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Induced Cat^1 -groups

Murat Alp

Abstract

In this paper we define the pullback cat^1 -group and show that this Pullback has a right adjoint which is the induced cat^1 -group. Later we show that this right adjoint is a pushout of category of cat^1 -groups. We calculate the Peiffer subgroups to find a finite group of the source of induced cat^1 -groups. The generating set of Peiffer subgroups are also given in this paper. All results are corrected by a GAP[13] program package in [4]. This paper also contains the some computational examples which are the calculation-induced cat^1 -group and comparative times between the induced crossed modules and induced cat^1 -groups.

Key Words: Pullback, Crossed module, Cat^1 -group, induced crossed modules, induced cat^1 -groups, Peiffer commutators, cocomplete, GAP.

1. Introduction

We begin by considering the possibility of implementing functions for doing calculations with crossed modules, derivations, actor crossed modules, cat^1 -groups, sections, induced crossed modules and induced cat^1 -groups in GAP[13].

To this end, we first explain the importance of crossed modules. The general points are:

- crossed modules may be thought of as 2-dimensional groups;
- a number of phenomena in group theory are better seen from a crossed module point of view;
- crossed modules occur geometrically as $\pi_2(X, A) \rightarrow \pi_1 A$ when A is a subspace of X or as $\pi_1 F \rightarrow \pi_1 E$ where $F \rightarrow E \rightarrow B$ is a fibration;

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- crossed modules are usefully related to forms of double groupoids.

Particular constructions, such as induced crossed modules, are important for the applications of the 2-dimensional Van-Kampen Theorem of Brown and Higgins [6], and so for the computation of homotopy 2-types.

For all these reasons, the facilitation of the computations with crossed modules should be advantageous. It should help to solve specific problems, and it should make it easier to construct examples and see relations with better known theories.

The powerful computer algebra system GAP[13] provides a high level programming language with several advantages for the coding of new mathematical structures. The GAP system has been developed over the last 15 years at RWTH in Aachen. Some of its most exciting features are:

- it has a highly developed, easy to understand programming language incorporated;
- it is especially powerful for group theory;
- it is portable to a wide variety of operating systems on many hardware platforms.
- it is public domain and it has a lively forum, with open discussion. These make it increasingly used by the mathematical community.

On the other hand, GAP has some disadvantages, too:

- the built in programming language is an interpreted language, which makes GAP programs relatively slow compared to compiled languages such as C or Pascal. GAP source can not be compiled.
- the demands on system resources are quite high for serious calculations.

However, the advantages outweigh the disadvantages, and so GAP was chosen.

Our aim in this paper is to describe a share package XMOD [4] for the GAP group theory language which enables computations with the equivalent notions of finite, permutation *crossed modules* and *cat1-groups*.

The term crossed module was introduced by J. H. C. Whitehead in [15]. Most of crossed modules references state the axioms of a crossed module using left actions, but we shall use right actions since this is the convention used by most computational group packages.

A crossed module $\mathcal{X} = (\partial : S \rightarrow R)$ consists of a group homomorphism ∂ , called the *boundary* of \mathcal{X} , together with an action $\alpha : R \rightarrow \text{Aut}(S)$ satisfying, for all $s, s' \in S$ and $r \in R$,

$$\begin{aligned} \mathbf{XMod\ 1:} \quad \partial(s^r) &= r^{-1}(\partial s)r \\ \mathbf{XMod\ 2:} \quad s^{\partial s'} &= s'^{-1}ss'. \end{aligned}$$

The kernel of ∂ is abelian.

Standard constructions for crossed modules include the following:

1. A *conjugation crossed module* is an inclusion of a normal subgroup $S \trianglelefteq R$, where R acts on S by conjugation.
2. A *central extension crossed module* has as boundary a surjection $\partial : S \rightarrow R$ with central kernel, where $r \in R$ acts on S by conjugation with $\partial^{-1}r$.
3. An *automorphism crossed module* has as range a subgroup R of the automorphism group $\text{Aut}(S)$ of S which contains the inner automorphism group of S . The boundary maps $s \in S$ to the inner automorphism of S by s .
4. An *R-Module crossed module* has an R -module as source and ∂ is the zero map.
5. The direct product $\mathcal{X}_1 \times \mathcal{X}_2$ of two crossed modules has source $S_1 \times S_2$, range $R_1 \times R_2$ and boundary $\partial_1 \times \partial_2$, with R_1, R_2 acting trivially on S_2, S_1 , respectively.

A morphism between two crossed modules \mathcal{X}_1 and \mathcal{X}_2 is a pair (σ, ρ) , where $\sigma : S_1 \rightarrow S_2$ and $\rho : R_1 \rightarrow R_2$ are homomorphisms satisfying

$$\partial_2 \sigma = \rho \partial_1, \quad \sigma(s^r) = (\sigma s)^{\rho r}.$$

When $\mathcal{X}_2 = \mathcal{X}_1$ and σ, ρ are automorphisms then (σ, ρ) is an automorphism of \mathcal{X}_1 . The group of automorphisms is denoted by $\text{Aut}(\mathcal{X}_1)$.

