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On the Centroid of the Prime Gamma Rings II

M. Ali Öztürk and Young Bae Jun

Abstract

The aim of this paper is to study the properties of the extended centroid of the prime Γ -rings. Main results are the following theorems: (1) Let M be a simple Γ -ring with unity. Suppose that for some $a \neq 0$ in M we have $a\gamma_1x\gamma_2a\beta_1y\beta_2a = a\beta_1y\beta_2a\gamma_1x\gamma_2a$ for all $x, y \in M$ and $\gamma_1, \gamma_2, \beta_1, \beta_2 \in \Gamma$. Then M is isomorphic onto the Γ -ring $D_{n,m}$, where $D_{n,m}$ is the additive abelian group of all rectangular matrices of type $n \times m$ over a division ring D and Γ is a nonzero subgroup of the additive abelian group of all rectangular matrices of type $m \times n$ over a division ring D . Furthermore M is the Γ -ring of all $n \times n$ matrices over the field C_Γ . (2) Let M be a prime Γ -ring and C_Γ the extended centroid of M . If a and b are non-zero elements in $S = M\Gamma C_\Gamma$ such that $a\gamma x\beta b = b\beta x\gamma a$ for all $x \in M$ and $\beta, \gamma \in \Gamma$, then a and b are C_Γ -dependent. (3) Let M be prime Γ -ring, Q quotient Γ -ring of M and C_Γ the extended centroid of M . If q is non-zero element in Q such that $q\gamma_1x\gamma_2q\beta_1y\beta_2q = q\beta_1y\beta_2q\gamma_1x\gamma_2q$ for all $x, y \in M$, $\gamma_1, \gamma_2, \beta_1, \beta_2 \in \Gamma$ then S is a primitive Γ -ring with minimal right (left) ideal such that $e\Gamma S$, where e is idempotent and $C_\Gamma\Gamma e$ is the commuting ring of S on $e\Gamma S$.

Key Words: Γ -division ring, Γ -field, extended centroid, central closure.

1. Introduction

Nobusawa [11] introduced the notion of a Γ -ring, more general than a ring. Barnes [1] weakened slightly the conditions in the definition of Γ -ring in the sense of Nobusawa.

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Barnes [1], Luh [7] and Kyuno [4] studied the structure of Γ -rings and obtained various generalizations analogous of corresponding parts in ring theory. Öztürk and Jun [12] studied the extended centroid of a prime Γ -ring. As a continuation of [12], in this paper, we study further properties of the extended centroid of the prime Γ -rings.

2. Preliminaries

Let M and Γ be two abelian groups. If for all $x, y, z \in M$ and all $\alpha, \beta \in \Gamma$ the conditions

- (i) $x\alpha y \in M$,
- (ii) $(x + y)\alpha z = x\alpha z + y\alpha z, x(\alpha + \beta)z = x\alpha z + x\beta z, x\alpha(y + z) = x\alpha y + x\alpha z,$
- (iii) $(x\alpha y)\beta z = x\alpha(y\beta z)$

are satisfied, then we call M a Γ -ring. By a *right* (resp. *left*) *ideal* of a Γ -ring M we mean an additive subgroup U of M such that $U\Gamma M \subseteq U$ (resp. $M\Gamma U \subseteq U$). If U is both a right and a left ideal, then we say that U is an *ideal* of M . For each a of a Γ -ring M the smallest right ideal containing a is called the *principal right ideal generated by a* and is denoted by $\langle a \rangle_r$. Similarly we define $\langle a \rangle_l$ (resp. $\langle a \rangle$), the *principal left* (resp. *two sided ideal generated by a*). An ideal P of a Γ -ring M is said to be *prime* if for any ideals A and B of M , $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. An ideal Q of a Γ -ring M is said to be *semi-prime* if for any ideal U of M , $U\Gamma U \subseteq Q$ implies $U \subseteq Q$. A Γ -ring M is said to be *prime* (resp. *semi-prime*) if the zero ideal is prime (resp. semi-prime).

