Turkish Journal of Mathematics

Volume 25 | Number 3 Article 2

1-1-2001

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Turk J Math
25 (2001) , 367 – 377.


# On the Centroid of the Prime Gamma Rings II

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**Abstract**

The aim of this paper is to study the properities of the extended centroid of the prime $\Gamma$-rings. Main results are the following theorems: (1) Let $M$ be a simple $\Gamma$-ring with unity. Suppose that for some $a \neq 0$ in $M$ we have $a\gamma_1 x\gamma_2 a\beta_1 y\beta_2 a = a\beta_1 y\beta_2 a\gamma_1 x\gamma_2 a$ for all $x, y \in M$ and $\gamma_1, \gamma_2, \beta_1, \beta_2 \in \Gamma$. Then $M$ is isomorphic onto the $\Gamma$-ring $D_{n,m}$, where $D_{n,m}$ is the additive abelian group of all rectangular matrices of type $n \times m$ over a division ring $D$ and $\Gamma$ is a nonzero subgroup of the additive abelian group of all rectangular matrices of type $m \times n$ over a division ring $D$. Furthermore $M$ is the $\Gamma$-ring of all $n \times n$ matrices over the field $C_\Gamma$. (2) Let $M$ be a prime $\Gamma$-ring and $C_\Gamma$ the extended centroid of $M$. If $a$ and $b$ are non-zero elements in $S = M\Gamma C_\Gamma$ such that $a\gamma x\beta b = b\beta x\gamma a$ for all $x \in M$ and $\beta, \gamma \in \Gamma$, then $a$ and $b$ are $C_\Gamma$-dependent. (3) Let $M$ be prime $\Gamma$-ring, $Q$ quotient $\Gamma$-ring of $M$ and $C_\Gamma$ the extended centroid of $M$. If $q$ is non-zero element in $Q$ such that $q\gamma_1 x\gamma_2 q\beta_1 y\beta_2 q = q\beta_1 y\beta_2 q\gamma_1 x\gamma_2 q$ for all $x, y \in M$, $\gamma_1, \gamma_2, \beta_1, \beta_2 \in \Gamma$ then $S$ is a primitive $\Gamma$-ring with minimal right ( left ) ideal such that $e\Gamma S$, where $e$ is idempotent and $C_\Gamma \Gamma e$ is the commuting ring of $S$ on $e\Gamma S$.

**Key Words:** $\Gamma$-division ring, $\Gamma$-field, extented centroid, central closure.

## 1. Introduction

Nobusawa [11] introduced the notion of a $\Gamma$-ring, more general than a ring. Barnes [1] weakened slightly the conditions in the definition of $\Gamma$-ring in the sense of Nobusawa.

2000 *Mathematics Subject Classification.* Primary 16N60, 16Y30, 16A76, 16Y99.

Barnes [1], Luh [7] and Kyuno [4] studied the structure of $\Gamma$-rings and obtained various generalizations analogous of corresponding parts in ring theory. Öztürk and Jun [12] studied the extended centroid of a prime $\Gamma$-ring. As a continuation of [12], in this paper, we study further properities of the extended centroid of the prime $\Gamma$-rings.

## 2. Preliminaries

Let $M$ and $\Gamma$ be two abelian groups. If for all $x, y, z \in M$ and all $\alpha,\ \beta \in \Gamma$ the conditions

(i) $x\alpha y \in M$,

(ii) $(x+y)\alpha z = x\alpha z + y\alpha z, x(\alpha+\beta)z = x\alpha z + x\beta z, x\alpha(y+z) = x\alpha y + x\alpha z$,

