-1-l)} \gamma_{i_k}^l (Q\psi_k, \psi_k)_1 \right\}.$$

This limit will be called the  $n^{th}$  regularized trace of the operator  $L$ .

**Theorem 1.** *If the conditions 1)-4) are satisfied then*

$$\sum_{q=1}^{\infty} \sum_{i=1}^{\infty} \left| \gamma_i^{n-1} \int_0^1 2x \frac{J_{\nu}^2(\beta_q x)}{J_{\nu+1}^2(\beta_q)} (Q(x)\varphi_i, \varphi_i) dx \right| < \infty.$$

**Proof.** Let  $f_i(x) = (Q(x)\varphi_i, \varphi_i)$ . Then

$$\begin{aligned} \int_0^1 2x \frac{J_{\nu}^2(\beta_q x)}{J_{\nu+1}^2(\beta_q)} (Q(x)\varphi_i, \varphi_i) dx &= \int_0^1 2x \frac{J_{\nu}^2(\beta_q x)}{J_{\nu+1}^2(\beta_q)} f_i(x) dx \\ \int_0^1 2x \frac{J_{\nu}^2(\beta_q x)}{J_{\nu+1}^2(\beta_q)} [f_i(x) - f_i(0)] dx + f_i(0) &= \int_0^{\beta_q^{-1+\varepsilon}} 2x \frac{J_{\nu}^2(\beta_q x)}{J_{\nu+1}^2(\beta_q)} [f_i(x) - f_i(0)] dx \\ + \int_{\beta_q^{-1+\varepsilon}}^1 2x \frac{J_{\nu}^2(\beta_q x)}{J_{\nu+1}^2(\beta_q)} [f_i(x) - f_i(0)] dx + f_i(0) \end{aligned} \quad (14)$$

where  $\varepsilon \in (0, \frac{1}{2})$ . It has been shown in [8] that the asymptotic formula

$$2x \frac{J_{\nu}^2(\beta_q x)}{J_{\nu+1}^2(\beta_q)} = 1 + \sin(2\beta_q x - \nu\pi) - \frac{\cos(2\beta_q x - \nu\pi)}{8\beta_q x} (4\nu^2 - 1) + O((\beta_q x)^{-2}) \quad (15)$$

is satisfied for the large value of  $q$ , where  $x \in [\beta_q^{-1+\varepsilon}, 1)$ .

From (14) and (15) we write

$$\begin{aligned} \sum_{q=1}^{\infty} \sum_{i=1}^{\infty} \gamma_i^{n-1} \left| \int_0^1 2x \frac{J_{\nu}^2(\beta_q x)}{J_{\nu+1}^2(\beta_q)} f_i(x) dx \right| &\leq \text{const.} \sum_{q=1}^{\infty} \sum_{i=1}^{\infty} \left[ \left| \int_0^{\beta_q^{-1+\varepsilon}} |f_i(x) - f_i(0)| 2x \frac{J_{\nu}^2(\beta_q x)}{J_{\nu+1}^2(\beta_q)} dx \right| + \right. \\ &\left| \int_{\beta_q^{-1+\varepsilon}}^1 (f_i(x) - f_i(0))(1 + \sin(2\beta_q x - \nu\pi)) dx + f_i(0) \right| + \\ &\left. \left| \int_{\beta_q^{-1+\varepsilon}}^1 (f_i(x) - f_i(0)) \frac{\cos(2\beta_q x - \nu\pi)}{\beta_q x} dx \right| + \int_{\beta_q^{-1+\varepsilon}}^1 \frac{|f_i(x) - f_i(0)|}{(\beta_q x)^2} dx \right] \gamma_i^{n-1}. \end{aligned} \quad (16)$$

By using the inequality  $|\sqrt{x}J_{\nu}(x)| < C$  and conditions 1), 4) for the first integral on the right hand side of (16), we get

$$\begin{aligned}
 & \left| \gamma_i^{n-1} \int_0^{\beta_q^{-1+\varepsilon}} (f_i(x) - f_i(0)) 2x \frac{J_{\nu}^2(\beta_q x)}{J_{\nu+1}^2(\beta_q)} dx \right| \leq \text{Const.} \gamma_i^{n-1} \int_0^{\beta_q^{-1+\varepsilon}} |f_i(x) - f_i(0)| dx \\
 & = \text{Const.} \gamma_i^{n-1} \int_0^{\beta_q^{-1+\varepsilon}} |f'_i(\xi x)| x dx = \text{Const.} \int_0^{\beta_q^{-1+\varepsilon}} |(A^{n-1} Q'(\xi x) \varphi_i, \varphi_i)| x dx \\
 & \leq \text{Const.} |(B\varphi_i, \varphi_i)| \beta_q^{-2+2\varepsilon}
 \end{aligned} \tag{17}$$