Theorem 2.1 ([4, Theorem 4]). *If M is a Γ -ring, the following conditions are equivalent:*

- (i) M is a prime Γ -ring.
- (ii) If $a, b \in M$ and $a\Gamma M\Gamma b = (0)$, then $a = 0$ or $b = 0$.
- (iii) If $\langle a \rangle$ and $\langle b \rangle$ are principal ideals in M such that $\langle a \rangle\Gamma\langle b \rangle = (0)$, then $a = 0$ or $b = 0$.
- (iv) If A and B are right ideals in M such that $A\Gamma B = (0)$, then $A = (0)$ or $B = (0)$.
- (v) If A and B are left ideals in M such that $A\Gamma B = (0)$, then $A = (0)$ or $B = (0)$.

A Γ -ring M is said to be *simple* if $M\Gamma M \neq 0$ and M has no ideals other 0 and M itself. When a Γ -ring M has the descending (resp. ascending) chain condition for right ideals, it is abbreviated to M has *min-r condition* (resp. *max-r condition*). The terms *min-l condition* or *max-l condition* on a Γ -ring M are likewise defined. Let M be a Γ -ring and let F be the free group generated by $\Gamma \times M$. Then

$$A = \left\{ \sum_i n_i(\gamma_i, x_i) \in F \mid a \in M \Rightarrow \sum_i n_i a \gamma_i x_i = 0 \right\}$$

is a subgroup of F . Let $R = F/A$ be the factor group, and denote the coset $(\gamma, x) + A$ by $[\gamma, x]$. Clearly, every element of R can be expressed as a finite sum $\sum_i [\gamma_i, x_i]$. Also it can be verified easily that $[\alpha, x] + [\alpha, y] = [\alpha, x + y]$ and $[\alpha, x] + [\beta, x] = [\alpha + \beta, x]$ for all $\alpha, \beta \in \Gamma$ and $x, y \in M$. We define a multiplication on R by

$$\sum_i [\alpha_i, x_i] \sum_j [\beta_j, y_j] = \sum_{i,j} [\alpha_i, x_i \beta_j y_j].$$

Then R forms a ring. If we define a composition on $M \times R$ into M by

$$a \sum_i [\gamma_i, x_i] = \sum_i a \gamma_i x_i, \forall a \in M, \forall \sum_i [\gamma_i, x_i] \in R$$

then M is a right R -module, and we call R the *right operator ring* of M . Similarly, we can define the *left operator ring* L of M . A Γ -ring M is said to be *right* (resp. *left*) *primitive* if it satisfies:

- (i) the right (resp. left) operator ring of M is a right (resp. left) primitive ring
- (ii) $M\Gamma x = 0$ (resp. $x\Gamma M = 0$) implies $x = 0$.

A Γ -ring M is said to be *two-sided primitive* (or simply, *primitive*) if it is both right and left primitive.

Theorem 2.2 ([7, Theorem 3.4]). *If M is a Γ -ring possessing minimal left (resp. right) ideal, then M is primitive if and only if it is prime.*

Theorem 2.3 ([7, Theorem 3.6]). *For a Γ -ring M with min-l condition, the following are equivalent:*

- (i) M is prime,

(ii) M is primitive,

(iii) M is simple.

Theorem 2.4 ([7, Theorem 4.2]). *If M is a simple Γ -ring possessing minimal left (resp. right) ideals, then M is a direct sum of minimal left (resp. right) ideals.*

Theorem 2.5 ([5, Theorem 3.23]). *Let M be a semi-prime Γ -ring with min- r condition and let $M = I_1 \oplus I_2 \oplus \cdots \oplus I_m = J_1 \oplus J_2 \oplus \cdots \oplus J_n$, where $I_1, I_2, \dots, I_m, J_1, J_2, \dots, J_n$ are minimal right ideals. Then $m = n$.*

The integer $m = n$ in Theorem 2.5 is called the *right dimension* of the semi-prime Γ -ring with min- r condition and denoted by $\dim(M_R)$. One can define the left dimension of a Γ -ring in a similar way. If M is simple, then M is semi-prime (see [5]). For an additive group G , denote by $G_{m,n}$ the additive group of all matrices over G . Let M be a Γ -ring M and let $M_{m,n}$ and $\Gamma_{n,m}$ denote, respectively, the sets of $m \times n$ matrices with entries from M and of $n \times m$ matrices with entries from Γ . For $(a_{ij}), (b_{ij}) \in M_{m,n}$ and $(\gamma_{ij}) \in \Gamma_{n,m}$, define $(a_{ij})(\gamma_{ij})(b_{ij}) = (c_{ij})$, where $c_{ij} = \sum_p \sum_q a_{ip} \gamma_{pq} b_{qj}$. Then $M_{m,n}$ forms a $\Gamma_{n,m}$ -ring.

