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A. SOFO

S. S. DRAGOMIR

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A. Sofo, S. S. Dragomir

Abstract

A generalisation of a perturbed version of the Ostrowski inequality for twice differentiable mappings is studied. It is shown that the error bounds are better than those obtained previously. Applications for general quadrature formulae are also given.

Key Words: Ostrowski Integral Inequality, Quadrature Formulae.

1. Introduction

The following theorem was proved by Ostrowski [9, p. 469] in 1938.

Theorem 1 *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative is bounded on (a,b) and denote $\|f'\|_\infty=\sup_{t\in(a,b)}|f'(t)|<\infty$. Then for all $x\in[a,b]$, we have the integral inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty . \qquad (1)$$

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

Dragomir and Wang [3] applied the Ostrowski inequality (1) to numerical quadrature problems and special means and extended the inequality (1) to include differentiable mappings whose derivatives belong to L_p spaces [4] and L_1-spaces [5].

Dragomir [6] also generalised the Ostrowski inequality for k points $x_1,\dots,x_k$ and obtained the following theorem.

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Theorem 2 *Let $I_k : a = x_0 < x_1 < ... < x_{k-1} < x_k = b$ be a division of the interval $[a,b]$ and α_i $(i = 0, ..., k+1)$ be "$k+2$" points so that $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1, ..., k)$ and $\alpha_{k+1} = b$. If $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$, then we have the inequality:*

$$\left| \int_a^b f(x)\,dx - \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) \right| \tag{2}$$

$$\leq \left[\frac{1}{4} \sum_{i=0}^{k-1} h_i^2 + \sum_{i=0}^{k-1} \left(\alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right)^2 \right] \|f'\|_\infty$$

$$\leq \frac{1}{2} \|f'\|_\infty \sum_{i=0}^{k-1} h_i^2$$

$$\leq \frac{1}{2} (b-a) \nu(h) \|f'\|_\infty ,$$

where $\nu(h) := \max\{h_i | i = 0, ..., k-1\}$, $h_i := x_{i+1} - x_i$ $(i = 0, ..., k-1)$, and $\|f'\|_\infty := ess \sup_{t \in [a,b]} |f'(t)| < \infty$.

The constant $\frac{1}{4}$ in the first inequality and the constant $\frac{1}{2}$ in the second and third inequalities are the best possible.

Note: If in Theorem 2 we take $k = 2$, $\alpha_1 = a$, $\alpha_2 = b$, $x_1 = x$ and upon division by $(b-a)$ we obtain Theorem 1.

The inequality (2) has also been extended to include results for the $L_1[a,b]$ −norm [7] and the $L_p[a,b]$-norm [8].

Cerone, Dragomir and Roumeliotis [1] then extended the Ostrowski result (1) by considering mappings whose second derivatives are bounded. They obtained:

Theorem 3 *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, twice differentiable on $[a,b]$ and $f'' \in L_1[a,b]$. Then we have the integral inequality:*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt - \left(x - \frac{a+b}{2} \right) f'(x) \right| \tag{3}$$

$$\leq \left[\frac{(b-a)^2}{24} + \frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 \right] \|f'\|_\infty$$

$$\leq \frac{(b-a)^2}{6} \|f'\|_\infty$$

for all $x \in [a, b]$.

Theorem 3 may be thought of as a perturbed version of the result (1). The inequality (3) can also be extended to the cases $f'' \in L_p[a, b]$ and $f'' \in L_1[a, b]$.

The main aim of this paper is to give an analogous theorem a perturbed version, for k points $x_1, ..., x_k$, of Theorem 3, for twice differentiable mappings which belong to the usual three norms. We also apply our theorem to some quadrature formulae. We state the main theorem as follows.

2. A Perturbed Ostrowski Theorem

Theorem 4 *Let* $I_k : a = x_0 < x_1 < ... < x_{k-1} < x_k = b$ *be a division of the interval* $[a, b]$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1, ..., k)$ *and* $\alpha_{k+1} = b$. *If* $f : [a, b] \to \mathbb{R}$ *is absolutely continuous, then we have the inequality*

$$\left| \int_a^b f(t)\,dt + \frac{1}{2}\sum_{i=0}^{k-1}\left\{ (x_{i+1} - \alpha_{i+1})^2 f'(x_{i+1}) - (x_i - \alpha_{i+1})^2 f'(x_i) \right\} \right. \tag{4}$$
$$\left. - \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) \right|$$

$$\leq \begin{cases} \frac{\|f''\|_\infty}{6} \sum_{i=0}^{k-1} \left[(\alpha_{i+1} - x_i)^3 + (x_{i+1} - \alpha_{i+1})^3 \right] \\ \quad \leq \frac{\|f''\|_\infty}{6} \sum_{i=0}^{k-1} h_i^3 \leq \frac{\|f''\|_\infty}{6} (b-a)\,\nu^2(h)\,, \text{ where } f'' \in L_\infty[a, b]\,; \\ \\ \frac{\|f''\|_p}{2(2q+1)^{\frac{1}{q}}} \left(\sum_{i=0}^{k-1} \left[(\alpha_{i+1} - x_i)^{2q+1} + (x_{i+1} - \alpha_{i+1})^{2q+1} \right] \right)^{\frac{1}{q}} \\ \quad \leq \frac{\|f''\|_p}{2(2q+1)^{\frac{1}{q}}} \left(\sum_{i=0}^{k-1} h_i^{2q+1} \right)^{\frac{1}{q}} \leq \frac{\|f''\|_p}{2(2q+1)^{\frac{1}{q}}} (b-a)^{\frac{1}{q}} \nu^2(h)\,, \\ \quad \text{where } f'' \in L_p[a, b]\,,\ p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \left[\frac{\nu^2(h)}{16} + \frac{\rho^2(\delta)}{4} + \frac{\nu(h)}{2} \max_{i=0,\dots,k-1} |\delta_i| \right] \|f''\|_1 \leq \frac{3}{8}\nu^2(h)\,\|f''\|_1\,, \\ \quad \text{where } f'' \in L_1[a, b]\,. \end{cases}$$

Here $h_i := x_{i+1} - x_i$,

$$\nu(h) := \max\{h_i | i = 0, ..., k-1\},$$
$$\delta_i := \left(\alpha_{i+1} - \frac{x_i + x_{i+1}}{2}\right),$$
$$\rho(\delta) := \max\{\delta_i | i = 0, ..., k-1\}.$$

Proof. Consider the kernel $K : [a,b] \to \mathbb{R}$ given by

$$K(t) := \begin{cases} \frac{(t-\alpha_1)^2}{2}, & t \in [a, x_1) \\ \frac{(t-\alpha_2)^2}{2}, & t \in [x_1, x_2) \\ \vdots & \\ \frac{(t-\alpha_{k-1})^2}{2}, & t \in [x_{k-2}, x_{k-1}) \\ \frac{(t-\alpha_k)^2}{2}, & t \in [x_{k-1}, b]. \end{cases}$$

Successively integrating by parts, we have that

$$\begin{aligned} & \int_a^b K(t) f''(t)\,dt \\ = & \int_a^b f(t)\,dt + \frac{1}{2}\sum_{i=0}^{k-1}\left\{(x_{i+1} - \alpha_{i+1})^2 f'(x_{i+1}) - (x_i - \alpha_{i+1})^2 f'(x_i)\right\} \\ & - \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) \end{aligned}$$

and

$$\begin{aligned} & \left| \int_a^b f(t)\,dt + \frac{1}{2}\sum_{i=0}^{k-1}\left\{(x_{i+1} - \alpha_{i+1})^2 f'(x_{i+1}) - (x_i - \alpha_{i+1})^2 f'(x_i)\right\} \right. \qquad (5) \\ & \left. - \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) \right| \\ = & \left| \int_a^b K(t) f''(t)\,dt \right|. \end{aligned}$$

In the first case, consider $f'' \in L_\infty[a,b]$, hence

$$\left|\int_a^b K(t) f''(t)\,dt\right| \le \|f''\|_\infty \int_a^b |K(t)|\,dt.$$

$$\begin{aligned}\int_a^b |K(t)|\,dt &= \int_a^{x_1} \frac{(t-\alpha_1)^2}{2}dt + ... + \int_{x_{k-1}}^b \frac{(t-\alpha_k)^2}{2}dt \\ &= \frac{1}{6}\left\{\sum_{i=0}^{k-1}\left[(\alpha_{i+1}-x_i)^3 + (x_{i+1}-\alpha_{i+1})^3\right]\right\}.\end{aligned}$$