From the conditions  $1^0$ - $4^0$ ) and by using the asymptotic formula  $\beta_q = (q + \frac{\nu}{2} - \frac{1}{4})\pi + O(q^{-1})$  we obtain

$$\begin{aligned}
 & \left| \gamma_i^{n-1} \int_{\beta_q^{-1+\varepsilon}}^1 (f_i(x) - f_i(0)) (1 + \sin(2\beta_q x - \nu\pi)) dx + f_i(0) \right| = \gamma_i^{n-1} \left| \int_0^1 (f_i(x) - f_i(0)) dx \right. \\
 & \left. - \int_0^{\beta_q^{-1+\varepsilon}} |f_i(x) - f_i(0)| dx + \int_{\beta_q^{-1+\varepsilon}}^1 (f_i(x) - f_i(0)) \sin(2\beta_q x - \nu\pi) dx + f_i(0) \right| \\
 & \leq \gamma_i^{n-1} \int_0^{\beta_q^{-1+\varepsilon}} |f'_i(\xi x)| x dx + \gamma_i^{n-1} \left| \frac{\cos(2\beta_q x - \nu\pi)(f_i(1) - f_i(0))}{2\beta_q} \right. \\
 & \left. - \frac{\cos(2\beta_q^\varepsilon - \nu\pi)(f_i(\beta_q^{-1+\varepsilon}) - f_i(0))}{2\beta_q} - \frac{1}{2\beta_q} \int_{\beta_q^{-1+\varepsilon}}^1 f'_i(x) \cos(2\beta_q x - \nu\pi) dx \right| \\
 & \leq \text{Const.} [(B\varphi_i, \varphi_i)| \beta_q^{-2+2\varepsilon} + \gamma_i^{n-1} \beta_q^{-2} (|f_i(0)| + |f_i(1)|) + |(B\varphi_i, \varphi_i)| \beta_q^{-2+2\varepsilon}] \\
 & + \frac{\gamma_i^{n-1}}{2\beta_q^2} |f'_i(1) \sin(2\beta_q - \nu\pi) - f'_i(\beta_q^{-1+\varepsilon}) \sin(2\beta_q^\varepsilon - \nu\pi) \\
 & - \int_{\beta_q^{-1+\varepsilon}}^1 f''_i(x) \sin(2\beta_q x - \nu\pi) dx| \leq \text{Const} [(B\varphi_i, \varphi_i)| \beta_q^{-2+2\varepsilon} \\
 & + \gamma_i^{n-1} \beta_q^{-2} (|f_i(0)| + |f_i(1)| + |f'_i(1)|) + \gamma_i^{n-1} \beta_q^{-2} \int_0^1 |f''_i(x)| dx].
 \end{aligned} \tag{18}$$

Now, let us give a bound the third integral on the right hand side of (16). First we write

$$\begin{aligned}
 & \left| \int_{\beta_q^{-1+\varepsilon}}^1 (f_i(x) - f_i(0)) \frac{\cos(2\beta_q x - \nu\pi)}{\beta_q x} dx \right| = \left| \int_{\beta_q^{-1+\varepsilon}}^\delta (f_i(x) - f_i(0)) \frac{\cos(2\beta_q x - \nu\pi)}{\beta_q x} dx + \right. \\
 & \left. + \int_\delta^1 (f_i(x) - f_i(0)) \frac{\cos(2\beta_q x - \nu\pi)}{\beta_q x} dx \right|.
 \end{aligned}$$