In [11] Loday reformulated the notion of a crossed module as a cat1-group, namely a group G with a pair of homomorphisms $t, h : G \rightarrow G$ having a common image R and satisfying certain axioms. We find it convenient to define a cat1-group $\mathcal{C} = (e; t, h : G \rightarrow R)$ as having source group G , range group R , and three homomorphisms: two surjections $t, h : G \rightarrow R$ and an embedding $e : R \rightarrow G$ satisfying:

$$\begin{aligned} \mathbf{Cat\ 1:} \quad & te = he = \text{id}_R, \\ \mathbf{Cat\ 2:} \quad & [\ker t, \ker h] = \{1_G\}. \end{aligned}$$

The maps t, h are usually referred to as the *source* and *target*, but we choose to call them the *tail* and *head* of \mathcal{C} , because *source* is the GAP term for the domain of a function.

A morphism $\mathcal{C}_1 \rightarrow \mathcal{C}_2$ of cat1-groups is a pair (γ, ρ) where $\gamma : G_1 \rightarrow G_2$ and $\rho : R_1 \rightarrow R_2$ are homomorphisms satisfying

$$h_2 \gamma = \rho h_1, \quad t_2 \gamma = \rho t_1, \quad e_2 \rho = \gamma e_1. \tag{1}$$

Induced crossed modules were introduced by Brown and Higgins in [6]. Later, induced cat^n -group structures were defined by Brown and Loday in [7]. In this paper, they gave some applications of cat^n -groups. They also showed that the H -module f_*M induced from a G -module M by a morphism $f : G \rightarrow H$ is given by $f_*M = M \otimes_{ZG} ZH$. Thus f_* is a left adjoint to the pull-back functor

$$f^* : (H - \text{modules}) \rightarrow (G - \text{modules}).$$

In the case of crossed modules, the inducing functor

$$f_* : (\text{crossed } H - \text{modules}) \rightarrow (\text{crossed } G - \text{modules})$$

is the left adjoint of the pull-back functor

$$f^* : (\text{crossed } H - \text{modules}) \rightarrow (\text{crossed } G - \text{modules}).$$

In addition, they used the notion of induced crossed square and the corresponding notion for cat^n -groups. See the historical background given by Brown in [5] for further details. The pull-back crossed modules were defined in [9] and [10]. In section 2, we describe the construction of pullback crossed modules and induced crossed modules. We also define pullback and induced cat^1 -groups. Section 3 contains an outline algorithm for computing induced cat^1 -groups and a table of sample execution times.

The author wishes to thank R. Brown and T. Porter for many profitable discussions concerning the mathematics. Considerable help with the implementation has been given by many members of the GAP team at Aachen, led by J. Neubüser, and by D. F. Holt who is the appointed editor for the package.

2. Pull-back crossed modules and Pull-back Cat^1 -groups

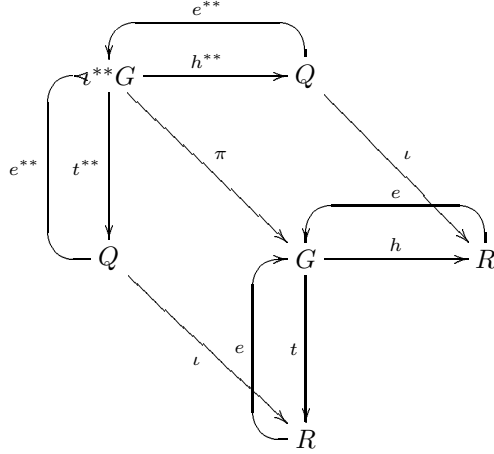
Let $\mathcal{X} = (\partial : S \rightarrow R)$ be a crossed R -module and $\iota : Q \rightarrow R$ be a morphism of groups. Then $\iota^{**}\mathcal{X} = (\partial^\bullet : \iota^{**}S \rightarrow Q)$ is the pullback of \mathcal{X} by ι , where $\iota^{**}S = \{(q, s) \in Q \times S \mid \iota q = \partial s\}$ and $\partial^\bullet(q, s) = q$. The action of Q on $\iota^{**}S$ is given by

$$(q_1, s)^q = (q^{-1}q_1q, s^{tq}). \quad (2)$$

Proposition 2.1 [6] The functor $\iota^{**} : \mathcal{XM}/R \rightarrow \mathcal{XM}/Q$ has a right adjoint ι_{**} .

The universal property of induced crossed modules and the proof of above proposition can be found in [9].

A pullback cat^1 -group is defined as follows.