Using the inequality $(A-B)^n + (C-A)^n \le (C-B)^n$,

$$\begin{aligned}\int_a^b |K(t)|\,dt &\le \frac{1}{6}\sum_{i=0}^{k-1}(x_{i+1}-x_i)^3 = \frac{1}{6}\sum_{i=0}^{k-1}h_i^3 \le \frac{1}{6}\nu^2(h)\sum_{i=0}^{k-1}h_i \\ &\le \frac{1}{6}(b-a)\,\nu^2(h),\end{aligned}$$

where $h_i := x_{i+1} - x_i$. Hence

$$\begin{aligned}\left|\int_a^b K(t) f''(t)\,dt\right| &\le \frac{\|f''\|_\infty}{6}\nu^2(h)\sum_{i=0}^{k-1}h_i \\ &\le \frac{\|f''\|_\infty}{6}\nu^2(h)(b-a) \quad \text{where } f'' \in L_\infty[a,b]\end{aligned}$$

and combining this with (5) gives us our first inequality in (4).
In the second case, consider $f'' \in L_p \in [a,b]$, hence by Hölder's inequality

$$\left|\int_a^b K(t) f''(t)\,dt\right| \le \left(\int_a^b |K(t)|^q\,dt\right)^{\frac{1}{q}} \|f''\|_p\,.$$

$$\int_a^b |K(t)|^q \, dt = \int_a^{\alpha_1} \frac{(\alpha_1 - t)^{2q}}{2^q} dt + \int_{\alpha_1}^{x_1} \frac{(t-\alpha_1)^{2q}}{2^q} dt + ... + \int_{x_{k-1}}^{b} \frac{(t-\alpha_k)^{2q}}{2^q} dt$$

$$= \frac{(\alpha_1 - a)^{2q+1}}{2^q(2q+1)} + \frac{(x_1 - \alpha_1)^{2q+1}}{2^q(2q+1)} + ... + \frac{(\alpha_k - x_{k-1})^{2q+1}}{2^q(2q+1)} + \frac{(x_k - \alpha_k)^{2q+1}}{2^q(2q+1)}$$

$$= \frac{1}{2^q(2q+1)} \sum_{i=0}^{k-1} \left\{ (\alpha_{i+1} - x_i)^{2q+1} + (x_{i+1} - \alpha_{i+1})^{2q+1} \right\}$$

$$\leq \frac{1}{2^q(2q+1)} \sum_{i=0}^{k-1} h_i^{2q+1} \leq \frac{1}{2^q(2q+1)} \max_{i=0,\ldots,k-1} \left\{ h_i^{2q} \right\} \sum_{i=0}^{k-1} h_i$$

$$= \frac{1}{2^q(2q+1)} (b-a) \nu^{2q}(h).$$

Thus, from this result we have

$$\left| \int_a^b K(t) f''(t) \, dt \right| \leq \frac{\|f''\|_p}{2(2q+1)^{\frac{1}{q}}} \left(\sum_{i=0}^{k-1} h_i^{2q+1} \right)^{\frac{1}{q}} \leq \frac{\|f''\|_p \, \nu^2(h)}{2(2q+1)^{\frac{1}{q}}} (b-a)^{\frac{1}{q}}$$

and from (5) we deduce the second part of the inequality (4).

Finally, consider $f'' \in L_1[a,b]$.

$$\left| \int_a^b K(t) f''(t) \, dt \right| \leq \int_a^b |K(t)| \, |f''(t)| \, dt.$$

$$\left| \int_a^b K(t) f''(t) \, dt \right| = \left| \sum_{i=0}^{k-1} \int_{xli}^{x_{i+1}} K(t) f''(t) \, dt \right| \leq \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} |K(t)| \, |f''(t)| \, dt$$

$$= \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} \left| \frac{(t - \alpha_{i+1})^2}{2} \right| |f''(t)| \, dt =: W.$$

Now

$$\begin{aligned}
&\int_{x_i}^{x_{i+1}} \left| \frac{(t-\alpha_{i+1})^2}{2} \right| |f''(t)|\, dt \\
&\leq \sup_{t\in[x_i,x_{i+1}]} \left| \frac{(t-\alpha_{i+1})^2}{2} \right| \int_{x_i}^{x_{i+1}} |f''(t)|\, dt \\
&= \max_{i=0,\dots,k-1} \left\{ \frac{(\alpha_{i+1}-x_i)^2}{2}, \frac{(x_{i+1}-\alpha_{i+1})^2}{2} \right\} \int_{x_i}^{x_{i+1}} |f''(t)|\, dt.
\end{aligned}$$

We may now observe that

$$\max_{i=0,\dots,k-1} \frac{1}{2} \left\{ (\alpha_{i+1}-x_i)^2, (x_{i+1}-\alpha_{i+1})^2 \right\}$$

$$= \max_{i=0,\dots,k-1} \frac{1}{2} \left\{ \frac{(x_{i+1}-\alpha_{i+1})^2 + (\alpha_{i+1}-x_i)^2}{2} + \frac{\left| (\alpha_{i+1}-x_i)^2 - (x_{i+1}-\alpha_{i+1})^2 \right|}{2} \right\}.$$

Using the identity

$$A^2 + B^2 = \left(\frac{A+B}{2}\right)^2 + \left(\frac{A-B}{2}\right)^2$$

reduces the previous line to

$$\begin{aligned}
&\max_{i=0,\dots,k-1} \frac{1}{2} \left[\frac{(x_{i+1}-x_i)^2}{8} + \frac{1}{2}\left(\alpha_{i+1} - \frac{x_{i+1}+x_i}{2}\right)^2 \right. \\
&\left. + (x_{i+1}-x_i) \left| \alpha_{i+1} - \frac{x_{i+1}+x_i}{2} \right| \right] \\
&= \max_{i=0,\dots,k-1} \left[\frac{h_i^2}{16}\right] + \max_{i=0,\dots,k-1} \left[\frac{\delta_i^2}{4}\right] + \max_{i=0,\dots,k-1} \left[\frac{h_i}{2} |\delta_i|\right] \\
&= \frac{\nu^2(h)}{16} + \frac{\rho^2(\delta)}{4} + \frac{\nu(h)}{2} \max_{i=0,\dots,k-1} |\delta_i|. \qquad (6)
\end{aligned}$$

Combining (6) with W gives us the first part of the last line in the inequality (4).

We note that from

$$|\delta_i| = \left|\alpha_{i+1} - \frac{x_{i+1}+x_i}{2}\right| \leq \frac{h_i}{2},$$

$$\delta_i^2 \leq \left(\frac{h_i}{2}\right)^2,$$

and so from (6)

$$\leq \frac{\nu^2(h)}{16} + \frac{\nu^2(h)}{16} + \frac{\nu^2(h)}{4} = \frac{3}{8}\nu^2(h).$$

Thus

$$\left|\int_a^b K(t) f''(t)\, dt\right| \leq \frac{3}{8}\nu^2(h)\, \|f''\|_1,$$

which gives us the last part of the final line in the inequality (4), therefore Theorem 4 is proved. □

If in Theorem 4 we take $k = 2$, in which case $x_0 = a$, $x_1 = x$, $x_2 = b$, $\alpha_0 = a$, $\alpha_1 = a$, $\alpha_2 = b$ and $\alpha_3 = b$, then upon dividing (4) by $(b - a)$ we obtain the result (3).

The real benefit of our new inequality (4) is that it is of order 2, while the inequality (2) is of order 1.

If we now assume that the points of the division I_k are fixed, the best inequality we can obtain from Theorem 4 is embodied in the following corollary.

Corollary 1 *Let $I_k : a = x_0 < x_1 < ... < x_{k-1} < x_k = b$ be a division of the interval $[a, b]$. If f is as above, then we have the inequality*

$$\left|\int_a^b f(t)\, dt + \frac{1}{8}\left[-(x_1 - a)^2 f'(a) + \right.\right. \tag{7}$$

$$\left.\sum_{i=1}^{k-1} (x_{i+1} - x_{i-1})(2x_i - x_{i-1} - x_{i+1}) f'(x_i) + (b - x_{k-1})^2 f'(b)\right]$$

$$\left.-\frac{1}{2}\left[(x_1 - a) f(a) + \sum_{i=1}^{k-1} (x_{i+1} - x_{i-1}) f(x_i) + (b - x_{k-1}) f(b)\right]\right|$$

$$\leq \begin{cases} \frac{\|f''\|_\infty}{24} \sum\limits_{i=0}^{k-1} h_i^3 \leq \frac{\|f''\|_\infty}{6} (b-a) \nu^2(h), & \textit{where} \quad f'' \in L_\infty[a,b]; \\ \frac{\|f''\|_p}{8(2q+1)^{\frac{1}{q}}} \left(\sum\limits_{i=0}^{k-1} h_i^{2q+1} \right)^{\frac{1}{q}} \leq \frac{\|f''\|_p}{2(2q+1)^{\frac{1}{q}}} (b-a)^{\frac{1}{q}} \nu^2(h), & \textit{where} \quad f'' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{3}{8}\nu^2(h)\|f''\|_1 & \textit{where} \quad f'' \in L_1[a,b]. \end{cases}$$

Proof. Choose in Theorem 4,

$$\alpha_0 = a, \quad \alpha_1 = \frac{a+x_1}{2}, \quad \alpha_2 = \frac{x_1+x_2}{2}, \ \ldots$$
$$\alpha_{k-1} = \frac{x_{k-2}+x_{k-1}}{2}, \quad \alpha_k = \frac{x_{k-1}+x_k}{2} \quad \text{and} \quad \alpha_{k+1} = b.$$