From (1),(2) and (4) we have

$$\begin{aligned}
 & \gamma_i^{n-1} \left| \int_{\beta_q^{-1+\varepsilon}}^1 (f_i(x) - f_i(0)) \frac{\cos(2\beta_q x - \nu\pi)}{\beta_q x} dx \right| \leq \left| \frac{\gamma_i^{n-1}}{\beta_q^2} \frac{\sin(2\beta_q x - \nu\pi)}{x} (f_i(x) - f_i(0)) \right|_{\beta_q^{-1+\varepsilon}}^\delta \\
 & - \frac{\gamma_i^{n-1}}{\beta_q^2} \int_{\beta_q^{-1+\varepsilon}}^\delta \left( \frac{f'_i(x)}{x} - \frac{f_i(x) - f_i(0)}{x^2} \right) \sin(2\beta_q x \nu\pi) dx \\
 & + \left| \frac{\gamma_i^{n-1}}{\beta_q^2} \frac{\sin(2\beta_q x - \nu\pi)}{x} (f_i(x) - f_i(0)) \right|_\delta^1 - \frac{\gamma_i^{n-1}}{\beta_q^2} \int_\delta^1 \left( \frac{f'_i(x)}{x} - \frac{f_i(x) - f_i(0)}{x^2} \right) \sin(2\beta_q x - \nu\pi) dx \\
 & \leq \text{Const.} |(B\varphi_i, \varphi_i)| \beta_q^{-2} + \text{Const.} \gamma_i^{n-1} \beta_q^{-2} \left[ |f_i(0)| + |f_i(1)| + \int_0^1 (|f_i(x)| + |f'_i(x)|) dx \right].
 \end{aligned} \tag{19}$$

Finally let us give a bound for the forth integral on the right hand side of (16).

$$\begin{aligned}
 & \gamma_i^{n-1} \int_{\beta_q^{-1+\varepsilon}}^1 \frac{|f_i(x) - f_i(0)|}{(\beta_q x)^2} dx = \gamma_i^{n-1} \int_{\beta_q^{-1+\varepsilon}}^\delta \frac{|f_i(x) - f_i(0)|}{(\beta_q x)^2} dx \\
 & + \gamma_i^{n-1} \int_\delta^1 \frac{|f_i(x) - f_i(0)|}{(\beta_q x)^2} dx \leq |(B\varphi_i, \varphi_i)| \beta_q^{-2} \ln(\delta \beta_q^{1-\varepsilon}) \\
 & + \text{Const.} \gamma_i^{n-1} \beta_q^{-2} \left[ |f_i(0)| + \int_0^1 |f_i(x)| dx \right] \\
 & \leq \text{Const.} \gamma_i^{n-1} \beta_q^{-2+\varepsilon} + \text{Const.} \gamma_i^{n-1} \beta_q^{-2} \left[ |f_i(0)| + \int_0^1 |f_i(x)| dx \right].
 \end{aligned} \tag{20}$$

From the relations (16),(17),(18),(19) and (20) we obtain

$$\begin{aligned}
 & \sum_{q=1}^\infty \sum_{i=1}^\infty \left| \gamma_i^{n-1} \int_0^1 2x \frac{J_\nu^2(\beta_q x)}{J_{\nu+1}^2(\beta_q)} f_i(x) dx \right| \leq \text{Const.} \left( \sum_{q=1}^\infty \beta_q^{-2+2\varepsilon} \right) \sum_{i=1}^\infty |(B\varphi_i, \varphi_i)| \\
 & + \text{Const.} \sum_{q=1}^\infty \beta_q^{-2} \sum_{i=1}^\infty \gamma_i^{n-1} \left[ |f_i(0)| + |f_i(1)| + |f'_i(1)| + \int_0^1 (|f_i(x)| + |f'_i(x)| + |f''_i(x)|) dx \right].
 \end{aligned} \tag{21}$$

By assumption, since  $B \in \sigma_1(H)$  and  $\|A^{n-1}Q^{(j)}(x)\|_{\sigma_1(H)} \leq \text{Const.}, (j = 0, 1, 2)$ , we have

$$\sum_{i=1}^\infty |(B\varphi_i, \varphi_i)| \leq \|B\|_{\sigma_1(H)}. \tag{22}$$

and

$$\begin{aligned}
 & \sum_{i=1}^{\infty} \gamma_i^{n-1} \int_0^1 (|f_i(x)| + |f'_i(x)| + |f''_i(x)|) dx \\
 &= \lim_{N \rightarrow \infty} \int_0^1 \left[ \sum_{i=1}^N \gamma_i^{n-1} (|f_i(x)| + |f'_i(x)| + |f''_i(x)|) \right] dx \tag{23} \\
 &\leq \int_0^1 \left[ \sum_{i=1}^{\infty} \gamma_i^{n-1} \left( \sum_{j=0}^2 |f_i^{(j)}(x)| \right) \right] dx \leq \int_0^1 \left[ \sum_{j=0}^2 \|A^{n-1}Q^{(j)}(x)\|_{\sigma_1(H)} \right] dx < Const.
 \end{aligned}$$

If we take into account  $\lim_{q \rightarrow \infty} \frac{\beta_q}{\pi q} = 1$ , then from (21),(22) and (23) we obtain

$$\sum_{q=1}^{\infty} \sum_{i=1}^{\infty} \left| \gamma_i^{n-1} \int_0^1 2x \frac{J_{\nu}^2(\beta_q x)}{J_{\nu+1}^2(\beta_q)} (Q(x)\varphi_i, \varphi_i) dx \right| < Const.$$

Thus the theorem is proved. □

The main result of this work is given in the following theorem.