(iii) $(x\alpha y)\beta z = x\alpha(y\beta z)$

are satisfied, then we call $M$ a $\Gamma$-*ring.* By a *right* (resp. *left*) *ideal* of a $\Gamma$-ring $M$ we mean an additive subgroup $U$ of $M$ such that $U\Gamma M \subseteq U$ (resp. $M\Gamma U \subseteq U$). If $U$ is both a right and a left ideal, then we say that $U$ is an *ideal* of $M$. For each $a$ of a $\Gamma$-ring $M$ the smallest right ideal containing $a$ is called the *principal right ideal generated* by $a$ and is denoted by $\langle a\rangle_r$. Similarly we define $\langle a\rangle_l$ (resp. $\langle a\rangle$), the *principal left* (resp. *two sided*) *ideal generated* by $a$. An ideal $P$ of a $\Gamma$-ring $M$ is said to be *prime* if for any ideals $A$ and $B$ of $M$, $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. An ideal $Q$ of a $\Gamma$-ring $M$ is said to be *semi-prime* if for any ideal $U$ of $M$, $U\Gamma U \subseteq Q$ implies $U \subseteq Q$. A $\Gamma$-ring $M$ is said to be *prime* (resp. *semi-prime*) if the zero ideal is prime (resp. semi-prime).

**Theorem 2.1** ([4, Theorem 4]). *If $M$ is a $\Gamma$-ring, the following conditions are equivalent:*

(i) *$M$ is a prime $\Gamma$-ring.*

(ii) *If $a, b \in M$ and $a\Gamma M\Gamma b = (0)$, then $a = 0$ or $b = 0$.*

(iii) *If $\langle a\rangle$ and $\langle b\rangle$ are principal ideals in $M$ such that $\langle a\rangle\Gamma\langle b\rangle = (0)$, then $a = 0$ or $b = 0$.*

(iv) *If $A$ and $B$ are right ideals in $M$ such that $A\Gamma B = (0)$, then $A = (0)$ or $B = (0)$.*

(v) *If $A$ and $B$ are left ideals in $M$ such that $A\Gamma B = (0)$, then $A = (0)$ or $B = (0)$.*

A $\Gamma$-ring $M$ is said to be *simple* if $M\Gamma M \neq 0$ and $M$ has no ideals other 0 and $M$ itself. When a $\Gamma$-ring $M$ has the descending (resp. ascending) chain condition for right ideals, it is abbreviated to $M$ has *min-r condition* (resp. *max-r condition*). The terms *min-l condition* or *max-l condition* on a $\Gamma$-ring $M$ are likewise defined. Let $M$ be a $\Gamma$-ring and let $F$ be the free group generated by $\Gamma \times M$. Then

$$A = \{\sum_i n_i(\gamma_i, x_i) \in F \mid a \in M \Rightarrow \sum_i n_i a\gamma_i x_i = 0\}$$

is a subgroup of $F$. Let $R = F/A$ be the factor group, and denote the coset $(\gamma, x) + A$ by $[\gamma, x]$. Clearly, every element of $R$ can be expressed as a finite sum $\sum_i [\gamma_i, x_i]$. Also it can be verified easily that $[\alpha, x] + [\alpha, y] = [\alpha, x + y]$ and $[\alpha, x] + [\beta, x] = [\alpha + \beta, x]$ for all $\alpha, \beta \in \Gamma$ and $x, y \in M$. We define a multiplication on $R$ by

$$\sum_i [\alpha_i, x_i] \sum_j [\beta_j, y_j] = \sum_{i,j} [\alpha_i, x_i\beta_j y_j].$$

Then $R$ forms a ring. If we define a composition on $M \times R$ into $M$ by

$$a \sum_i [\gamma_i, x_i] = \sum_i a\gamma_i x_i, \ \forall a \in M, \ \forall \sum_i [\gamma_i, x_i] \in R$$

then $M$ is a right $R$-module, and we call $R$ the *right operator ring* of $M$. Similarly, we can define the *left operator ring* $L$ of $M$. A $\Gamma$-ring $M$ is said to be *right* (resp. *left*) *primitive* if it satisfies:

(i) the right (resp. left) operator ring of $M$ is a right (resp. left) primitive ring

(ii) $M\Gamma x = 0$ (resp. $x\Gamma M = 0$) implies $x = 0$.

A $\Gamma$-ring $M$ is said to be *two-sided primitive* (or simply, *primitive*) if it is both right and left primitive.