The term

$$\begin{aligned} &\sum_{i=0}^{k} (\alpha_{i+1}-\alpha_i) f(x_i) \\ &= \left(\frac{a+x_1}{a} - a \right) f(a) + \left(\frac{x_1+x_2}{2} - \frac{a+x_1}{a} \right) f(x_1) + \ldots \\ &\quad + \left(\frac{x_{k-1}+b}{2} - \frac{x_{k-2}+x_{k-1}}{2} \right) f(x_{k-1}) + \left(b - \frac{x_{k-1}+b}{2} \right) f(b) \\ &= \frac{1}{2} \left[(x_1-a) f(a) + \sum_{i=1}^{k-1} (x_{i+1}-x_{i-1}) f(x_i) + (b-x_{k-1}) f(b) \right], \end{aligned} \tag{8}$$

and the term

$$\begin{aligned} &\sum_{i=0}^{k-1} \left\{ (x_{i+1}-\alpha_{i+1})^2 f'(x_{i+1}) - (x_i-\alpha_{i+1})^2 f'(x_i) \right\} \\ &= \frac{1}{4} \sum_{i=0}^{k-1} \left\{ (x_{i+1}-x_i)^2 f'(x_{i+1}) - (x_{i+1}-x_i)^2 f'(x_i) \right\} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\sum_{i=0}^{k-1} h_i^2\left(f'\left(x_{i+1}\right)-f'\left(x_i\right)\right), \text{ where } h_i = x_{i+1}-x_i,\\
&= \frac{1}{4}\Big[\left(x_1-a\right)^2\left(f'\left(x_1\right)-f'\left(a\right)\right)+\left(x_2-x_1\right)^2\left(f'\left(x_2\right)-f'\left(x_1\right)\right)+\dots\\
&\quad+\left(x_{k-1}-x_{k-2}\right)^2\left(f'\left(x_{k-1}\right)-f'\left(x_{k-2}\right)\right)+\left(x_k-x_{k-1}\right)^2\left(f'\left(x_k\right)-f'\left(x_{k-1}\right)\right)\Big]\\
&= \frac{1}{4}\Big[-\left(x_1-a\right)^2 f'\left(a\right)\\
&\quad+\sum_{i=1}^{k-1}\Big\{\left(x_i-x_{i-1}\right)^2-\left(x_{i+1}-x_i\right)^2\Big\} f'\left(x_i\right)+\left(b-x_{k-1}\right) f'\left(b\right)\Big]\\
&= \frac{1}{4}\Big[-\left(x_1-a\right)^2 f'\left(a\right)+\\
&\quad\sum_{i=1}^{k-1}\left(x_{i+1}-x_{i-1}\right)\left(2x_i-x_{i-1}-x_{i+1}\right) f'\left(x_i\right)+\left(b-x_{k-1}\right)^2 f'\left(b\right)\Big]. \qquad (9)
\end{aligned}
$$

Substituting (8) and (9) into (4), we obtain the result (7). □

The case of equidistant partitioning is important in practice, and with this in mind we obtain the following corollary.

Corollary 2 *Let*

$$I_k : x_i = a + i \cdot \frac{b-a}{k} \quad (i = 0, ..., k)$$

be an equidistant partitioning of $[a, b]$*, then we have the inequality*

$$\left| \int_a^b f(t)\,dt + \frac{(b-a)^2}{8k^2}\{f'(b) - f'(a)\} \right. \tag{10}$$

$$\left. - \frac{(b-a)}{k}\left\{\frac{f(a)+f(b)}{2} + \sum_{i=1}^{k-1} f\left(\frac{(k-i)a+ib}{k}\right)\right\}\right|$$

$$\leq \begin{cases} \|f''\|_\infty \frac{(b-a)^3}{24k^2}, \\ \frac{\|f''\|_p (b-a)^{2+\frac{1}{q}}}{8(2q+1)^{\frac{1}{q}} k^2}, \\ \frac{3(b-a)^2}{8k^2}\|f''\|_1 . \end{cases}$$

Proof. The proof follows directly from Corollary 1 by noting that after substituting

$$x_i = a + i\left(\frac{b-a}{k}\right), \quad i = 0, ..., k,$$

we have from (7)

$$\frac{1}{2}\left[(x_1 - a) f(a) + \sum_{i=1}^{k-1} (x_{i+1} - x_{i-1}) f(x_i) + (b - x_{k-1}) f(b)\right]$$

$$= \frac{b-a}{k}\left\{\frac{f(a)+f(b)}{2} + \sum_{i=1}^{k-1} f\left(\frac{(k-i)a+ib}{k}\right)\right\},$$

$$\frac{1}{8}\left[-(x_1 - a)^2 f'(a) + \right.$$

$$\left. \sum_{i=1}^{k-1} (x_{i+1} - x_{i-1})(2x_i - x_{i-1} - x_{i+1}) f'(x_i) + (b - x_{k-1})^2 f'(b)\right]$$

$$= \frac{(b-a)^2}{8k^2}(f'(b) - f'(a));$$

hence the inequality (10) follows. □

3. The Convergence of a General Quadrature Formula

Let $\Delta_n : a = x_0^{(n)} < x_1^{(n)} < ... < x_{n-1}^{(n)} < x_n^{(n)} = b$ be a sequence of divisions of $[a, b]$ and consider the sequence of real numerical integration formulae

$$\begin{aligned} & I_n\left(f, f', \Delta_n, w_n\right) \qquad (11) \\ := & \sum_{j=0}^{n} w_j^{(n)} f\left(x_j^{(n)}\right) + \frac{1}{2} \sum_{j=0}^{n-1} \left[\left(x_j^{(n)} - a - \sum_{r=0}^{j} w_r^{(n)} \right)^2 f'\left(x_j^{(n)}\right) \right. \\ & \left. - \left(x_{j+1}^{(n)} - a - \sum_{r=0}^{j} w_r^{(n)} \right)^2 f'\left(x_{j+1}^{(n)}\right) \right], \end{aligned}$$

where w_j $(j = 0, ..., n)$ are the quadrature weights.

The following theorem contains a sufficient condition for the weights $w_j^{(n)}$ so that $I_n(f, f', \Delta_n, w_n)$ approximates the integral $\int_a^b f(t)\,dt$.

Theorem 5 *Let $f : [a, b] \to \mathbb{R}$ be a continuous mapping on $[a, b]$. If the quadrature weights, $w_j^{(n)}$ satisfy the condition*

$$x_i^{(n)} - a \le \sum_{j=0}^{i} w_j^{(n)} \le x_{i+1}^{(n)} - a \quad \text{for all} \quad i = 0, ..., n-1, \qquad (12)$$

then we have the estimation

$$\left| I_n\left(f, f', \Delta_n, w_n\right) - \int_a^b f(t)\,dt \right| \qquad (13)$$

$$\leq \begin{cases} \frac{\|f''\|_\infty}{6} \sum_{i=0}^{n-1} \left[\left(a + \sum_{j=0}^{i} w_j^{(n)} - x_i^{(n)} \right)^3 + \left(x_{i+1}^{(n)} - a - \sum_{j=0}^{i} w_j^{(n)} \right)^3 \right] \\ \qquad \leq \frac{\|f''\|_\infty}{6} \sum_{i=0}^{n-1} \left(h_i^{(n)} \right)^3 \leq \frac{\|f''\|_\infty}{6} \nu^2 \left(h^{(n)} \right) (b-a), \text{ where } f'' \in L_\infty [a,b]; \\ \\ \frac{\|f''\|_p}{2(2q+1)^{\frac{1}{q}}} \left(\sum_{i=0}^{n-1} \left[\left(a + \sum_{j=0}^{i} w_j^{(n)} - x_i^{(n)} \right)^{2q+1} + \left(x_{i+1}^{(n)} - a - \sum_{j=0}^{i} w_j^{(n)} \right)^{2q+1} \right] \right)^{\frac{1}{q}} \\ \qquad \leq \frac{\|f''\|_p}{2(2q+1)^{\frac{1}{q}}} \left(\sum_{i=0}^{n-1} \left(h_i^{(n)} \right)^{2q+1} \right)^{\frac{1}{q}} \leq \frac{\|f''\|_p}{2(2q+1)^{\frac{1}{q}}} (b-a)^{\frac{1}{q}} \nu^2 \left(h^{(n)} \right) \\ \qquad\qquad \text{where } f'' \in L_p [a,b], \ p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \left[\frac{\nu^2\left(h^{(n)}\right)}{16} + \frac{\rho^2\left(\delta^{(n)}\right)}{4} + \frac{\nu\left(h^{(n)}\right)}{2} \max_{i=0,\ldots,n-1} \left| \delta_i^{(n)} \right| \right] \|f''\|_1 \leq \frac{3}{8} \nu^2 \left(h^{(n)} \right) \|f''\|_1 \\ \qquad\qquad \text{where } f'' \in L_1 [a,b]. \end{cases}$$

Also,

$$\nu \left(h^{(n)} \right) = \max_{i=0,\ldots,n-1} \left\{ h_i^{(n)} \right\}, \quad h_i^{(n)} := x_{i+1}^{(n)} - x_i^{(n)},$$

$$\rho \left(\delta^{(n)} \right) := \max_{i=0,\ldots,n-1} \left\{ \delta_i^{(n)} \right\} \quad \text{and} \quad \delta_i^{(n)} := \alpha_{i+1}^{(n)} - \frac{x_{i+1}^{(n)} + x_i^{(n)}}{2}.$$