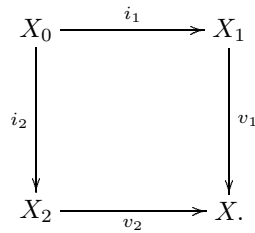
Let $\mathcal{C} = (e; t, h : G \rightarrow R)$ be a cat^1 -group and let $\iota : Q \rightarrow R$ be a group homomorphism. Define $\iota^{**}\mathcal{C} = (e^{**}; t^{**}, h^{**} : \iota^{**}G \rightarrow Q)$ to be the pullback of G where

$$\iota^{**}G = \{(q_1, g, q_2) \in Q \times G \times Q \mid \iota q_1 = tg, \iota q_2 = hg\},$$

$t^{**}(q_1, g, q_2) = q_1$, $h^{**}(q_1, g, q_2) = q_2$ and $e^{**}(q) = (q, e\iota q, q)$. Multiplication in $\iota^{**}G$ is componentwise. The pair (π, ι) is a morphism of cat^1 -groups where $\pi : \iota^{**}G \rightarrow G$, $(q_1, g, q_2) \mapsto g$. The verifications of the Cat^1 -group axioms and the universal property of induced Cat^1 -groups can be found in [1].

Proposition 2.2 The category of cat^1 -groups is cocomplete.

We recall the definition of pushouts in a general category. Suppose we are given a commutative diagram of morphisms in a category \mathbf{C} . Then



Recall [12] that (v_1, v_2) is pushout of (i_1, i_2) , and also that the above square is a pushout square, if the following property holds: if $f_1 : X_1 \rightarrow H$, $f_2 : X_2 \rightarrow H$ are morphisms such that $f_1 i_1 = f_2 i_2$ then there is a unique map $f : X \rightarrow H$ such that $f v_1 = f_1$, $f v_2 = f_2$.

As usual, this property characterizes the pair (v_1, v_2) up to an automorphism of X . For this reason, it is common to coin an abuse of language and refer to X as the pushout of (i_1, i_2) . In such case, we write

$$X = X_2 *_{X_0} X_1,$$

where $*_{X_0}$ denotes a free product with amalgamation over X_0 .

Proposition 2.3 The functor $\iota^{**} : \text{Cat}^1 \text{Grp}/U \rightarrow \text{Cat}^1 \text{Grp}/R$ has a left adjoint $\iota_{**} : \text{Cat}^1 \text{Grp}/R \rightarrow \text{Cat}^1 \text{Grp}/U$.

Proof. The proof of proposition can be found in [2]. \square

We now include some basic properties of commutators which we shall need to obtain some relations between the Peiffer subgroup $P = [\ker t, \ker h]$ and $P_{**} = [\ker t_{**}, \ker h_{**}]$.

Proposition 2.4 The following identities are easily verified.

$$\begin{aligned} \text{(ia)} \quad [ab, c] &= [a, c]^b [b, c] \\ \text{(ib)} \quad [a, bc] &= [a, c][a, b]^c \\ \text{(iia)} \quad [a_1 a_2 \dots a_n, c] &= [a_1, c]^{a_2 \dots a_n} [a_2, c]^{a_3 \dots a_n} \dots [a_{n-1}, c]^{a_n} [a_n, c] \\ \text{(iib)} \quad [a, c_1 c_2 \dots c_n] &= [a, c_n]^{c_1 \dots c_{n-1}} \dots [a, c_2]^{c_3 \dots c_n} [a, c_1]^{c_2 \dots c_n} \end{aligned}$$

Proposition 2.5 The Peiffer subgroup $P = [\ker t, \ker h]$ is normal in G and R -invariant.

Proof. If $a \in \ker t$, $c \in \ker h$ and $g \in G$ then $[a, c]^g = [a^g, c^g] \in P$. Since $r \in R$ acts on G by conjugation with er , it follows that $[a, c]^r \in P$. \square

Proposition 2.6 Let X_t, X_h be generating sets for $\ker t, \ker h$, closed under conjugation in G . The Peiffer subgroup $[\ker t, \ker h]$ of G has generating set

$$\{[x, y] \mid x \in X_t, y \in X_h\}.$$

Proof. An element of $[\ker t, \ker h]$ has the form

$$z = \prod_i [a_i, c_i],$$

where $a_i = x_{i1}x_{i2} \dots x_{ir_i} \in \ker t$, $x_{ij} \in X_t$, and $c_i = y_{i1}y_{i2} \dots y_{is_i} \in \ker h$, $y_{ij} \in X_h$, so

$$z = \prod_i [x_{i1}x_{i2} \dots x_{ir_i}, y_{i1}y_{i2} \dots y_{is_i}].$$

From Proposition 2.4, z is a product of generating commutators. \square

To any pre-cat¹-group \mathcal{P} there is a canonically associated a cat¹-group \mathcal{C} , obtained by quotienting the source group by the Peiffer subgroup $[\ker t, \ker h]$. The corresponding functor is denoted

$$\mathbf{ass} : (\text{pre-cat}^1\text{-groups}) \rightarrow (\text{cat}^1\text{-groups}) \quad (3)$$

and is clearly the identity when restricted to cat¹-groups [7].