**Theorem 2.** *If the conditions 1)-4) are satisfied and  $\lim_{i \rightarrow \infty} \frac{\gamma_i}{a i^{\alpha}} = 1$  ( $a > 0, \alpha > 2$ ), then for the  $n^{th}$  regularized trace of the operator  $L$  the formula*

$$\begin{aligned}
 & \lim_{p \rightarrow \infty} \sum_{k=1}^{m_p} \left\{ \lambda_k^n - \mu_k^n - n \sum_{j=2}^N (-1)^j j^{-1} Res_{\lambda=\mu_k} [\lambda^{n-1} tr(QR_{\lambda}^0)^j] \right. \\
 & \left. - n \sum_{l=0}^{n-2} C_{n-1}^l \beta_{q_k}^{2(n-1-l)} \gamma_{i_k}^l (Q\psi_k, \psi_k)_1 \right\} \\
 &= -\frac{n}{4} [2\nu \cdot tr(A^{n-1}Q(0)) + tr(A^{n-1}Q(1))]
 \end{aligned}$$

is satisfied, where  $N = \left\lceil \frac{2\alpha n + \alpha + 6}{\alpha - 2} \right\rceil + 1$ .

**Proof.** From theorem 1, we observe that the sequence  $\sum_{k=1}^{\infty} \gamma_{i_k}^{n-1} (Q\psi_k, \psi_k)_1$  is convergent and

$$\sum_{k=1}^{\infty} \gamma_{i_k}^{n-1} (Q\psi_k, \psi_k)_1 = \sum_{i=1}^{\infty} \sum_{q=1}^{\infty} \gamma_i^{n-1} \int_0^1 2x \cdot \frac{J_{\nu}^2(\beta_q x)}{J_{\nu+1}^2(\beta_q)} (Q(x)\varphi_i, \varphi_i)_1 dx.$$

From these and the equality

$$\sum_{q=1}^{\infty} \int_0^1 2x \frac{J_{\nu}^2(\beta_q x)}{J_{\nu+1}^2(\beta_q)} (Q(x)\varphi_i, \varphi_i) dx = \frac{1}{4} [2\nu(Q(0)\varphi_i, \varphi_i) + (Q(1)\varphi_i, \varphi_i)]$$

proved in [8], we obtain

$$\begin{aligned} \sum_{k=1}^{\infty} \gamma_{i_k}^{n-1} (Q\psi_k, \psi_k)_1 &= -\frac{1}{4} \left[ 2\nu \sum_{i=1}^{\infty} \gamma_i^{n-1} (Q(0)\varphi_i, \varphi_i) + \sum_{i=1}^{\infty} \gamma_i^{n-1} (Q(1)\varphi_i, \varphi_i) \right] \\ &= -\frac{1}{4} [2\nu \operatorname{tr}(A^{n-1}Q(0)) + \operatorname{tr}(A^{n-1}Q(1))] \end{aligned} \quad (24)$$

By making use of inequality (2), it can be proved that the inequalities

$$\|R_{\lambda}^0\|_{\sigma_1(H_1)} \leq \operatorname{Const}.m_p^{1-\delta}$$

$$\|R_{\lambda}^0\|_1 \leq \operatorname{Const}.m_p^{-\delta}$$

$$\|R_{\lambda}\|_1 \leq \operatorname{Const}.m_p^{-\delta}$$

are satisfied on the circle  $|\lambda| = b_p$  where  $\delta = \frac{\alpha-2}{\alpha+2}$ . Now, let us give a bound for  $B_p^{(N)}$  by using these inequalities. From the (8), we find

$$\begin{aligned} |B_p^{(N)}| &\leq \frac{1}{2\pi} \int_{|\lambda|=b_p} |\lambda|^n |\operatorname{tr}[R_{\lambda}(QR_{\lambda}^0)^{N+1}]| |d\lambda| \\ &\leq b_p^n \int_{|\lambda|=b_p} \|R_{\lambda}(QR_{\lambda}^0)^{N+1}\|_{\sigma_1(H_1)} |d\lambda| \end{aligned}$$

$$\begin{aligned}
 &\leq b_p^n \int_{|\lambda|=b_p} \|R_\lambda\|_1 \|(QR_\lambda^0)^{N+1}\|_{\sigma_1(H_1)} |d\lambda| \\
 &\leq b_p^n \int_{|\lambda|=b_p} \|R_\lambda\|_1 \|(QR_\lambda^0)^N\|_1 \|QR_\lambda^0\|_{\sigma_1(H_1)} |d\lambda| \\
 &\leq b_p^n \int_{|\lambda|=b_p} \|R_\lambda\|_1 \|Q\|_1^N \|R_\lambda^0\|_1^N \|Q_1\| \|R_\lambda^0\|_{\sigma_1(H_1)} |d\lambda| \\
 &\leq Const. b_p^{n+1} m_p^{-(N+1)\delta} m_p^{1-\delta}.
 \end{aligned}$$