In particular, if $\|f''\|_{\infty,p,1} < \infty$*, then*

$$\lim_{\nu\left(h^{(n)}\right) \to 0} I_n \left(f, f', \Delta_n, w_n \right) = \int_a^b f(t)\, dt$$

uniformly by rapport of the weight w_n*.*

Proof. Define the sequence of real numbers

$$\alpha_{i+1}^{(n)} := a + \sum_{j=0}^{i} w_j^{(n)}, \quad i = 0, \ldots, n.$$

Note that

$$\alpha_{n+1}^{(n)} := a + \sum_{j=0}^{n} w_j^{(n)} = a + b - a = b.$$

By the assumption (12), we have

$$\alpha_{i+1}^{(n)} \in \left[x_i^{(n)}, x_{i+1}^{(n)}\right] \quad \text{for all} \quad i = 0, \ldots, n-1.$$

Define $\alpha_0^{(n)} = a$ and compute

$$\alpha_1^{(n)} - \alpha_0^{(n)} = w_0^{(n)},$$

$$\alpha_{i+1}^{(n)} - \alpha_i^{(n)} = a + \sum_{j=0}^{i} w_j^{(n)} - a - \sum_{j=0}^{i-1} w_j^{(n)} = w_i^{(n)} \quad (i = 0, \ldots, n-1)$$

and

$$\alpha_{n+1}^{(n)} - \alpha_n^{(n)} = a + \sum_{j=0}^{n} w_j^{(n)} - a - \sum_{j=0}^{n-1} w_j^{(n)} = w_n^{(n)}.$$

Consequently,

$$\sum_{i=0}^{n} \left(\alpha_{i+1}^{(n)} - \alpha_i^{(n)}\right) f\left(x_i^{(n)}\right) = \sum_{i=0}^{n} w_i^{(n)} f\left(x_i^{(n)}\right),$$

and let

$$\sum_{i=0}^{n} w_i^{(n)} f\left(x_i^{(n)}\right) + \frac{1}{2} \sum_{i=0}^{n} \left[\left(x_i^{(n)} - a + \sum_{j=0}^{i} w_j^{(n)}\right)^2 f'\left(x_i^{(n)}\right)\right.$$

$$\left. - \left(x_{i+1}^{(n)} - a - \sum_{j=0}^{i} w_j^{(n)}\right)^2 f'\left(x_{i+1}^{(n)}\right)\right] := I_n\left(f, f', \Delta_n, w_n\right).$$

Applying the inequality (4) we obtain the estimate (13).

The uniform convergence by rapport of quadrature weights $w_j^{(n)}$ is obvious by the last terms in (13). □

The case when the partitioning is equidistant is important in practice. Consider the equidistant partition

$$E_n : x_i^{(n)} := a + i \cdot \frac{b-a}{n} \quad (i = 0, ..., n)$$

and define the sequence of numerical quadrature formulae

$$\begin{aligned} I_n(f, f', w_n) :=& \sum_{i=0}^{n} w_j^{(n)} f\left(a + i \cdot \frac{(b-a)}{n}\right) \\ &+ \frac{1}{2} \sum_{i=0}^{n} \left[\left(i\left(\frac{b-a}{n}\right) - \sum_{j=0}^{i} w_j^{(n)} \right)^2 f'\left(a + i\left(\frac{b-a}{n}\right)\right) \right. \\ &\left. - \left((i+1)\left(\frac{b-a}{n}\right) - \sum_{j=0}^{i} w_j^{(n)} \right)^2 f'\left(a + (i+1)\left(\frac{b-a}{n}\right)\right) \right]. \end{aligned}$$

The following result follows.

Corollary 3 *Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous on $[a, b]$. If the quadrature weights $w_j^{(n)}$ satisfy the condition*

$$\frac{i}{n} \le \frac{1}{b-a} \sum_{j=0}^{i} w_j^{(n)} \le \frac{i+1}{n}, \quad i = 0, \ldots, n-1,$$

then the following bound holds:

$$\left| I_n \left(f, f', w_n\right) - \int_a^b f(t)\, dt \right|$$

$$\leq \begin{cases} \frac{\|f''\|_\infty}{6} \sum_{i=0}^{n-1} \left[\left(\sum_{j=0}^{i} w_j^{(n)} - i \left(\frac{b-a}{n}\right) \right)^3 + \left((i+1)\left(\frac{b-a}{n}\right) - \sum_{j=0}^{i} w_j^{(n)} \right)^3 \right] \\ \qquad \leq \frac{\|f''\|_\infty}{6n^3} (b-a)^3 ; \quad \text{where } f'' \in L_\infty [a,b] ; \\ \frac{\|f''\|_p}{2(2q+1)^{\frac{1}{q}}} \left(\sum_{i=0}^{n-1} \left[\left(\sum_{j=0}^{i} w_j^{(n)} - i \left(\frac{b-a}{n}\right) \right)^{2q+1} \right.\right. \\ \qquad \left.\left. + \left((i+1)\left(\frac{b-a}{n}\right) - \sum_{j=0}^{i} w_j^{(n)} \right)^{2q+1} \right] \right)^{\frac{1}{q}} \leq \frac{\|f''\|_p}{2(2q+1)^{\frac{1}{q}}} \left(\frac{b-a}{n}\right)^{2+\frac{1}{q}} , \\ \qquad \text{where } f'' \in L_p [a,b] , \ p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{3}{8} \left(\frac{b-a}{n}\right)^2 \|f''\|_1 , \qquad \text{where } f'' \in L_1 [a,b] . \end{cases}$$

In particular, if $\|f''\|_{\infty,p,1} < \infty$*, then*

$$\lim_{n \to \infty} I_n \left(f, f', w_n\right) = \int_a^b f(t)\, dt$$

uniformly by rapport of the weight w_n*.*

4. Some Particular Integral Inequalities

In this section we utilize the results of the previous sections to point out some particular inequalities and generalize some classical results such as the: Rectangle inequality, Trapezoid inequality, Ostrowski's Inequality, Midpoint inequality and Simpson's inequality.

Proposition 1 *Let* $f : [a,b] \to \mathbb{R}$ *be absolutely continuous on* $[a,b]$ *and* $\alpha \in [a,b]$*. Then* *we have the inequality:*

$$\left|\int_a^b f(t)\,dt + \frac{(b-\alpha)^2}{2} f'(b) - \frac{(a-\alpha)^2}{2} f'(a) - (b-\alpha) f(b) - (\alpha - a) f(a)\right| \quad (14)$$

$$\leq \begin{cases} \frac{\|f''\|_\infty}{6} \left[(\alpha-a)^3 + (b-\alpha)^3\right] \leq \frac{\|f''\|_\infty}{6} (b-a)^3; \\ \frac{\|f''\|_p}{2(2q+1)^{\frac{1}{q}}} \left(\left[(\alpha-a)^{2q+1} + (b-\alpha)^{2q+1}\right]\right)^{\frac{1}{q}} \leq \frac{\|f''\|_p}{2(2q+1)^{\frac{1}{q}}} (b-a)^{2+\frac{1}{q}}; \\ \frac{3}{8}(b-a)^2 \|f''\|_1 . \end{cases}$$

Proof. From Theorem 4, we choose $x_0 = a$, $\alpha_0 = a$, $x_1 = b$, $\alpha_2 = b$ and $\alpha_1 = \alpha \in [a, b]$. Hence $\nu(h) = \max(h_i | i = 0, ..., k-1) = b - a$ and the inequality (14) follows. □

Remark 1 (*a*) *If in (14) we substitute $\alpha = b$, then we obtain the perturbed 'left rectangle inequality'*

$$\left|\int_a^b f(t)\,dt - \frac{(b-a)^2}{2} f'(a) - (b-a) f(a)\right| \leq \begin{cases} \frac{(b-a)^3}{6} \|f''\|_\infty; \\ \leq \frac{(b-a)^{2+\frac{1}{q}}}{2(2q+1)^{\frac{1}{q}}} \|f''\|_p; \\ \frac{3}{8}(b-a)^2 \|f''\|_1 . \end{cases} \quad (15)$$

(*b*) *If $\alpha = a$, from (14) we obtain the perturbed 'right rectangle inequality'*

$$\left|\int_a^b f(t)\,dt + \frac{(b-a)^2}{2} f'(b) - (b-a) f(b)\right| \leq \begin{cases} \frac{(b-a)^3}{6} \|f''\|_\infty; \\ \leq \frac{(b-a)^{2+\frac{1}{q}}}{2(2q+1)^{\frac{1}{q}}} \|f''\|_p; \\ \frac{3}{8}(b-a)^2 \|f''\|_1 . \end{cases} \quad (16)$$

(*c*) *In the case that $\alpha = \frac{a+b}{2}$, then from (14) we obtain the best estimate, a 'perturbed* *trapezoid inequality'*

$$\left|\int_a^b f(t)\,dt + \frac{(b-a)^2}{8}\left[f'(b)-f'(a)\right] - \frac{(b-a)}{2}\left[f(b)+f(a)\right]\right| \tag{17}$$

$$\leq \begin{cases} \frac{(b-a)^3}{24}\left\|f''\right\|_\infty ; \\ \frac{(b-a)^{2+\frac{1}{q}}}{8(2q+1)^{\frac{1}{q}}}\left\|f''\right\|_p ; \\ \frac{3}{8}(b-a)^2\left\|f''\right\|_1 . \end{cases}$$

The following result produces another integral inequality with many applications.