Theorem 2.6 ([6, Theorem 4.2]). *Let M be a simple Γ -ring with min- r and min- l conditions and $\Gamma_0 = \Gamma/\kappa$, where $\kappa := \{\gamma \in \Gamma \mid M\gamma M = 0\}$. Then the Γ_0 -ring M is isomorphic to the Γ' -ring $D_{n,m}$, where $D_{n,m}$ is the additive abelian group of all rectangular matrices of type $n \times m$ over a division ring D and Γ' is a nonzero subgroup of the additive abelian group of all rectangular matrices of type $m \times n$ over a division ring D and $m = \dim(M_L)$ and $n = \dim(M_R)$.*

Lemma 2.7 ([12, Lemma 3]). *Let M be a prime Γ -ring such that $M\Gamma M \neq M$ and quotient Γ -ring Q of M . Then, for each non-zero $q \in Q$ there is a non-zero ideal U of M such that $q(U) \subset M$.*

Lemma 2.8 ([12, p. 476]). *Let M be a prime Γ -ring such that $M\Gamma M \neq M$ and C_Γ the extended centroid of M . If a_i and b_i are non-zero elements of M such that $\sum a_i \gamma_i x \beta_i b_i = 0$ for all $x \in M$ and $\gamma_i, \beta_i \in \Gamma$, then the a_i 's (also a_i 's) are linearly dependent over C_Γ . Moreover, if $a\gamma x \beta b = b\gamma x \beta a$ for all $x \in M$ and $\gamma, \beta \in \Gamma$ where $a (\neq 0), b \in M$ are fixed, then there exists $\lambda \in C_\Gamma$ such that $b = \lambda \alpha a$ for all $\alpha \in \Gamma$.*

3. Centroids

Let M be a prime Γ -ring such that $M\Gamma M \neq M$. Denote

$$\mathcal{M} := \{(U, f) \mid U(\neq 0) \text{ is an ideal of } M \text{ and} \\ f : U \rightarrow M \text{ is a right } M\text{-module homomorphism}\}.$$

Define a relation \sim on \mathcal{M} by $(U, f) \sim (V, g)$ if and only if $\exists W(\neq 0) \subset U \cap V$ such that $f = g$ on W . Since M is a prime Γ -ring, it is possible to find a non-zero W and so “ \sim ” is an equivalence relation. This gives a chance for us to get a partition of \mathcal{M} . We then denote the equivalence class by $Cl(U, f) = \hat{f}$, where $\hat{f} := \{g : V \rightarrow M \mid (U, f) \sim (V, g)\}$, and denote by Q the set of all equivalence classes. Now we define an addition “+” on Q as follows:

$$\hat{f} + \hat{g} = Cl(U, f) + Cl(V, g) = Cl(U \cap V, f + g)$$

where $f + g : U \cap V \rightarrow M$ is a right M -module homomorphism. Then Q is an additive abelian group (see [12]). Since $M\Gamma M \neq M$ and since M is a prime Γ -ring, $M\Gamma M (\neq 0)$ is an ideal of M . We can take the homomorphism $1_{M\Gamma} : M\Gamma M \rightarrow M$ as a unit M -module homomorphism. Note that $M\beta M \neq 0$ for all $0 \neq \beta \in \Gamma$ so that $1_{M\beta} : M\beta M \rightarrow M$ is non-zero M -module homomorphism. Denote

$$\mathcal{N} := \{(M\beta M, 1_{M\beta}) \mid 0 \neq \beta \in \Gamma\},$$

and define a relation “ \approx ” on \mathcal{N} by $(M\beta M, 1_{M\beta}) \approx (M\gamma M, 1_{M\gamma})$ if and only if $\exists W := M\alpha M(\neq 0) \subset M\beta M \cap M\gamma M$ such that $1_{M\beta} = 1_{M\gamma}$ on W . We can easily check that “ \approx ” is an equivalence relation on \mathcal{N} . Denote by $Cl(M\beta M, 1_{M\beta}) = \hat{\beta}$ the equivalence class containing $(M\beta M, 1_{M\beta})$ and by $\hat{\Gamma}$ the set of all equivalence classes of \mathcal{N} with respect to \approx , that is,