**Theorem 2.2** ([7, Theorem 3.4])**.** *If $M$ is a $\Gamma$-ring possessing minimal left (resp. right) ideal, then $M$ is primitive if and only if it is prime.*

**Theorem 2.3** ([7, Theorem 3.6])**.** *For a $\Gamma$-ring $M$ with min-l condition, the following are equivalent:*

(i) *$M$ is prime,*

(ii) *$M$ is primitive,*

(iii) *$M$ is simple.*

**Theorem 2.4** ([7, Theorem 4.2]). *If $M$ is a simple $\Gamma$-ring possessing minimal left (resp. right) ideals, then $M$ is a direct sum of minimal left (resp. left) ideals.*

**Theorem 2.5** ([5, Theorem 3.23]). *Let $M$ be a semi-prime $\Gamma$-ring with min-r condition and let $M = I_1 \oplus I_2 \oplus \cdots \oplus I_m = J_1 \oplus J_2 \oplus \cdots \oplus J_n$, where $I_1, I_2, \cdots, I_m, J_1, J_2, \cdots, J_n$ are minimal right ideals. Then $m = n$.*

The integer $m = n$ in Theorem 2.5 is called the *right dimension* of the semi-prime $\Gamma$-ring with min-$r$ condition and denoted by $dim(M_R)$. One can define the left dimension of a $\Gamma$-ring in a similar way. If $M$ is simple, then $M$ is semi-prime (see [5]). For an additive group $G$, denote by $G_{m,n}$ the additive group of all matrices over $G$. Let $M$ be a $\Gamma$-ring $M$ and let $M_{m,n}$ and $\Gamma_{n,m}$ denote, respectively, the sets of $m \times n$ matrices with entries from $M$ and of $n \times m$ matrices with entries from $\Gamma$. For $(a_{ij}), (b_{ij}) \in M_{m,n}$ and $(\gamma_{ij}) \in \Gamma_{n,m}$, define $(a_{ij})(\gamma_{ij})(b_{ij}) = (c_{ij})$, where $c_{ij} = \sum_p \sum_q a_{ip}\gamma_{pq}b_{qj}$. Then $M_{m,n}$ forms a $\Gamma_{n,m}$-ring.

**Theorem 2.6** ([6, Theorem 4.2]). *Let $M$ be a simple $\Gamma$-ring with min-r and min-l conditions and $\Gamma_0 = \Gamma/\kappa$, where $\kappa := \{\gamma \in \Gamma \mid M\gamma M = 0\}$. Then the $\Gamma_0$-ring $M$ is isomorphic to the $\Gamma'$-ring $D_{n,m}$, where $D_{n,m}$ is the additive abelian group of all rectangular matrices of type $n \times m$ over a division ring $D$ and $\Gamma'$ is a nonzero subgroup of the additive abelian group of all rectangular matrices of type $m \times n$ over a division ring $D$ and $m = dim(M_L)$ and $n = dim(M_R)$.*

**Lemma 2.7** ([12, Lemma 3]). *Let $M$ be a prime $\Gamma$-ring such that $M\Gamma M \neq M$ and quotient $\Gamma$-ring $Q$ of $M$. Then, for each non-zero $q \in Q$ there is a non-zero ideal $U$ of $M$ such that $q(U) \subset M$.*

**Lemma 2.8** ([12, p. 476]). *Let $M$ be a prime $\Gamma$-ring such that $M\Gamma M \neq M$ and $C_\Gamma$ the extended centroid of $M$. If $a_i$ and $b_i$ are non-zero elements of $M$ such that $\sum a_i\gamma_i x\beta_i b_i = 0$ for all $x \in M$ and $\gamma_i, \beta_i \in \Gamma$, then the $a_i$'s ( also $a_i$'s ) are linearly dependent over $C_\Gamma$. Moreover, if $a\gamma x\beta b = b\gamma x\beta a$ for all $x \in M$ and $\gamma, \beta \in \Gamma$ where $a(\neq 0), b \in M$ are fixed, then there exists $\lambda \in C_\Gamma$ such that $b = \lambda\alpha a$ for all $\alpha \in \Gamma$.*

## 3. Centroids

Let $M$ be a prime $\Gamma$-ring such that $M\Gamma M \neq M$. Denote

$$\mathcal{M} := \{(U,f) \mid \begin{array}{l} U(\neq 0) \text{ is an ideal of } M \text{ and} \\ f: U \to M \text{ is a right } M\text{-module homomorphism}\}. \end{array}$$