Proposition 2 *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous mapping on $[a,b]$ and $a \leq x_1 \leq b$, $a \leq \alpha_1 \leq x_1 \leq \alpha_2 \leq b$. Then we have*

$$\left|\int_a^b f(t)\,dt + \left(\frac{(x_1-\alpha_1)^2}{2} - \frac{(x_1-\alpha_2)^2}{2}\right) f'(x_1) + \frac{(b-\alpha_2)^2}{2} f'(b) \right. \tag{18}$$

$$\left. -\frac{(\alpha_1-a)^2}{2} f'(a) - (\alpha_2-\alpha_1) f(x_1) - (\alpha_1-a) f(a) - (b-\alpha_2) f(b) \right|$$

$$\leq \begin{cases} \frac{\left\|f''\right\|_\infty}{6}\left[(\alpha_1-a)^3 + (x_1-\alpha_1)^3 + (\alpha_2-x_1)^3 + (b-\alpha_2)^3\right] \\ \leq \frac{\left\|f''\right\|_\infty}{6}\left((x_1-a)^3 + (b-x_1)^3\right) \leq \frac{\left\|f''\right\|_\infty}{6}(b-a)^3 ; \\ \frac{\left\|f''\right\|_p}{2(2q+1)^{\frac{1}{q}}}\left((\alpha_1-a)^{2q+1} + (x_1-\alpha_1)^{2q+1} + (\alpha_2-x_1)^{2q+1} + (b-\alpha_2)^{2q+1}\right)^{\frac{1}{q}} \\ \leq \frac{\left\|f''\right\|_p}{2(2q+1)^{\frac{1}{q}}}(b-a)^{2+\frac{1}{q}} ; \\ \frac{3}{8}(b-a)^2\left\|f''\right\|_1 . \end{cases}$$

Proof. Consider the division $a = x_0 \leq x_1 \leq x_2 = b$ and the numbers $\alpha_0 = a$, $\alpha_1 \in [a, x_1)$, $\alpha_2 \in [x_1, b]$, $\alpha_3 = b$. Utilizing these choices in Theorem 4, we obtain the desired result (18). □

Corollary 4 *Let f be as above and $x_1 \in [a,b]$. Then we have the perturbed Ostrowski's integral inequality for twice differentiable functions:*

$$\left|\int_a^b f(t)\,dt + (b-a)\left[\left(x_1 - \frac{a+b}{2}\right) f'(x_1) - f(x_1)\right]\right| \tag{19}$$

$$\leq \begin{cases} \frac{\|f''\|_\infty}{6}\left[(x_1-a)^3 + (x_1-\alpha)^3\right] \\ = \|f''\|_\infty (b-a)\left[\frac{(b-a)^2}{24} + \frac{1}{2}\left(x_1 - \frac{a+b}{2}\right)^2\right] \leq \frac{\|f''\|_\infty}{6}(b-a)^3; \\ \\ \frac{\|f''\|_p}{2(2q+1)^{\frac{1}{q}}}\left(\left[(x_1-a)^{2q+1} + (b-x_1)^{2q+1}\right]\right)^{\frac{1}{q}} \leq \frac{\|f''\|_p}{2(2q+1)^{\frac{1}{q}}}(b-a)^{2+\frac{1}{q}}; \\ \\ \frac{\|f''\|_1}{2}\left(\frac{b-a}{2} + \left|x_1 - \frac{a+b}{2}\right|\right)^2 \leq \frac{3}{8}(b-a)^2\|f''\|_1. \end{cases}$$

Proof. Making the choice $\alpha_1 = a$, $\alpha_2 = b$ and substituting into (18) gives us the desired result (19). The result (19) has previously been obtained by Cerone, Dragomir and Roumeliotis in [1], [2]. □

Remark 2 *If we choose $x_1 = \frac{a+b}{2}$ in (19) we obtain the best estimate, the 'midpoint inequality'*

$$\left|\int_a^b f(t)\,dt - (b-a) f\left(\frac{a+b}{2}\right)\right| \leq \begin{cases} \frac{\|f''\|_\infty}{24}(b-a)^3; \\ \\ \frac{\|f''\|_p}{8(2q+1)^{\frac{1}{q}}}(b-a)^{2+\frac{1}{q}}; \\ \\ \frac{3}{8}\|f''\|_1 (b-a)^2. \end{cases} \tag{20}$$

Simpson's quadrature inequality may now be generalised as follows.

Corollary 5 *Let f be as above and $x_1 \in \left[\frac{5a+b}{6}, \frac{a+5b}{6}\right]$. Then we have the inequality*

$$\left|\int_a^b f(t)\,dt + \frac{2(b-a)}{3}\left[\left(x_1 - \frac{a+b}{2}\right) f'(x_1) + \frac{(b-a)}{48}\left(f'(b) - f'(a)\right)\right] \right. \tag{21}$$

$$\left. -\frac{(b-a)}{3}\left[\frac{f(b)+f(a)}{2} + 2f(x_1)\right]\right|$$

$$\leq \begin{cases} \|f''\|_\infty \frac{(b-a)}{3}\left[\left(x_1 - \frac{a+b}{2}\right)^2 + \frac{1}{24}(b-a)^2\right] \leq \frac{\|f''\|_\infty}{6}(b-a)^3; \\ \frac{\|f''\|_p}{2(2q+1)^{\frac{1}{q}}}\left(\frac{(b-a)^{2q+1}}{3\cdot 6^{2q}} + \left(x_1 - \frac{5a+b}{6}\right)^{2q+1} + \left(\frac{a+5b}{6} - x_1\right)^{2q+1}\right)^{\frac{1}{q}}; \\ \frac{3}{8}(b-a)^2\|f''\|_1. \end{cases}$$

The proof follows directly from Proposition 2 by putting $\alpha_1 = \frac{5a+b}{6}$ and $\alpha_2 = \frac{a+5b}{6}$.

Remark 3 *From Corollary 5, if we choose $x_1 = \frac{a+b}{2}$, we obtain the perturbed Simpson inequality*

$$\left|\int_a^b f(t)\,dt + \frac{(b-a)}{72}\left(f'(b) - f'(a)\right) \right. \tag{22}$$

$$\left. -\frac{(b-a)}{3}\left[\frac{f(b)+f(a)}{2} + 2f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq \begin{cases} \|f''\|_\infty \frac{(b-a)^3}{72}; \\ \frac{\|f''\|_p}{72}\left[\frac{1+2^{2q+1}}{3(2q+1)}\right]^{\frac{1}{q}}(b-a)^{2+\frac{1}{q}}; \\ \frac{3}{8}(b-a)^2\|f''\|_1. \end{cases}$$

Corollary 6 *Let f be as above and $a \leq \alpha_1 \leq \frac{a+b}{2} \leq \alpha_2 \leq b$. Then we have the following* *inequality*

$$\left| \int_a^b f(t)\,dt + (\alpha_2 - \alpha_1)(a + b - \alpha_2 - \alpha_1) f'\left(\frac{a+b}{2}\right) \right. \tag{23}$$

$$+ \frac{(b-\alpha_2)^2}{2} f'(b) - \frac{(\alpha_1 - a)^2}{2} f'(a) - (\alpha_2 - \alpha_1) f\left(\frac{a+b}{2}\right)$$

$$\left. - (\alpha_1 - a) f(a) - (b - \alpha_2) f(b) \right|$$

$$\leq \begin{cases} \frac{\|f''\|_\infty}{6}\left((\alpha_1 - a)^3 + \left(\frac{a+b}{2} - \alpha_1\right)^3 + \left(\alpha_2 - \frac{a+b}{2}\right)^3 + (b-\alpha_2)^3\right) \\ \leq \frac{\|f''\|_\infty}{24}(b-a)^3; \\ \frac{\|f''\|_p}{2(2q+1)^{\frac{1}{q}}}\left((\alpha_1 - a)^{2q+1} + \left(\frac{a+b}{2} - \alpha_1\right)^{2q+1} + \left(\alpha_2 - \frac{a+b}{2}\right)^{2q+1} + (b-\alpha_2)^{2q+1}\right)^{\frac{1}{q}} \\ \leq \frac{\|f''\|_p}{2(2q+1)^{\frac{1}{q}}}(b-a)^{2+\frac{1}{q}}; \\ \frac{3}{8}(b-a)^2 \|f''\|_1 . \end{cases}$$

The proof of the corollary follows directly from Proposition 2.

Utilizing the above inequality (23) we have the following remark.

Remark 4 *Choose in (23)* $\alpha_1 = \frac{3a+b}{4}$ *and* $\alpha_2 = \frac{a+3b}{4}$ *such that we have*

$$\left| \int_a^b f(t)\,dt + \frac{(b-a)^2}{32}\left[f'(b) - f'(a)\right] \right.$$

$$\left. - \frac{(b-a)}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right] \right|$$

$$\leq \begin{cases} \frac{\|f''\|_\infty}{96}(b-a)^3; \\ \frac{\|f''\|_p}{32}(b-a)^2\left(\frac{b-a}{2q+1}\right)^{\frac{1}{q}} \\ \frac{3}{8}\|f''\|_1 (b-a)^2 . \end{cases}$$

5. Composite Quadrature Fomulae

Consider the partitioning of the interval $[a,b]$ given by $\Delta_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ and put $h_i := x_{i+1} - x_i$ $(i = 0, ..., n-1)$ and $\nu(h) := \max(h_i | i = 0, ..., n-1)$. The following theorem holds.