Our aim now is to find a convenient generating set for $[\ker t_{**}, \ker h_{**}]$. To this end we define, for an arbitrary pre-cat¹-group $\mathcal{P} = (e; t, h, : G \rightarrow R)$, projections

$$\begin{aligned} \pi_t : G &\rightarrow \ker t, & g &\mapsto (etg^{-1})g, \\ \pi_h : G &\rightarrow \ker h, & g &\mapsto (ehg^{-1})g. \end{aligned}$$

The maps π_t, π_h are respectively derivations for the conjugation crossed modules $(\text{inc} : \ker t \rightarrow G)$ and $(\text{inc} : \ker h \rightarrow G)$.

Since $\pi_t g = g$ when $g \in \ker t$ and $\pi_h g = g$ when $g \in \ker h$, both π_t and π_h are surjective. The following proposition gives values for π_t, π_h and their inverses in some special cases.

Proposition 2.7 The following identities are easily verified:

$$\begin{aligned} \text{(ia)} \quad \pi_t(ab) &= (\pi_t a)^{tb}(\pi_t b) \\ \text{(ib)} \quad \pi_h(cd) &= (\pi_h c)^{hd}(\pi_h d) \\ \text{(iia)} \quad (\pi_t a)^{-1} &= (\pi_t a^{-1})^{ta} \\ \text{(iib)} \quad (\pi_h c)^{-1} &= (\pi_h c^{-1})^{hc} \\ \text{(iiia)} \quad \pi_t(a_1 \dots a_n) &= (\pi_t a_1)^{t(a_2 \dots a_n)} (\pi_t a_2)^{t(a_3 \dots a_n)} \dots (\pi_t a_{n-1})^{t a_n} (\pi_t a_n) \\ \text{(iiib)} \quad \pi_h(c_1 \dots c_n) &= (\pi_h c_1)^{h(c_2 \dots c_n)} (\pi_h c_2)^{h(c_3 \dots c_n)} \dots (\pi_h c_{n-1})^{h c_n} (\pi_h c_n). \end{aligned}$$

Proof. We first verify (ia) :

$$\begin{aligned} \pi_t(ab) &= et(ab)^{-1}ab \\ &= (etb^{-1})(eta^{-1})ab \\ &= (etb^{-1})(\pi_t a)(etb)(etb^{-1})b \end{aligned}$$

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$$\begin{aligned} &= (\pi_t a)^{etb} (\pi_t b) \\ &= (\pi_t a)^{tb} (\pi_t b) \end{aligned}$$

by definition of the action of R on G . Then (iia) follows by setting $a = b^{-1}$ and (iiia) follows by induction. Identities (ib), (iib) and (iiib) can be proved in a similar way. \square

A second set of identities, but with the order of the factors reversed, may be proved in a similar way.

Proposition 2.8

$$\begin{aligned} (ia) \quad \pi_t(ab) &= (\pi_t b)(\pi_t a)^b \\ (ib) \quad \pi_h(bc) &= (\pi_h c)(\pi_h b)^c \\ (iia) \quad (\pi_t a)^{-1} &= (\pi_t a^{-1})^a \\ (iib) \quad (\pi_h c)^{-1} &= (\pi_h c^{-1})^c \\ (iiia) \quad \pi_t(a_1 \dots a_n) &= (\pi_t a_n)(\pi_t a_{n-1})^{a_n} \dots (\pi_t a_1)^{a_2 \dots a_n} \\ (iiib) \quad \pi_h(c_1 \dots c_n) &= (\pi_h c_n)(\pi_h c_{n-1})^{c_n} \dots (\pi_h c_1)^{c_2 \dots c_n}. \end{aligned}$$

We now obtain two pairs of identities expanding commutators containing terms $\pi_t(ab)$ or $\pi_h(bc)$.

Proposition 2.9

$$\begin{aligned} (i) \quad [\pi_t(ab), \pi_h c] &= [(\pi_t a)^{tb}, (\pi_h c)]^{\pi_t b} [\pi_t b, \pi_h c] \\ (ii) \quad [\pi_t a, \pi_h(bc)] &= [\pi_t a, \pi_h c] [\pi_t a, (\pi_h b)^{(hc)}]^{\pi_h c}. \end{aligned}$$

Proof. Using the Proposition 2.7,

(i)

$$\begin{aligned} [\pi_t(ab), (\pi_h c)] &= [(\pi_t a)^{tb} (\pi_t b), (\pi_h c)] \\ &= (\pi_t b)^{-1} ((\pi_t a)^{tb})^{-1} (\pi_h c)^{-1} (\pi_t a)^{tb} (\pi_t b) (\pi_h c) \\ &= (\pi_t b)^{-1} ((\pi_t a)^{tb})^{-1} (\pi_h c)^{-1} (\pi_t a)^{tb} (\pi_h c) (\pi_t b) (\pi_t b)^{-1} (\pi_h c)^{-1} (\pi_t b) (\pi_h c) \\ &= [(\pi_t a)^{tb}, (\pi_h c)]^{(\pi_t b)} [(\pi_t b), (\pi_h c)] \\ &= [(\pi_t a)^{(tb)(\pi_t b)}, (\pi_h c)^{(\pi_t b)}] [(\pi_t b), (\pi_h c)] \\ &= [(\pi_t a)^b, (\pi_h c)^{(\pi_t b)}] [(\pi_t b), (\pi_h c)]. \end{aligned}$$