Define a relation $\sim$ on $\mathcal{M}$ by $(U,f) \sim (V,g)$ if and only if $\exists W(\neq 0) \subset U \cap V$ such that $f = g$ on $W$. Since $M$ is a prime $\Gamma$-ring, it is possible to find a non-zero $W$ and so "$\sim$" is an equivalence relation. This gives a chance for us to get a partition of $\mathcal{M}$. We then denote the equivalence class by $Cl(U,f) = \hat{f}$, where $\hat{f} := \{g : V \to M | (U,f) \sim (V,g)\}$, and denote by $Q$ the set of all equivalence classes. Now we define an addition "+" on $Q$ as follows:

$$\hat{f} + \hat{g} = Cl(U,f) + Cl(V,g) = Cl(U \cap V, f+g)$$

where $f + g : U \cap V \to M$ is a right $M$-module homomorphism. Then $Q$ is an additive abelian group (see [12]). Since $M\Gamma M \neq M$ and since $M$ is a prime $\Gamma$-ring, $M\Gamma M$ $(\neq 0)$ is an ideal of $M$. We can take the homomorphism $1_{M\Gamma} : M\Gamma M \to M$ as a unit $M$-module homomorphism. Note that $M\beta M \neq 0$ for all $0 \neq \beta \in \Gamma$ so that $1_{M\beta} : M\beta M \to M$ is non-zero $M$-module homomorphism. Denote

$$\mathcal{N} := \{(M\beta M, 1_{M\beta}) \mid 0 \neq \beta \in \Gamma\},$$

and define a relation "$\approx$" on $\mathcal{N}$ by $(M\beta M, 1_{M\beta}) \approx (M\gamma M, 1_{M\gamma})$ if and only if $\exists W := M\alpha M(\neq 0) \subset M\beta M \cap M\gamma M$ such that $1_{M\beta} = 1_{M\gamma}$ on $W$. We can easily check that "$\approx$" is an equivalence relation on $\mathcal{N}$. Denote by $Cl(M\beta M, 1_{M\beta}) = \hat{\beta}$ the equivalence class containing $(M\beta M, 1_{M\beta})$ and by $\hat{\Gamma}$ the set of all equivalence classes of $\mathcal{N}$ with respect to $\approx$, that is,

$$\hat{\beta} := \{1_{M\gamma} : M\gamma M \to M \mid (M\beta M, 1_{M\beta}) \approx (M\gamma M, 1_{M\gamma})\}$$

and $\hat{\Gamma} := \{\hat{\beta} \mid 0 \neq \beta \in \Gamma\}$. Define an addition "+" on $\hat{\Gamma}$ as follows:

$$\begin{aligned} \hat{\beta} + \hat{\delta} &= Cl(M\beta M, 1_{M\beta}) + Cl(M\delta M, 1_{M\delta}) \\ &= Cl(M\beta M \cap M\delta M, 1_{M\beta} + 1_{M\delta}) \end{aligned}$$

for every $\beta(\neq 0)$, $\delta(\neq 0) \in \Gamma$. Then $(\hat{\Gamma}, +)$ is an abelian group. Now we define a mapping $(-,-,-) : Q \times \hat{\Gamma} \times Q \to Q$, $(\hat{f}, \hat{\beta}, \hat{g}) \mapsto \hat{f}\hat{\beta}\hat{g}$, as follows:

$$\begin{aligned}\hat{f}\hat{\beta}\hat{g} &= Cl(U, f)Cl(M\beta M, 1_{M\beta})Cl(V, g)\\ &= Cl(V\Gamma M\beta M\Gamma U, f1_{M\beta}g)\end{aligned}$$

where

$$V\Gamma M\beta M\Gamma U = \{\sum v_i\gamma_i m_i\beta n_i\alpha_i u_i \mid v_i \in V, u_i \in U, m_i, n_i \in M \text{ and } \alpha_i, \gamma_i \in \Gamma\}$$

is an ideal of $M$ and $f1_{M\beta}g : V\Gamma M\beta M\Gamma U \to M$ which is given by

$$f1_{M\beta}g(\sum v_i\gamma_i m_i\beta n_i\alpha_i u_i) = f(\sum g(v_i)\gamma_i m_i\beta n_i\alpha_i u_i)$$

is a right $M$-module homomorphism. Then $Q$ is a $\hat{\Gamma}$-ring with unity. Noticing that the mapping $\varphi : \Gamma \to \hat{\Gamma}$ defined by $\varphi(\beta) = \hat{\beta}$ for every $0 \neq \beta \in \Gamma$ is an isomorphism, we know that the $\hat{\Gamma}$-ring $Q$ is a $\Gamma$-ring (see [12]). For purposes of convenience, we use $q$ instead of $\hat{q} \in Q$.