Theorem 6 *Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous on $[a,b]$ and $k \geq 1$. Then we have the composite quadrature formula*

$$\int_a^b f(t)\,dt = A_k(\Delta_n, f) + R_k(\Delta_n, f), \tag{24}$$

where

$$\begin{aligned} &A_k(\Delta_n, f) \\ &:= \frac{1}{k}\left[T(\Delta_n, f) - \frac{1}{k}U(\Delta_n, f) + \sum_{i=0}^{n}\sum_{j=1}^{k-1} f\left[\frac{(k-j)\,x_i + jx_{i+1}}{k}\right]h_i\right], \end{aligned} \tag{25}$$

$$T(\Delta_n, f) := \frac{1}{2}\sum_{i=0}^{n-1}\left[f(x_{i+1}) + f(x_i)\right]h_i, \tag{26}$$

and

$$U(\Delta_n, f) := \frac{1}{8}\sum_{i=0}^{n-1}\left(f'(x_{i+1}) - f'(x_i)\right)h_i^2 \tag{27}$$

is a perturbed quadrature formula. The remainder $R_k(\Delta_n, f)$ satisfies the estimation

$$\begin{aligned} &|R_k(\Delta_n, f)| \\ &\leq \begin{cases} \frac{\|f''\|_\infty}{6k^2}\sum\limits_{i=0}^{n-1} h_i^3, & \text{where } f'' \in L_\infty[a,b]; \\ \frac{\|f''\|_p}{2k^2(2q+1)^{\frac{1}{q}}}\left(\sum\limits_{i=0}^{n-1} h_i^{2q+1}\right)^{\frac{1}{q}}, & \text{where } f'' \in L_p[a,b], \\ & \frac{1}{p}+\frac{1}{q}=1,\ p>1; \\ \frac{3\|f''\|_1}{8k^2}\sum\limits_{i=0}^{n-1} h_i^2, & \text{where } f'' \in L_1(a,b). \end{cases} \end{aligned} \tag{28}$$

Proof. Consider the second term in the inequality (28) and apply Corollary 2 on the interval $[x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ such that

$$\left|\int_{x_i}^{x_{i+1}} f(t)\,dt + \frac{h_i^2}{8k^2}\left\{f'(x_{i+1}) - f'(x_i)\right\} - \frac{h_i}{k}\left\{\frac{f(x_{i+1}) + f(x_i)}{2} + \sum_{j=1}^{k-1} f\left[\frac{(k-j)x_i + jx_{i+1}}{k}\right]\right\}\right|$$

$$\leq \frac{h_i^{2+\frac{1}{q}}}{2(2q+1)^{\frac{1}{q}}k^2}\left(\int_{x_i}^{x_{i+1}} |f''(t)|^p\,dt\right)^{\frac{1}{p}}.$$

Summing over i from 0 to $n-1$ and using the generalised triangle inequality, we have

$$|R_k(\Delta_n, f)| \tag{29}$$

$$\leq \sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}} f(t)\,dt + \frac{h_i^2}{8k^2}\left\{f'(x_{i+1}) - f'(x_i)\right\} - \frac{h_i}{k}\left\{\frac{f(x_{i+1}) + f(x_i)}{2} + \sum_{j=1}^{k-1} f\left[\frac{(k-j)x_i + jx_{i+1}}{k}\right]\right\}\right|$$

$$\leq \frac{1}{2k^2(2q+1)^{\frac{1}{q}}}\sum_{i=0}^{n-1} h_i^{2+\frac{1}{q}}\left(\int_{x_i}^{x_{i+1}} |f''(t)|^p\,dt\right)^{\frac{1}{p}}.$$

By Hölder's discrete inequality, we have that

$$\sum_{i=0}^{n-1} h_i^{2+\frac{1}{q}}\left(\int_{x_i}^{x_{i+1}} |f''(t)|^p\,dt\right)^{\frac{1}{p}} \tag{30}$$

$$\leq \left[\sum_{i=0}^{n-1}\left(h_i^{2+\frac{1}{q}}\right)^q\right]^{\frac{1}{q}}\left[\sum_{i=0}^{n-1}\left(\left(\int_{x_i}^{x_{i+1}} |f''(t)|^p\,dt\right)^{\frac{1}{p}}\right)^p\right]^{\frac{1}{p}}$$

$$= \left[\sum_{i=0}^{n-1} h_i^{2q+1}\right]^{\frac{1}{q}}\|f''\|_p.$$

Putting (30) into (29) proves the second term in the inequality (28).

The first and third terms in the inequality (28) are proved in a similar fashion and will not be done here. Thus the proof of the theorem is complete. □

Corollary 7 *Let f and Δ_n be as above. Then we have the quadrature formula:*

$$\int_a^b f(t)\,dt = \frac{1}{2}\left[T(\Delta_n, f) + M(\Delta_n, f)\right] - \frac{1}{4}U(\Delta_n, f) + R_2(\Delta_n, f)$$

where $M(\Delta_n, f)$ is the midpoint rule, namely

$$M(\Delta_n, f) := \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) h_i,$$

$$\begin{aligned} &T(\Delta_n, f) \ \textit{ is defined by (26) and} \\ &U(\Delta_n, f) \ \textit{ is defined by (27).} \end{aligned}$$

The remainder $R_2(\Delta_n, f)$ satisfies the estimate

$$|R_2(\Delta_n, f)| \le \begin{cases} \frac{\|f''\|_\infty}{24} \sum\limits_{i=0}^{n-1} h_i^3; \\ \\ \frac{\|f''\|_p}{8(2q+1)^{\frac{1}{q}}} \left(\sum\limits_{i=0}^{n-1} h_i^{2q+1}\right)^{\frac{1}{q}}; \\ \\ \frac{3}{32}\|f''\|_1 \sum\limits_{i=0}^{n-1} h_i^2. \end{cases}$$

The proof of the corollary follows directly from Theorem 6.

Again utilizing Theorem 6, the following corollary holds.

Corollary 8 *If f and Δ_n are defined as above, then we have the equality*

$$\begin{aligned} &\int_a^b f(t)\,dt \qquad (31) \\ &= \frac{1}{3}\left[T(\Delta_n, f) + \sum_{i=0}^{n-1}\left[f\left(\frac{2x_i + x_{i+1}}{3}\right) + f\left(\frac{x_i + 2x_{i+1}}{3}\right)\right] h_i\right] \\ &-\frac{1}{9}U(\Delta_n, f) + R_3(\Delta_n, f), \end{aligned}$$

where the remainder $R_3(\Delta_n, f)$ satisfies the bound

$$|R_3(\Delta_n, f)| \leq \begin{cases} \frac{\|f''\|_\infty}{54} \sum\limits_{i=0}^{n-1} h_i^3; \\ \frac{\|f''\|_p}{18(2q+1)^{\frac{1}{q}}} \left(\sum\limits_{i=0}^{n-1} h_i^{2q+1}\right)^{\frac{1}{q}}; \\ \frac{\|f''\|_1}{24} \sum\limits_{i=0}^{n-1} h_i^2. \end{cases}$$

Theorem 7 *Let f and Δ_n be defined as above and suppose that $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$. Then we have the equality*

$$\begin{aligned} &\int_a^b f(t)\,dt \qquad (32) \\ &= \sum_{i=0}^{n-1} \Big[(x_{i+1} - \xi_i) f(x_{i+1}) + (\xi_i - x_i) f(x_i) + \frac{1}{2}(x_i - \xi_i)^2 f'(x_i) \\ &\quad - (x_{i+1} - \xi_i)^2 f'(x_{i+1}) \Big] + R(\xi, \Delta_n, f). \end{aligned}$$

The remainder $R(\xi, \Delta_n, f)$ satisfies the estimate

$$\begin{aligned} &|R(\xi, \Delta_n, f)| \qquad (33) \\ &\leq \begin{cases} \frac{\|f''\|_\infty}{6} \sum\limits_{i=0}^{n-1} \left[(\xi_i - x_i)^3 + (x_{i+1} - \xi_i)^3 \right] \leq \frac{\|f''\|_\infty}{6} \sum\limits_{i=0}^{n-1} h_i^3; \\ \frac{\|f''\|_p}{2(2q+1)^{\frac{1}{q}}} \left[\sum\limits_{i=0}^{n-1} (\xi_i - x_i)^{2q+1} + \sum\limits_{i=0}^{n-1} (x_{i+1} - \xi_i)^{2q+1} \right]^{\frac{1}{q}} \\ \qquad \leq \frac{\|f''\|_p}{2(2q+1)^{\frac{1}{q}}} \left[\sum\limits_{i=0}^{n-1} h_i^{2q+1} \right]^{\frac{1}{q}}; \\ \frac{3}{8} \|f''\|_1 \sum\limits_{i=0}^{n-1} h_i^2. \end{cases} \end{aligned}$$

Proof. Apply Proposition 1, utilizing the second inequality in (14), on the intervals $[x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ such that we obtain

$$\left|\int_{x_i}^{x_{i+1}} f(t)\,dt - \frac{1}{2}\left\{(x_i-\xi_i)^2 f'(x_i) - (x_{i+1}-\xi_i)^2 f'(x_{i+1})\right\}\right.$$
$$\left. - (x_{i+1}-\xi_i) f(x_{i+1}) - (\xi_i - x_i) f(x_i)\right|$$
$$\leq \frac{1}{2(2q+1)^{\frac{1}{q}}}\left[(\xi_i-x_i)^{2q+1} + (x_{i+1}-\xi_i)^{2q+1}\right]^{\frac{1}{q}}\left(\int_{x_i}^{x_{i+1}}|f''(t)|^p\,dt\right)^{\frac{1}{p}}.$$