(ii)

$$\begin{aligned}
[\pi_t(a), \pi_h(bc)] &= [(\pi_t a), (\pi_h b)^{hc}(\pi_h c)] \\
&= (\pi_t a)^{-1}(\pi_h c)^{-1}(\pi_h b)^{hc-1}(\pi_t a)(\pi_h b)^{hc}(\pi_h c) \\
&= (\pi_t a)^{-1}(\pi_h c)^{-1}(\pi_t a)(\pi_h c)(\pi_h c)^{-1}(\pi_t a)(\pi_h b)^{hc-1}(\pi_t a)(\pi_h b)^{hc}(\pi_h c) \\
&= [(\pi_t a), (\pi_h c)][(\pi_t a), (\pi_h b)^{(hc)}]^{(\pi_h c)}.
\end{aligned}$$

□

Proposition 2.10

$$\begin{aligned}
(i) \quad [\pi_t(ab), \pi_h c] &= [\pi_t b, \pi_h c]^{(\pi_t a)^b} [(\pi_t a)^b, \pi_h c] \\
(ii) \quad [\pi_t a, \pi_h(bc)] &= [\pi_t a, (\pi_h b)^c][\pi_t a, \pi_h c]^{(\pi_h b)^c}.
\end{aligned}$$

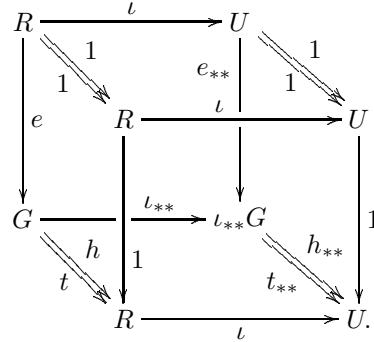
Proof. The proof is similar to the previous proof, but uses Proposition 2.8. □**Proposition 2.11** The maps π_t and π_h preserve the action of U .**Proof.**

$$\begin{aligned}
\pi_t(g^u) &= (et(g^u))^{-1}(g^u) \\
&= (et((eu)^{-1}g(eu)))^{-1}(g^{eu}) \\
&= ((eteu)^{-1}(etg)(eteu))^{-1}(eu)^{-1}g(eu) \\
&= (eu)^{-1}(etg)^{-1}g(eu) \\
&= ((etg)^{-1}g)^{eu} \\
&= (\pi_t g)^u
\end{aligned}$$

The proof for π_h is similar. □

Let $\mathcal{C} = (e; t, h : G \rightarrow R)$ be a cat^1 -group and let $\iota : R \rightarrow U$ be an inclusion. Denote by \mathcal{U}, \mathcal{R} the identity cat^1 -groups on U and R . Let $\iota_{**}\mathcal{C} = (e_{**}; t_{**}, h_{**} : \iota_{**}G \rightarrow U)$ be the pushout of the pre- cat^1 -group morphisms $(\iota, \iota) : \mathcal{R} \rightarrow \mathcal{U}$ and $(e, 1) : \mathcal{R} \rightarrow \mathcal{C}$. The elements of $\iota_{**}G$ are words of the form $\kappa = g_1 u_1 g_2 u_2 \dots g_k u_k$, and the identity element

is the empty word λ . We make no notational distinction between elements $g \in G$, $u \in U$ and their images under the inclusions $\iota_{**} : G \rightarrow \iota_{**}G$, $e_{**} : U \rightarrow \iota_{**}G$. Thus

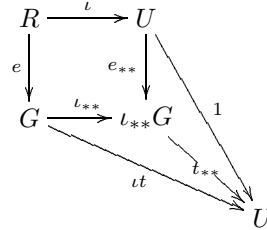


Proposition 2.12 The pushout $\iota_{**}\mathcal{C} = (e_{**}; t_{**}, h_{**} : \iota_{**}G \rightarrow U)$ is a pre-cat¹-group, with tail t_{**} and head h_{**} given on words of length 1 by

$$h_{**}g = \iota hg, \quad h_{**}u = u, \quad t_{**}g = \iota tg, \quad t_{**}u = u,$$

and extended componentwise to longer words, while $e_{**}u = u$.

Proof.



Since the above diagram is commutative and a pushout of groups then there is a unique homomorphism t_{**} such that $t_{**}e_{**} = 1$. Similarly, there is a unique h_{**} such that $h_{**}e_{**} = 1$. So the condition is satisfied. \square

To turn this pre-cat¹-group into a cat¹-group we must find the Peiffer subgroup $P_{**} = [\ker t_{**}, \ker h_{**}]$. For this situation we denote the maps π_t and π_h by $\pi_{t_{**}}$ and $\pi_{h_{**}}$ so

$$\begin{aligned} \pi_{t_{**}} : G *_R U &\rightarrow G *_R U, & \kappa &\mapsto (e_{**}t_{**}\kappa^{-1})\kappa \\ \pi_{h_{**}} : G *_R U &\rightarrow G *_R U, & \kappa &\mapsto (e_{**}h_{**}\kappa^{-1})\kappa. \end{aligned}$$

The following proposition gives values for $\pi_{t_{**}}$ and $\pi_{h_{**}}$ in some special cases.