**Definition 3.1.** Let $M$ be a $\Gamma$-ring with unity. An element $u$ in $M$ is called a *unit* of $M$ if it has a multiplicative inverse in $M$. If every nonzero emenet of $M$ is a unit, we say that $M$ is a $\Gamma$-*division ring*. A $\Gamma$-ring $M$ is called a $\Gamma$-*field* if it is a commutative $\Gamma$-division ring.

**Definition 3.2.** The set

$$C_\Gamma := \{g \in Q \mid g\gamma f = f\gamma g \text{ for all } f \in Q \text{ and } \gamma \in \Gamma\}$$

is called the *extended centroid* of a $\Gamma$-ring $M$.

**Lemma 3.3.** *Let $M$ be a prime $\Gamma$-ring. Then the extended centroid $C_\Gamma$ of $M$ is a $\Gamma$-field.*

**Proof.** Noticing that $C_\Gamma$ is a commutative ring with unity, it is sufficient to show that every nonzero element of $C_\Gamma$ is invertible. If $c(\neq 0) \in C_\Gamma$, then $c = Cl(U, \mu)$. Thus, by Lemma 2.7., there is a nonzero ideal $U$ of $M$ such that $\mu(U) \subset M$. Clearly, $0 \neq V = \mu(U)$ is an ideal of $M$. Since $U\Gamma M \subset U$, therefore $\mu(U)\Gamma M \subset \mu(U)$. Hence we can define a mapping $f : \mu(U) \to M$ by $f(\mu(u)) = u$ for all $u \in U$, and this is a right $M$-module homomorphism. In fact, let $v_1, v_2 \in V = \mu(U)$ and so there exists $u_1, u_2 \in U$ such that $v_1 = \mu(u_1)$ and $v_2 = \mu(u_2)$. It follows that

$$\begin{aligned} f(v_1 + v_2) &= f(\mu(u_1) + \mu(u_2)) \\ &= f(\mu(u_1 + u_2)) = u_1 + u_2 \\ &= f(\mu(u_1)) + f(\mu(u_2)) \\ &= f(v_1) + f(v_2). \end{aligned}$$

Now, for any $v \in V$, $m \in M$ and $\gamma \in \Gamma$, we have

$$f(v\gamma m) = f(\mu(u)\gamma m) = f(\mu(u\gamma m)) = u\gamma m = f(\mu(u))\gamma m = f(v)\gamma m.$$

Finally, considering $d = Cl(V, f)$, we get

$$\begin{aligned} d\gamma c &= Cl(V, f)Cl(M\gamma M, 1_{M\gamma})Cl(U, \mu) \\ &= Cl(U\Gamma M\gamma M\Gamma V, f1_{M\gamma}\mu) \\ &= Cl(U\Gamma M\gamma M\Gamma\mu(U), 1) = I. \end{aligned}$$

This completes the proof. □

**Definition 3.4.** For the extended centroid $C_\Gamma$ of a prime $\Gamma$-ring $M$, we say that $S := M\Gamma C_\Gamma$ is the *central closure* of $M$.

**Remark 3.5.** For $a, b \in S$, if $a\Gamma S\Gamma b = 0$ then $a\Gamma M\Gamma C_\Gamma b = 0$ and so $a\Gamma M\Gamma b\Gamma M\Gamma a\Gamma C_\Gamma b = 0$. Since $M$ is a prime $\Gamma$-ring, it follows that $a\Gamma M\Gamma b = 0$ or $a\Gamma C_\Gamma b = 0$ so $a = 0$ or $b = 0$. Thus $S$ is a prime $\Gamma$-ring.