Summing over i from 0 to $n-1$, using the generalised triangle inequality and Hölder's discrete inequality, we obtain

$$|R(\xi, \Delta_n, f)|$$
$$\leq \sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}} f(t)\,dt - \frac{1}{2}\left\{(x_i-\xi_i)^2 f'(x_i) - (x_{i+1}-\xi_i)^2 f'(x_{i+1})\right\}\right.$$
$$\left. - (x_{i+1}-\xi_i) f(x_{i+1}) - (\xi_i - x_i) f(x_i)\right|$$
$$\leq \frac{1}{2(2q+1)^{\frac{1}{q}}}\sum_{i=0}^{n-1}\left[(\xi_i-x_i)^{2q+1} + (x_{i+1}-\xi_i)^{2q+1}\right]^{\frac{1}{q}}$$
$$\times\left(\int_{x_i}^{x_{i+1}}|f''(t)|^p\,dt\right)^{\frac{1}{p}}$$
$$\leq \frac{1}{2(2q+1)^{\frac{1}{q}}}\left[\sum_{i=0}^{n-1}\left(\left[(\xi_i-x_i)^{2q+1} + (x_{i+1}-\xi_i)^{2q+1}\right]^{\frac{1}{q}}\right)^q\right]^{\frac{1}{q}}$$
$$\times\left[\sum_{i=0}^{n-1}\left(\left(\int_{x_i}^{x_{i+1}}|f''(t)|^p\,dt\right)^{\frac{1}{p}}\right)^p\right]^{\frac{1}{p}}$$
$$= \frac{1}{2(2q+1)^{\frac{1}{q}}}\left[\sum_{i=0}^{n-1}(\xi_i-x_i)^{2q+1} + \sum_{i=0}^{n-1}(x_{i+1}-\xi_i)^{2q+1}\right]^{\frac{1}{q}}\|f''\|_p \tag{34}$$

and since we know that

$$(\xi_i-x_i)^{2q+1} + (x_{i+1}-\xi_i)^{2q+1} \leq h_i^{2q+1},$$

(34) can be written as

$$\leq \frac{\|f''\|_p}{2\,(2q+1)^{\frac{1}{q}}} \left[\sum_{i=0}^{n-1} h_i^{2q+1}\right]^{\frac{1}{q}}.$$

Thus, these last two lines prove the second part of the inequality in (33). The first and third inequalities in (33) are proved in a similar manner. Hence, the proof of Theorem 7 is complete. □

The following corollary, a consequence of Theorem 7, contains some higher order estimates of particular quadrature formulae.

Corollary 9 *Let f and Δ_n be defined as above. The following estimates apply.*

1. *The 'higher order left rectangle rule'*

$$\int_a^b f(t)\,dt = \sum_{i=0}^{n-1} h_i \left[f(x_i) + \frac{h_i}{2} f'(x_i)\right] + R_l(\Delta_n, f);$$

2. *The 'higher order right rectangle rule'*

$$\int_a^b f(t)\,dt = \sum_{i=0}^{n-1} h_i \left[f(x_{i+1}) + \frac{h_i}{2} f'(x_{i+1})\right] + R_r(\Delta_n, f);$$

3. *The 'higher order trapezoidal rule'*

$$\int_a^b f(t)\,dt = \sum_{i=0}^{n-1} h_i \left[\frac{f(x_{i+1}) + f(x_i)}{2} - \frac{h_i}{4}\left(\frac{f'(x_{i+1}) - f'(x_i)}{2}\right)\right] + R_T(\Delta_n, f),$$

where

$$|R_l(\Delta_n, f)|, \ |R_r(\Delta_n, f)| \leq \begin{cases} \frac{\|f''\|_\infty}{6} \sum\limits_{i=0}^{n-1} h_i^3; \\ \\ \frac{\|f''\|_p}{2(2q+1)^{\frac{1}{q}}} \left(\sum\limits_{i=0}^{n-1} h_i^{2q+1}\right)^{\frac{1}{q}}; \\ \\ \frac{3}{8}\|f''\|_1 \sum\limits_{i=0}^{n-1} h_i^2, \end{cases}$$

and

$$|R_T(\Delta_n, f)| \leq \begin{cases} \frac{\|f''\|_\infty}{24} \sum\limits_{i=0}^{n-1} h_i^3; \\ \frac{\|f''\|_p}{8(2q+1)^{\frac{1}{q}}} \left(\sum\limits_{i=0}^{n-1} h_i^{2q+1} \right)^{\frac{1}{q}}; \\ \frac{3}{8} \|f''\|_1 \sum\limits_{i=0}^{n-1} h_i^2. \end{cases}$$

The following theorem holds.

Theorem 8 *Consider the interval* $x_i \leq \alpha_i^{(1)} \leq \xi_i \leq \alpha_i^{(2)} \leq x_{i+1}$ $(i = 0, ..., n-1)$ *and let* f *and* Δ_n *be defined as above. Then we have the equality*

$$\begin{aligned} &\int_a^b f(t)\, dt \\ &= \sum_{i=0}^{n-1} \left[\left(\alpha_i^{(2)} - \alpha_i^{(1)} \right) f(\xi_i) - \frac{1}{2} \left(\left(\xi_i - \alpha_i^{(1)} \right)^2 - \left(\xi_i - \alpha_i^{(2)} \right)^2 \right) f'(\xi_i) \right] \\ &\quad + \sum_{i=0}^{n-1} \left[\left(\alpha_i^{(1)} - x_i \right) f(x_i) + \left(x_{i+1} - \alpha_i^{(2)} \right) f(x_{i+1}) \right. \\ &\quad \left. + \frac{1}{2} \left(\left(\alpha_i^{(1)} - x_i \right)^2 f'(x_i) - \left(x_{i+1} - \alpha_i^{(2)} \right)^2 f'(x_{i+1}) \right) \right] \\ &\quad + R\left(\xi, \alpha_i^{(1)}, \alpha_i^{(2)}, \Delta_n, f \right), \end{aligned}$$

where the remainder satisfies the estimate

$$\left| R\left(\xi, \alpha_i^{(1)}, \alpha_i^{(2)}, \Delta_n, f \right) \right|$$

$$\le \begin{cases} \frac{\|f''\|_\infty}{6} \sum_{i=0}^{n-1} \left[\left(\alpha_i^{(1)} - x_i\right)^3 + \left(\xi_i - \alpha_i^{(1)}\right)^3 + \left(\alpha_i^{(2)} - \xi_i\right)^3 + \left(x_{i+1} - \alpha_i^{(2)}\right)^3 \right] \\ \le \frac{\|f''\|_\infty}{6} \sum_{i=0}^{n-1} h_i^3; \\ \\ \frac{\|f''\|_p}{2(2q+1)^{\frac{1}{q}}} \left(\sum_{i=0}^{n-1} \left[\left(\alpha_i^{(1)} - x_i\right)^{2q+1} + \left(\xi_i - \alpha_i^{(1)}\right)^{2q+1} \right.\right. \\ \qquad \left.\left. + \left(\alpha_i^{(2)} - \xi_i\right)^{2q+1} + \left(x_{i+1} - \alpha_i^{(2)}\right)^{2q+1} \right] \right)^{\frac{1}{q}} \le \frac{\|f''\|_p}{2(2q+1)^{\frac{1}{q}}} \left(\sum_{i=0}^{n-1} h_i^{2q+1} \right)^{\frac{1}{q}}; \\ \\ \frac{3}{8} \|f''\|_1 \sum_{i=0}^{n-1} h_i^2. \end{cases}$$

The proof follows directly upon application of Proposition 2 on the intervals $[x_i, x_{i+1}]$ $(i = 0, ..., n-1)$, hence the details are omitted.

The following corollary is a consequence of Theorem 8.

Corollary 10 *Choose $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ and let f and Δ_n be defined as above. The following Riemann type formula holds:*

$$\int_a^b f(t)\,dt = \sum_{i=0}^{n-1} [f(\xi_i) - \eta_i f'(\xi_i)]\, h_i + R_R(\xi, \Delta_n, f), \tag{35}$$

whence the remainder, $R_R(\xi, \Delta_n, f)$, satisfies the estimate

$$|R_R(\xi, \Delta_n, f)|$$

$$\le \begin{cases} \frac{\|f''\|_\infty}{6} \sum_{i=0}^{n-1} \left[(\xi_i - x_i)^3 + (x_{i+1} - \xi_i)^3 \right] \le \frac{\|f''\|_\infty}{6} \sum_{i=0}^{n-1} h_i^3; \\ \\ \frac{\|f''\|_p}{2(2q+1)^{\frac{1}{q}}} \left(\sum_{i=0}^{n-1} \left[(\xi_i - x_i)^{2q+1} + (x_{i+1} - \xi_i)^{2q+1} \right] \right)^{\frac{1}{q}} \\ \qquad \le \frac{\|f''\|_p}{2(2q+1)^{\frac{1}{q}}} \left(\sum_{i=0}^{n-1} h_i^{2q+1} \right)^{\frac{1}{q}}; \\ \\ \frac{3}{8} \|f''\|_1 \sum_{i=0}^{n-1} h_i^2, \end{cases}$$

and $\eta_i := \xi_i - \frac{x_{i+1} + x_i}{2}$.