Proposition 2.13 The following identities hold for any $g \in G, u \in U$:

$$\begin{aligned}
e_{**}t_{**}g &= etg \\
e_{**}h_{**}g &= ehg \\
\pi_{t_{**}}g &= \pi_tg \\
\pi_{t_{**}}u &= \lambda \\
\pi_{t_{**}}(gu) &= (\pi_tg)^u \\
\pi_{t_{**}}(ug) &= \pi_tg \\
\pi_{t_{**}}(g_1u_1g_2u_2) &= (\pi_tg_1)^{u_1(tg_2)u_2}(\pi_tg_2)^{u_2} \\
\pi_{t_{**}}(g_1u_1 \dots g_nu_n) &= (\pi_tg_1)^{u_1(tg_2)u_2 \dots (tg_n)u_n} \dots (\pi_tg_{n-1})^{u_{n-1}(tg_n)u_n} (\pi_tg_n)^{u_n}.
\end{aligned}$$

Proof. Using the Proposition 2.12 and $\iota_{**} : G \rightarrow \iota_{**}G$,

$$e_{**}t_{**}g = e_{**}\iota_{**}tg = \iota_{**}etg = etg.$$

Also using the definition of $\pi_{t_{**}}$,

$$\pi_{t_{**}}g = (e_{**}t_{**}g^{-1})g = (etg^{-1})g = \pi_tg,$$

$$\pi_{t_{**}}u = (e_{**}t_{**}u^{-1})u = u^{-1}u = \lambda,$$

$$\begin{aligned}
\pi_{t_{**}}(gu) &= (\pi_{t_{**}}g)^{(t_{**}u)}(\pi_{t_{**}}u) \\
&= (\pi_{t_{**}}g)^u \lambda \\
&= (\pi_tg)^u \\
\pi_{t_{**}}(ug) &= (\pi_{t_{**}}u)^{(t_{**}g)}(\pi_{t_{**}}g) \\
&= (\pi_{t_{**}}g) \\
&= \pi_tg.
\end{aligned}$$

Applying Proposition 2.7,

$$\begin{aligned}
\pi_{t_{**}}(g_1u_1g_2u_2) &= \pi_{t_{**}}(g_1u_1)^{(t_{**}g_2u_2)}(\pi_{t_{**}}g_2u_2) \\
&= [\pi_{t_{**}}(g_1)^{t_{**}u_1}\pi_{t_{**}}(u_1)]^{(t_{**}g_2u_2)}\pi_{t_{**}}(g_2)^{(t_{**}u_2)}\pi_{t_{**}}(u_2) \\
&= (\pi_tg_1)^{u_1(tg_2)u_2}(\pi_tg_2)^{u_2}.
\end{aligned}$$

The final identity follows from proposition 2.7 (iiia). □

Proposition 2.14 The following identities hold for any $g \in G, u \in U$:

$$\begin{aligned}\pi_{h^{**}}g &= \pi_h g, \\ \pi_{h^{**}}u &= \lambda, \\ \pi_{h^{**}}(gu) &= (\pi_h g)^u, \\ \pi_{h^{**}}(ug) &= \pi_h g, \\ \pi_{h^{**}}(g_1 u_1 g_2 u_2) &= (\pi_h g_1)^{u_1 (h g_2)^{u_2}} (\pi_h g_2)^{u_2}, \\ \pi_{h^{**}}(g_1 u_1 \dots g_n u_n) &= (\pi_h g_1)^{u_1 (h g_2)^{u_2} \dots (h g_n)^{u_n}} \dots (\pi_h g_n)^{u_n}.\end{aligned}$$

The following two pairs of identities, which expand commutators containing $\pi_{t^{**}}(ab)$ or $\pi_{h^{**}}(bc)$ follows immediately.

Corollary 2.15

$$\begin{aligned}[\pi_{t^{**}}(ab), (\pi_{h^{**}}c)] &= [(\pi_t a)^{tb}, (\pi_h c)]^{(\pi_t b)} [(\pi_t b), (\pi_h c)] \\ &= [(\pi_t b), (\pi_h c)]^{(\pi_t a)^b} [(\pi_t a)^b, (\pi_h c)]\end{aligned}$$

$$\begin{aligned}[\pi_{t^{**}}(a), \pi_{h^{**}}(bc)] &= [(\pi_t a), (\pi_h c)] [(\pi_t a), (\pi_h b)^{(hc)}]^{(\pi_h c)} \\ &= [(\pi_t a), (\pi_h b)^c] [(\pi_t a), (\pi_h c)]^{(\pi_h b)^c}.\end{aligned}$$

Let X_S be a generating set for $S = \ker t$, let $Y_S = X_S^R = \{g_1, \dots, g_n\}$ be the closure of X_S under the action of R , and let $T = \{c_1 = (), c_2, \dots, c_m\}$ be a transversal for the right cosets U/R .