$$\hat{\beta} := \{1_{M\gamma} : M\gamma M \rightarrow M \mid (M\beta M, 1_{M\beta}) \approx (M\gamma M, 1_{M\gamma})\}$$

and $\hat{\Gamma} := \{\hat{\beta} \mid 0 \neq \beta \in \Gamma\}$. Define an addition “+” on $\hat{\Gamma}$ as follows:

$$\begin{aligned} \hat{\beta} + \hat{\delta} &= Cl(M\beta M, 1_{M\beta}) + Cl(M\delta M, 1_{M\delta}) \\ &= Cl(M\beta M \cap M\delta M, 1_{M\beta} + 1_{M\delta}) \end{aligned}$$

for every $\beta(\neq 0), \delta(\neq 0) \in \Gamma$. Then $(\hat{\Gamma}, +)$ is an abelian group. Now we define a mapping $(-, -, -) : Q \times \hat{\Gamma} \times Q \rightarrow Q, (\hat{f}, \hat{\beta}, \hat{g}) \mapsto \hat{f}\hat{\beta}\hat{g}$, as follows:

$$\begin{aligned} \hat{f}\hat{\beta}\hat{g} &= Cl(U, f)Cl(M\beta M, 1_{M\beta})Cl(V, g) \\ &= Cl(V\Gamma M\beta M\Gamma U, f1_{M\beta}g) \end{aligned}$$

where

$$V\Gamma M\beta M\Gamma U = \left\{ \sum v_i \gamma_i m_i \beta n_i \alpha_i u_i \mid v_i \in V, u_i \in U, m_i, n_i \in M \text{ and } \alpha_i, \gamma_i \in \Gamma \right\}$$

is an ideal of M and $f1_{M\beta}g : V\Gamma M\beta M\Gamma U \rightarrow M$ which is given by

$$f1_{M\beta}g\left(\sum v_i \gamma_i m_i \beta n_i \alpha_i u_i\right) = f\left(\sum g(v_i) \gamma_i m_i \beta n_i \alpha_i u_i\right)$$

is a right M -module homomorphism. Then Q is a $\hat{\Gamma}$ -ring with unity. Noticing that the mapping $\varphi : \Gamma \rightarrow \hat{\Gamma}$ defined by $\varphi(\beta) = \hat{\beta}$ for every $0 \neq \beta \in \Gamma$ is an isomorphism, we know that the $\hat{\Gamma}$ -ring Q is a Γ -ring (see [12]). For purposes of convenience, we use q instead of $\hat{q} \in Q$.

Definition 3.1. Let M be a Γ -ring with unity. An element u in M is called a *unit* of M if it has a multiplicative inverse in M . If every nonzero element of M is a unit, we say that M is a Γ -division ring. A Γ -ring M is called a Γ -field if it is a commutative Γ -division ring.

Definition 3.2. The set

$$C_\Gamma := \{g \in Q \mid g\gamma f = f\gamma g \text{ for all } f \in Q \text{ and } \gamma \in \Gamma\}$$

is called the *extended centroid* of a Γ -ring M .

Lemma 3.3. *Let M be a prime Γ -ring. Then the extended centroid C_Γ of M is a Γ -field.*

Proof. Noticing that C_Γ is a commutative ring with unity, it is sufficient to show that every nonzero element of C_Γ is invertible. If $c(\neq 0) \in C_\Gamma$, then $c = Cl(U, \mu)$. Thus, by Lemma 2.7., there is a nonzero ideal U of M such that $\mu(U) \subset M$. Clearly, $0 \neq V = \mu(U)$ is an ideal of M . Since $U\Gamma M \subset U$, therefore $\mu(U)\Gamma M \subset \mu(U)$. Hence we can define a mapping $f : \mu(U) \rightarrow M$ by $f(\mu(u)) = u$ for all $u \in U$, and this is a right M -module

homomorphism. In fact, let $v_1, v_2 \in V = \mu(U)$ and so there exists $u_1, u_2 \in U$ such that $v_1 = \mu(u_1)$ and $v_2 = \mu(u_2)$. It follows that

$$\begin{aligned} f(v_1 + v_2) &= f(\mu(u_1) + \mu(u_2)) \\ &= f(\mu(u_1 + u_2)) = u_1 + u_2 \\ &= f(\mu(u_1)) + f(\mu(u_2)) \\ &= f(v_1) + f(v_2). \end{aligned}$$