Taking into account  $b_p \leq Const. m_p^{1+\delta}$ , one then obtains

$$|B_p^{(N)}| \leq Const. m_p^{(n+1)(1+\delta) - (N+1)\delta + 1 - \delta}.$$

Hence, if

$$N = \lceil \lceil \delta^{-1}(n + 2 + n\delta - \delta) \rceil \rceil + 1 = \left\lceil \left\lceil \frac{2\alpha n + \alpha + 6}{\alpha - 2} \right\rceil \right\rceil + 1$$

then

$$\lim_{p \rightarrow \infty} B_p^{(N)} = 0.$$

From (13), (24) and this last equality, we obtain

$$\begin{aligned}
 &\lim_{p \rightarrow \infty} \sum_{k=1}^{m_p} \left\{ \lambda_k^n - \mu_k^n - n \sum_{j=2}^N (-1)^j j^{-1} Res_{\lambda=\mu_k} [\lambda^{n-1} tr(QR_\lambda^0)^j] \right. \\
 &\quad \left. - n \sum_{l=0}^{n-2} C_{n-1}^l \beta_{q_k}^{2(n-1-l)} \gamma_{i_k}^l (Q\psi_k, \psi_k)_1 \right\} \\
 &= -\frac{n}{4} [2\nu. tr(A^{n-1}Q(0)) + tr(A^{n-1}Q(1))]
 \end{aligned}$$

Thus, the proof is done. □

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### References

- [1] Gelfand, I. M., Levitan, B.M., “İkinci mertebeden bir diferansiel operatörün özdeğerleri için bir formül hakkında”, Dokl. AN SSSR, 1953, T.88, No: 4, 593-596.
- [2] Dikiy, L.A., “Gelfand-Levitannın bir formülü hakkında”, Uspeki Matem Nauk,1953,T.8, No:2, 119-123.
- [3] Halberg, C.J. and Kramer, V.A., “A Generalization of the trace concept”, Duke Mathematical Journal, 1960, Vol. 27, No: 4, 607-618.
- [4] Levitan, B.M. and Sargsyan I.S., “Sturm-Liouville and Dirac operators”, Kluzer, Dordrechz, 1991.
- [5] Fulton, T.C., Pruess, S.A., “Eigenvalue and eigenfunction asymptotics for regular Sturm-Liouville problems”, *J. Math. Anal. Appl.* **188** (1994), 297-340.
- [6] Maksudov, F.G., Bayramoğlu M., Adıgüzelov E. E., “On a regularized trace of Sturm-Liouville operator on a finite interval with the unbounded operator coefficient”, Dokl. AN SSSR, English translation: *Soviet Math. Dokl.* **30**, 1984, No: 1, 169-173.
- [7] Gorbachuk, V.I., Gorbachuk, M.L., Ukrain. Mat. Zh 24 (1972), 291-305; English transl. in Ukrainian Math.J.24 (1972)
- [8] Heşimov I.F., “Sonlu aralıkta verilmiş tekilliğe sahip Sturm-Liouville operatör denkleminin düzenli izinin hesaplanması”, AN Azerb. SSR, Institut Matematiki i Mekaniki, Baku, 1989-VİNİTİ, No: 7340-V89.
- [9] Adıgüzelov, E.E., Bakşi Ö., “İkinci mertebeden operatör katsayılı bir diferansiyel operatörün düzenli izinin hesaplanması”, Doktora Tezi, 1999, YTÜ Fen-Bil. Enst.
- [10] Dubrovski, V.V., “Regularized traces of self-adjoint operators, Vestsi Akad., Navuk Belarusi, Ser. Fiz. Mat. Navuk, 1996, No:1, 124, 30-33.
- [11] Gohberg, I.C., Krein, M.G., “Introduction to the theory of linear non-selfadjoint operators”, Translation of Mathematical Monograph, Volume 18. Amer.Math.Sos., 1969, Providence, R.I.



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