Proposition 2.16 The kernels $\ker t^{**}$ and $\ker h^{**}$ have generating sets

$$Z_{t^{**}} = \{(1, g_i)^{c_j} \mid g_i \in Y_S, c_j \in T\},$$

$$Z_{h^{**}} = \{(\partial g_i^{-1}, g_i)^{c_j} \mid g_i \in Y_S, c_j \in T\},$$

where ∂ is an inclusion morphism.

Proof. In the cat^1 -group $(\mathcal{C} = e; t, h : G \rightarrow R)$, where $G = R \times S$, we have $t(r, s) = r$, $h(r, s) = r\partial s$, $e(r) = (r, 1)$, $\pi_t(r, s) = (1, s)$ and $\pi_h(r, s) = (\partial s^{-1}, s)$. Using Proposition 2.12, $t^{**}(r, s) = \iota t(r, s) = r$, $h^{**}(r, s) = \iota h(r, s) = r\partial s$ and $e^{**}r = (r, 1_S)$. We

also have $t_{**}u = h_{**}u = e_{**}u = u$, $\pi_{t_{**}}(r, s) = r^{-1}(r, s) = (1_R, s)$, $\pi_{t_{**}}u = \lambda$, $\pi_{h_{**}}(r, s) = \partial s^{-1}r^{-1}(r, s) = \partial s^{-1}(1_R, s)$ and $\pi_{h_{**}}u = \lambda$. Now $\pi_{t_{**}}$ is onto and, by Proposition 2.13,

$$\begin{aligned} \pi_{t_{**}}((r_1, s_1)u_1 \dots (r_\ell, s_\ell)u_\ell) &= (1, s_1)^{u_1(r_2, 1)u_2 \dots (r_\ell, 1)u_\ell} \dots (1, s_{\ell-1})^{u_{\ell-1}(r_\ell, 1)u_\ell} (1, s_\ell)^{u_\ell} \\ &= (1, s_1)^{u_1 r_2 u_2} \dots (1, s_\ell)^{u_\ell} \\ &= (1, s_1)^{u'_1} \dots (1, s_\ell)^{u'_\ell}, \end{aligned}$$

so every element of $\ker t_{**}$ has the form

$$(1, s_1)^{u_1} \dots (1, s_\ell)^{u_\ell}, \quad s_i \in S, u_i \in U.$$

Since

$$(1, s_1 s_2 \dots s_\ell)^u = ((1, s_1) \dots (1, s_\ell))^u = (1, s_1)^u \dots (1, s_\ell)^u,$$

we need only take a generating set for S . Furthermore, since $u = rc$ for some $r \in R$ and coset representation $c \in U$,

$$(1, s)^u = (1, s)^{rc} = (1, s^r)^c.$$

So $\ker t_{**}$ has a generating set

$$\{(1, g_i)^{c_j} \mid g_i \in Y_S, c_j \in T\}$$

and similarly $\ker h_{**}$ has a generating set

$$\{(\partial g_i^{-1}, g_i)^{c_j} \mid g_i \in Y_S, c_j \in T\}.$$

□

Proposition 2.17 The Peiffer commutator subgroup $P_{**} = [\ker t_{**}, \ker h_{**}]$ has normal generating set

$$Z_{P_{**}} = \{[(1, g_i)^{c_j}, (\partial g_k^{-1}, g_k)] \mid g_i, g_k \in Y_S, c_j \in T\}.$$

Proof. Since $\ker t_{**}$ is generated by $Z_{t_{**}}$ and $\ker h_{**}$ is generated by $Z_{h_{**}}$ it follows that P_{**} is normally generated by $\{[x, y] \mid x \in Z_{t_{**}}, y \in Z_{h_{**}}\}$. Also,

$$\begin{aligned} [x, y] &= [(1, g_i)^{c_j}, (\partial g_k^{-1}, g_k)^{c_\ell}] \\ &= [(1, g_i)^{c_j c_\ell^{-1}}, (\partial g_k^{-1}, g_k)]^{c_\ell} \\ &= [(1, g_i^r)^{c'_j}, (\partial g_k^{-1}, g_k)]^{c_\ell}, \end{aligned}$$

where $c_j c_\ell^{-1} = r c'$, $r \in R$, $c' \in T$. □

A smaller generating set may be obtained if we relax the requirement that the elements y are closed under the action of R .