Now, for any $v \in V$, $m \in M$ and $\gamma \in \Gamma$, we have

$$f(v\gamma m) = f(\mu(u)\gamma m) = f(\mu(u\gamma m)) = u\gamma m = f(\mu(u))\gamma m = f(v)\gamma m.$$

Finally, considering $d = Cl(V, f)$, we get

$$\begin{aligned} d\gamma c &= Cl(V, f)Cl(M\gamma M, 1_{M\gamma})Cl(U, \mu) \\ &= Cl(UTM\gamma M\Gamma V, f1_{M\gamma}\mu) \\ &= Cl(UTM\gamma M\Gamma\mu(U), 1) = I. \end{aligned}$$

This completes the proof. □

Definition 3.4. For the extended centroid C_Γ of a prime Γ -ring M , we say that $S := M\Gamma C_\Gamma$ is the *central closure* of M .

Remark 3.5. For $a, b \in S$, if $a\Gamma S\Gamma b = 0$ then $a\Gamma M\Gamma C_\Gamma b = 0$ and so $a\Gamma M\Gamma b\Gamma M\Gamma a\Gamma C_\Gamma b = 0$. Since M is a prime Γ -ring, it follows that $a\Gamma M\Gamma b = 0$ or $a\Gamma C_\Gamma b = 0$ so $a = 0$ or $b = 0$. Thus S is a prime Γ -ring.

If M has a unit element, then $C_\Gamma = Z(S)$, the centre of S . If M is a simple Γ -ring with unity, then $Q = S = M$. Because the only non-zero ideal of M is M itself. In this case; M is its own central closure.

Throughout, we shall use M as a prime Γ -ring such that $M\Gamma M \neq M$.

Theorem 3.6. *Let C_Γ be the extended centroid of a prime Γ -ring M . If a is a nonzero element of M such that $a\gamma_1 x\gamma_2 a\beta_1 y\beta_2 a = a\beta_1 y\beta_2 a\gamma_1 x\gamma_2 a$ for all $x, y \in M$, $\gamma_1, \gamma_2, \beta_1, \beta_2 \in \Gamma$ then $S = M\Gamma C_\Gamma$ is a primitive Γ -ring with minimal right (left) ideal and the commuting ring of S on this right (left) ideal is merely C_Γ itself.*

Proof. Let fixed $a\gamma_1 x\gamma_2 a$ element in the relation $(a\gamma_1 x\gamma_2 a)\beta_1 y\beta_2 a = a\beta_1 y\beta_2 (a\gamma_1 x\gamma_2 a) = 0$ then, from Lemma 2.8 we get $a\gamma_1 x\gamma_2 a = \lambda(x)\alpha a$, where $\lambda(x) \in C_\Gamma$ and $\alpha \in \Gamma$ and for

all $x \in M$. Similarly we also get $a\beta_1y\beta_2a = \lambda(y)\acute{a}a$, where $\lambda(y) \in C_\Gamma$ and $\acute{a} \in \Gamma$ and for all $y \in M$. Thus, since $a\beta_1y\beta_2a = \lambda(y)\acute{a}a \in C_\Gamma\Gamma a$ we get $a\Gamma S\Gamma a \subset C_\Gamma\Gamma a$. Since $a \neq 0$ and S is prime Γ -ring, there is some $y_o \in S$ such that $a\beta_1y_o\beta_2a \neq 0$ for some $\beta_1, \beta_2 \in \Gamma$. Thus, $a\beta_1y_o\beta_2a = \lambda(y_o)\acute{a}a$, where $0 \neq \lambda(y_o) \in C_\Gamma$. Similarly we get $a\gamma_1x_o\gamma_2a = \lambda(x_o)\alpha a$, where $0 \neq \lambda(x_o) \in C_\Gamma$. If $x_o = \lambda^{-1}(y_o)\alpha y_o$, then $a\gamma_1x_o\gamma_2a = a\gamma_1\lambda^{-1}(y_o)\alpha y_o\gamma_2a = \lambda^{-1}(y_o)\alpha a\gamma_1y_o\gamma_2a = \lambda^{-1}(y_o)\alpha\lambda(y_o)\alpha a = a$. Thus, let $e = a\gamma_1x_o$. $e\gamma_2e = (a\gamma_1x_o)\gamma_2(a\gamma_1x_o) = (a\gamma_1x_o\gamma_2a)\gamma_1x_o = a\gamma_1x_o = e$. From this we will have e idempotent. In this case; $e\Gamma S\Gamma e = (a\gamma_1x_o)\Gamma S\Gamma(a\gamma_1x_o) \subset C_\Gamma\Gamma(a\gamma_1x_o) = C_\Gamma\Gamma e$. Thus $e\Gamma S$ is a minimal right ideal of S and $C_\Gamma\Gamma e$ is the commuting ring of S on $e\Gamma S$ by Lemma 3.3. Since S is prime Γ -ring and has a minimal right ideal. S is primitive Γ -ring by Theorem 2.2. \square