The proof follows from Theorem 8 upon putting $\alpha_i^{(1)} = x_i$ and $\alpha_i^{(2)} = x_{i+1}$.

Remark 5 *If in (35) we choose the midpoint* $\xi_i = \frac{x_{i+1}+x_i}{2}$, *we obtain the midpoint quadrature formula*

$$\int_a^b f(t)\,dt = M(\Delta_n, f) + R_M(\Delta_n, f), \tag{36}$$

where

$$M(\Delta_n, f) := \sum_{i=0}^{n-1} f\left(\frac{x_{i+1}+x_i}{2}\right) h_i$$

and the best estimate for the remainder, $R_M(\Delta_n, f)$, *is given by*

$$|R_M(\Delta_n, f)| \le \begin{cases} \frac{\|f''\|_\infty}{24} \sum\limits_{i=0}^{n-1} h_i^3; \\ \\ \frac{\|f''\|_p}{8(2q+1)^{\frac{1}{q}}} \left(\sum\limits_{i=0}^{n-1} h_i^{2q+1}\right)^{\frac{1}{q}}; \\ \\ \frac{3}{8}\|f''\|_1 \sum\limits_{i=0}^{n-1} h_i^2. \end{cases}$$

The following corollary holds and is a consequence of Theorem 8.

Corollary 11 *Consider a set of points* ξ_i *such that* $\xi_i \in \left[\frac{5x_i+x_{i+1}}{6}, \frac{x_i+5x_{i+1}}{6}\right]$ $(i = 0, ..., n-1)$ *and let* f *and* Δ_n *be defined as above. Then the following equality holds:*

$$\int_a^b f(t)\,dt = \frac{1}{6}\sum_{i=0}^{n-1} h_i\left(\Delta f(\xi_i) - \eta_i f'(\xi_i)\right) \tag{37}$$
$$+\frac{1}{3}T(\Delta_n, f) - \frac{1}{9}U(\Delta_n, f) + R_S(\Delta_n, f),$$

where $T(\Delta_n, f)$ *and* $U(\Delta_n, f)$ *are defined respectively by (26) and (27), and* $\eta_i = \left(\xi_i - \frac{x_i+x_{i+1}}{2}\right)$.

The remainder $R_S(\xi, \Delta_n, f)$ satisfies the estimate:

$$|R_S(\xi, \Delta_n, f)|$$
$$\leq \begin{cases} \frac{\|f''\|_\infty}{6} \sum\limits_{i=0}^{n-1} \left[\frac{h_i^3}{108} + \left(\xi_i - \frac{5x_i + x_{i+1}}{6} \right)^3 + \left(\frac{x_i + 5x_{i+1}}{6} - \xi_i \right)^3 \right] \leq \frac{\|f''\|_\infty}{6} \sum\limits_{i=0}^{n-1} h_i^3; \\ \frac{\|f''\|_p}{2(2q+1)^{\frac{1}{q}}} \left(\sum\limits_{i=0}^{n-1} \left[2 \cdot \left(\frac{h_i}{6} \right)^{2q+1} \left(\xi_i - \frac{5x_i + x_{i+1}}{6} \right)^{2q+1} + \left(\frac{x_i + 5x_{i+1}}{6} - \xi_i \right)^{2q+1} \right] \right)^{\frac{1}{q}}; \\ \frac{3}{8} \|f''\|_1 \sum\limits_{i=0}^{n-1} h_i^2. \end{cases}$$

Remark 6 *If in (37) we choose the midpoint $\xi_i = \frac{x_i + x_{i+1}}{2}$, we obtain a perturbed Simpson's formula:*

$$\int_a^b f(t)\, dt = \frac{2}{3} \sum_{i=0}^{n-1} h_i f\left(\frac{x_i + x_{i+1}}{2} \right)$$
$$+ \frac{1}{3} T(\Delta_n, f) - \frac{1}{9} U(\Delta_n, f) + R_S(\Delta_n, f),$$

where the remainder $R_S(\Delta_n, f)$, satisfies the inequality

$$|R_S(\Delta_n, f)| \leq \begin{cases} \frac{\|f''\|_\infty}{72} \sum\limits_{i=0}^{n-1} h_i^3; \\ \frac{\|f''\|_p}{72} \left(\frac{1 + 2^{2q+1}}{3(2q+1)} \right)^{\frac{1}{q}} \left(\sum\limits_{i=0}^{n-1} h_i^{2q+1} \right)^{\frac{1}{q}}, \\ \frac{3}{8} \|f''\|_1 \sum\limits_{i=0}^{n-1} h_i^2. \end{cases}$$

The following corollary also holds.

Corollary 12 *Consider the intervals $x_i \leq \alpha_i^{(1)} \leq \frac{x_i + x_{i+1}}{2} \leq \alpha_i^{(2)} \leq x_{i+1}$, $(i = 0, \ldots, n -$*

1*), and let* f *and* Δ_n *be defined as above. The following equality is obtained:-*

$$
\begin{aligned}
&\int_a^b f(t)\,dt \\
&= \sum_{i=0}^{n-1}\left[\left(\alpha_i^{(2)}-\alpha_i^{(1)}\right) f\left(\frac{x_i+x_{i+1}}{2}\right)\right. \\
&\quad \left.-\frac{1}{2}\left(\left(\frac{x_i+x_{i+1}}{2}-\alpha_i^{(1)}\right)^2-\left(\frac{x_i+x_{i+1}}{2}-\alpha_i^{(2)}\right)^2\right) f'\left(\frac{x_i+x_{i+1}}{2}\right)\right] \\
&\quad +\sum_{i=0}^{n-1}\left[\left(\alpha_i^{(1)}-x_i\right) f(x_i)+\left(x_{i+1}-\alpha_i^{(2)}\right) f(x_{i+1})\right. \\
&\quad \left.+\frac{1}{2}\left(\left(\alpha_i^{(1)}-x_i\right)^2 f'(x_i)+\left(x_{i+1}-\alpha_i^{(2)}\right)^2 f'(x_{i+1})\right)\right] \\
&\quad +R_B\left(\alpha_i^{(1)},\alpha_i^{(2)},\Delta_n,f\right),
\end{aligned}
$$

where the remainder satisfies the estimate

$$
\left|R_B\left(\alpha_i^{(1)},\alpha_i^{(2)},\Delta_n,f\right)\right|
$$

$$
\leq \begin{cases}
\frac{\|f''\|_\infty}{6}\sum\limits_{i=0}^{n-1}\left[\left(\alpha_i^{(1)}-x_i\right)^3+\left(\frac{x_i+x_{i+1}}{2}-\alpha_i^{(1)}\right)^3\right. \\
\qquad\qquad \left.+\left(\alpha_i^{(2)}-\frac{x_i+x_{i+1}}{2}\right)^3+\left(x_{i+1}-\alpha_i^{(2)}\right)^3\right] \\
\qquad\qquad\qquad\qquad \leq \frac{\|f''\|_\infty}{6}\sum\limits_{i=0}^{n-1} h_i^3; \\
\frac{\|f''\|_p}{2(2q+1)^{\frac{1}{q}}}\left(\sum\limits_{i=0}^{n-1}\left[\left(\alpha_i^{(1)}-x_i\right)^{2q+1}+\left(\frac{x_i+x_{i+1}}{2}-\alpha_i^{(1)}\right)^{2q+1}\right.\right. \\
\qquad \left.\left.+\left(\alpha_i^{(2)}-\frac{x_i+x_{i+1}}{2}\right)^{2q+1}+\left(x_{i+1}-\alpha_i^{(2)}\right)^{2q+1}\right]\right)^{\frac{1}{q}} \\
\qquad\qquad\qquad\qquad \leq \frac{\|f''\|_p}{2(2q+1)^{\frac{1}{q}}}\left(\sum\limits_{i=0}^{n-1} h_i^{2q+1}\right)^{\frac{1}{q}}; \\
\frac{3}{8}\|f''\|_1\sum\limits_{i=0}^{n-1} h_i^2,
\end{cases}
$$

Finally, we have the following remark in relation to Corollary 12.

Remark 7 *If in Corollary 12, we choose* $\alpha_i^{(1)} = \frac{3x_i + x_{i+1}}{4}$ *and* $\alpha_i^{(2)} = \frac{x_i + 3x_{i+1}}{4}$*, we obtain the formula:*

$$\int_a^b f(t)\,dt = \frac{1}{2}\left[T(\Delta_n, f) + M(\Delta_n, f)\right] + R_B(\Delta_n, f).$$

The remainder, $R_B(\Delta_n, f)$*, satisfies the bounds*

$$|R_B(\Delta_n, f)| \le \begin{cases} \frac{\|f''\|_\infty}{96} \sum\limits_{i=0}^{n-1} h_i^3; \\ \frac{\|f''\|_p}{32(2q+1)^{\frac{1}{q}}} \left(\sum\limits_{i=0}^{n-1} h_i^{2q+1}\right)^{\frac{1}{q}}, \\ \frac{3}{8}\|f''\|_1 \sum\limits_{i=0}^{n-1} h_i^2. \end{cases}$$

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A. SOFO, S. S. DRAGOMIR
School of Communications and Informatics
Victoria University of Technology
PO Box 14428, Melbourne City MC
Victoria 8001, AUSTRALIA

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