Proposition 2.18 The Peiffer commutator subgroup P_{**} has a normal generating set

$$Z'_{P_{**}} = \{[(1, g_i)^{c_j}, (\partial g_k^{-1}, g_k)] \mid g_i \in Y_S, c_j \in T, g_k \in X_S\}.$$

Proof. Suppose $g_k = s_k^{r_k}$ where $s_k \in X_S, r_k \in R$. Then

$$\begin{aligned} [(1, g_i)^{c_j}, (\partial g_k^{-1}, g_k)^{c_\ell}] &= [(1, g_i)^{c_j}, ((\partial s_k^{-1})^{r_k}, s_k^{r_k})^{c_\ell}] \\ &= [(1, g_i)^{c_j r_k^{-1} c_\ell^{-1}}, (\partial s_k^{-1}, s_k)]^{r_k c_\ell} \\ &= [(1, g_i^{r'})^{c'}, (\partial s_k^{-1}, s_k)]^{r_k c_\ell}. \end{aligned}$$

□

3. Algorithm for Induced Cat^1 -groups

The induced cat^1 -group $\iota_{**}\mathcal{C}$ may be obtained by using **XModCat1** to construct \mathcal{X} , then **InducedXMod** to construct $\iota_{**}\mathcal{X}$ and then **Cat1XMod**. An alternative procedure is to calculate the induced cat^1 -group $\iota_{**}G = (G *_R U)/P_{**}$ directly. This has been implemented for the case when $\mathcal{C} = (e; t, h : G \rightarrow R)$ and $\iota : R \rightarrow U$ is an inclusion.

3.1. Record Structure for InducedCat1

The record structure for an induced cat^1 -group contains, in addition to the usual fields for a cat^1 -group,

IC.cat1 ,	the cat^1 -group \mathcal{C} ,
IC.name ,	written as $\text{IC}(\text{name of } \mathcal{C})$,
IC.morphism ,	the morphism $\langle \iota_{**}, \iota \rangle : \mathcal{C} \rightarrow \iota_{**}\mathcal{C}$,
IC.isInducedCat1 ,	a boolean flag normally true.

3.2. Algorithm for InducedCat1

The function **InducedCat1** is called as:

```
gap> InducedCat1( G, R, U );
gap> InducedCat1( C, iota );
```

The function requires as data a conjugation cat^1 -group $\mathcal{C} = (e; t, h : G \rightarrow R)$ and an inclusion morphism $\iota : R \rightarrow U$. The data may be specified using either of the two forms shown, where the first form requires $G \geq R \geq U$. As output, the function returns $\iota_{**}\mathcal{C} = (e_{**}; t_{**}, h_{**} : \iota_{**}G \rightarrow U)$ together with a morphism $\langle \iota_{**}, \iota \rangle : \mathcal{C} \rightarrow \iota_{**}\mathcal{C}$.

- Step 1** Check the argument: if the argument is a collection of groups then call **ConjugationCat1(G, R)**; to construct \mathcal{C} , and call **InclusionMorphism(R, U)**; to construct ι . Otherwise, **G:= C.source**; **R := C.range**; and **iota** is the second argument.
- Step 2** Obtain finitely presented groups G', U' isomorphic to G and U .
- Step 3** Construct a free F whose rank is the total length of the generating sets of G and U .
- Step 4** Map the relators for G', U' into words in F and, for each generator r of R , add the relation $(r \in G') = (r \in U')$.
- Step 5** Obtain coset representatives for U/R .
- Step 6** Construct generators for the Peiffer subgroup P_{**} .
- Step 7** Construct the finitely presented group $IG' = F/\text{rels}$ where **rels** is the set of relators constructed in steps 4 and 6.
- Step 8** Obtain a faithful permutation representation IG of IG' .
- Step 9** Construct homomorphisms **tstar**, **hstar** and **estar**.
- Step 10** Call **Cat1(IG, tstar, hstar, estar)**; to construct $\iota_{**}\mathcal{C}$.
- Step 11** Use **Cat1Morphism** to construct $\langle \iota_{**}, \iota \rangle$.
- Step 12** Add the fields described in section 3.1

3.3. Comparative timing

The following table gives timing in milliseconds for the calculation of some induced crossed modules and induced cat^1 -groups. Computations were performed on a Digital Alpha 64-bit workstation using 20M memory.

Conjugation cat ¹ -group crossed module	time		I	iota
	\mathcal{C}	\mathcal{IC}		
	\mathcal{X}	\mathcal{IX}		
k4 → c2	124	2605	36	c2 → s3
c2 → c2	59	1207	6	
a4a4 → a4	9429	107322	864	a4 → s4
a4 → a4	1705	7328	36	
c4c4 → c4	733	4395	128	c4 → d8
c4 → c4	152	2273	16	
c4c4 → c4	248	5135	128	c4 → c4c2
c4 → c4	150	1777	16	
c3c3 → c3	683	9789	288	c3 → a4
c3 → c3	116	3719	24	
d8d8 → d8	4131	89649	512	d8 → d8c2
d8 → d8	756	21317	32	
q8q8 → q8	4604	229838	512	q8 → q8c2
q8 → q8	1073	19663	32	
c4c2c4c2 → c4c2	723	6802340	1024	c4c2 → d8y4
c4c2 → c4c2	532	10183	64	
c3 ² c3 ² → c3 ²	726	6227880	1458	c3 ² → c3 ² × c2
c3 ² → c3 ²	551	10813	81	

Table 1. Sample timings for induced crossed modules and cat¹-groups

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