Theorem 3.7. *Let M be a simple Γ -ring with unity. Suppose that for some $a \neq 0$ in M we have $a\gamma_1x\gamma_2a\beta_1y\beta_2a = a\beta_1y\beta_2a\gamma_1x\gamma_2a$ for all $x, y \in M$ and $\gamma_1, \gamma_2, \beta_1, \beta_2 \in \Gamma$. Then M is isomorphic onto the Γ -ring $D_{n,m}$, where $D_{n,m}$ is the additive abelian group of all rectangular matrices of type $n \times m$ over a division ring D and Γ is a nonzero subgroup of the additive abelian group of all rectangular matrices of type $m \times n$ over a division ring D . Furthermore M is the Γ -ring of all $n \times n$ matrices over the field C_Γ .*

Proof. Since M is simple Γ -ring we have $M = S$ and from Theorem 3.6 we get M has a minimal right (left) ideal of M . In this case, M is the sum of minimal right (left) ideals by Theorem 2.4, that is, M is the sum of minimal right ideals N_i , where $N_i = x_i\Gamma N$ (N is a non-zero minimal right ideal of M) for some $x_i \in M$. Also, since M has unit ($1 \in M$), $1 \in N_1 + \dots + N_n$ for some n , we get $M = N_1 + \dots + N_n$ and so M is the sum of a finite number of minimal right ideals, each of which is an irreducible right M -module. Thus M , as a M - module, has a composition series. Thus M has min-condition and so M is primitive Γ -ring by Theorem 2.3. In this case, by Theorem 3.6, the commuting ring of M on an irreducible module is $C_\Gamma = Z(M)$, the center of M . Thus, this finishes the proof of the theorem by Theorem 2.6. \square

Theorem 3.8. *Let M be prime Γ -ring and C_Γ the extended centroid of M . If a and b are non-zero elements in $S = M\Gamma C_\Gamma$ such that $a\gamma x\beta b = b\beta x\gamma a$ for all $x \in M$ and $\gamma, \beta \in \Gamma$, then a and b are C_Γ -dependent.*

Proof. Firstly, we assume that $a \neq 0$ and $b \neq 0$. Let U be a non-zero ideal of

M such that $a\Gamma U \subseteq M$ and $a\Gamma U \subseteq M$, and set $V = U\Gamma a\Gamma U = \{\sum x_i \gamma_i a \beta_i y_i \mid x_i, y_i \in U, \gamma_i, \beta_i \in \Gamma\}$. We define a mapping $f : V \rightarrow M$ defined by $v \mapsto f(v) = f(\sum x_i \gamma_i a \beta_i y_i) = \sum x_i \gamma_i b \beta_i y_i$, for all $x_i, y_i \in U$ and $\gamma_i, \beta_i \in \Gamma$. We suppose that $\sum x_i \gamma_i a \beta_i y_i = 0$. Then,

$$\begin{aligned} 0 &= b\alpha_i m\sigma_i \sum x_i \gamma_i a \beta_i y_i = \sum b\alpha_i (m\sigma_i x_i) \gamma_i a \beta_i y_i \\ &= \sum a\alpha_i (m\sigma_i x_i) \gamma_i b \beta_i y_i = a\alpha_i m\sigma_i \sum x_i \gamma_i b \beta_i y_i \end{aligned}$$

Thus, we get, for all $x_i, y_i \in U$ and $\gamma_i, \beta_i \in \Gamma$

$$a\Gamma M\Gamma(\sum x_i \gamma_i b \beta_i y_i) = 0$$

and so since $a \neq 0$ and M is prime Γ -ring we get $\sum x_i \gamma_i b \beta_i y_i = 0$. Therefore, f is well defined. Also, specially $f((x\gamma a\beta y)\alpha m) = x\gamma b\beta y\alpha m = f(x\gamma a\beta y)\alpha m$ for all $x, y \in U$ and $m \in M$ and $\gamma, \beta, \alpha \in \Gamma$ and so f is a M -module homomorphism. Let q denote the element of Q determined by f , that is, $q = Cl(V, f)$. Let p be any element of Q with $p(W) \subseteq M$ for some non-zero ideal W of M by Lemma 2.7. In this case,

$$\begin{aligned} &(f1_{M\alpha p})(\sum w_i \gamma_i m_i \alpha n_i \beta_i x_i \gamma_i a \beta_i y_i) \\ &= f(\sum p(w_i) \gamma_i m_i \alpha n_i \beta_i x_i \gamma_i a \beta_i y_i) \\ &= \sum p(w_i) \gamma_i m_i \alpha n_i \beta_i x_i \gamma_i b \beta_i y_i \\ &= p(\sum w_i \gamma_i m_i \alpha n_i \beta_i x_i \gamma_i b \beta_i y_i) \\ &= p(1_{M\alpha} f(\sum w_i \gamma_i m_i \alpha n_i \beta_i x_i \gamma_i a \beta_i y_i)) \\ &= (p1_{M\alpha} f)(\sum w_i \gamma_i m_i \alpha n_i \beta_i x_i \gamma_i a \beta_i y_i) \end{aligned}$$

and so $q\alpha p = Cl(W\Gamma M\alpha M\Gamma V, f1_{M\alpha} P) = Cl(W\Gamma M\alpha M\Gamma V, P1_{M\alpha} f) = p\alpha q$. Thus, we get $q \in C_\Gamma$. For $\gamma, \beta, \alpha \in \Gamma$,

$$\begin{aligned} q\gamma(x\alpha a\beta y) &= Cl(V, f)Cl(M\gamma M, 1_{M\gamma})Cl(\acute{V}, x\alpha a\beta y) \\ &= Cl(\acute{V}\Gamma M\gamma M\Gamma V, f1_{M\gamma}(x\alpha a\beta y)) \\ &= Cl(\acute{V}\Gamma M\gamma M\Gamma V, x\alpha a\beta y) \\ &= x\alpha b\beta y, \end{aligned}$$

Hence we have $(x\gamma q\alpha a - x\alpha b)\beta y = 0$ for all $x, y \in U$ and $\gamma, \beta, \alpha \in \Gamma$. Therefore, since M is prime Γ -ring we get $x\gamma q\alpha a - x\alpha b = 0$ for all $x, y \in U$ and $\gamma, \alpha \in \Gamma$. Now writing $\alpha + \gamma$ for in the previous equation we get, $x\gamma(q\gamma a - b) = 0$ for all $\gamma, \in \Gamma$ and $x \in U$. Thus, since M is prime Γ -ring, we get, $q\gamma a = b$ for all $\gamma \in \Gamma$ and so this completes the proof. \square

Theorem 3.9. *Let M be prime Γ -ring, Q quotient Γ -ring of M and C_Γ the extended centroid of M . If q is non-zero element in Q such that $q\gamma_1 x \gamma_2 q \beta_1 y \beta_2 q = q \beta_1 y \beta_2 q \gamma_1 x \gamma_2 q$ for all $x, y \in M$, $\gamma_1, \gamma_2, \beta_1, \beta_2 \in \Gamma$ then S is a primitive Γ -ring with minimal right (left) ideal such that $e\Gamma S$, where e is idempotent and $C_\Gamma \Gamma e$ is the commuting ring of S on $e\Gamma S$.*

Proof. If $q \in M$, then the proof finishes from Theorem 3.6. If $q \in Q$ then one can pick $a \in M$ such that $\acute{q} = q\alpha a$ is a non-zero element of M by Lemma 2.7. Also, \acute{q} satisfies $\acute{q}\gamma_1 x \gamma_2 \acute{q} \beta_1 y \beta_2 \acute{q} = \acute{q} \beta_1 y \beta_2 \acute{q} \gamma_1 x \gamma_2 \acute{q}$ for all $x, y \in M$, $\gamma_1, \gamma_2, \beta_1, \beta_2 \in \Gamma$ and so this completes the proof. \square

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