If $M$ has a unit element, then $C_\Gamma = Z(S)$, the centre of $S$. If $M$ is a simple $\Gamma$-ring with unity, then $Q = S = M$. Because the only non-zero ideal of $M$ is $M$ itself. In this case; $M$ is its own central closure.

Throughout, we shall use $M$ as a prime $\Gamma$-ring such that $M\Gamma M \neq M$.

**Theorem 3.6.** *Let $C_\Gamma$ be the extended centroid of a prime $\Gamma$-ring $M$. If $a$ is a nonzero element of $M$ such that $a\gamma_1 x\gamma_2 a\beta_1 y\beta_2 a = a\beta_1 y\beta_2 a\gamma_1 x\gamma_2 a$ for all $x, y \in M$, $\gamma_1, \gamma_2, \beta_1, \beta_2 \in \Gamma$ then $S = M\Gamma C_\Gamma$ is a primitive $\Gamma$-ring with minimal right (left) ideal and the commuting ring of $S$ on this right (left) ideal is merely $C_\Gamma$ itself.*

**Proof.** Let fixed $a\gamma_1 x\gamma_2 a$ element in the relation $(a\gamma_1 x\gamma_2 a)\beta_1 y\beta_2 a = a\beta_1 y\beta_2(a\gamma_1 x\gamma_2 a) = 0$ then, from Lemma 2.8 we get $a\gamma_1 x\gamma_2 a = \lambda(x)\alpha a$, where $\lambda(x) \in C_\Gamma$ and $\alpha \in \Gamma$ and for all $x \in M$. Similarly we also get $a\beta_1 y\beta_2 a = \lambda(y)\acute{\alpha}a$, where $\lambda(y) \in C_\Gamma$ and $\acute{\alpha} \in \Gamma$ and for all $y \in M$. Thus, since $a\beta_1 y\beta_2 a = \lambda(y)\acute{\alpha}a \in C_\Gamma\Gamma a$ we get $a\Gamma S\Gamma a \subset C_\Gamma\Gamma a$. Since $a \neq 0$ and $S$ is prime $\Gamma$-ring, there is some $y_o \in S$ such that $a\beta_1 y_o\beta_2 a \neq 0$ for some $\beta_1, \beta_2 \in \Gamma$. Thus, $a\beta_1 y_o\beta_2 a = \lambda(y_o)\acute{\alpha}a$, where $0 \neq \lambda(y_o) \in C_\Gamma$. Similarly we get $a\gamma_1 x_o\gamma_2 a = \lambda(x_o)\alpha a$, where $0 \neq \lambda(x_o) \in C_\Gamma$. If $x_o = \lambda^{-1}(y_o)\alpha y_o$, then $a\gamma_1 x_o\gamma_2 a = a\gamma_1\lambda^{-1}(y_o)\alpha y_o\gamma_2 a = \lambda^{-1}(y_o)\alpha a\gamma_1 y_o\gamma_2 a = \lambda^{-1}(y_o)\alpha\lambda(y_o\alpha a = a$. Thus, let $e = a\gamma_1 x_o$. $e\gamma_2 e = (a\gamma_1 x_o)\gamma_2(a\gamma_1 x_o) = (a\gamma_1 x_o\gamma_2 a)\gamma_1 x_o = a\gamma_1 x_o = e$. From this we will have $e$ idempotent. In this case; $e\Gamma S\Gamma e = (a\gamma x_o)\Gamma S\Gamma(a\gamma x_o) \subset C_\Gamma\Gamma(a\gamma x_o) = C_\Gamma\gamma e$. Thus $e\Gamma S$ is a minimal right ideal of $S$ and $C_\Gamma\Gamma e$ is the commuting ring of $S$ on $e\Gamma S$ by Lemma 3.3. Since $S$ is prime $\Gamma$-ring and has a minimal right ideal. $S$ is primitive $\Gamma$-ring by Theorem 2.2. $\square$

**Theorem 3.7.** *Let $M$ be a simple $\Gamma$-ring with unity. Suppose that for some $a \neq 0$ in $M$ we have $a\gamma_1 x\gamma_2 a\beta_1 y\beta_2 a = a\beta_1 y\beta_2 a\gamma_1 x\gamma_2 a$ for all $x, y \in M$ and $\gamma_1, \gamma_2, \beta_1, \beta_2 \in \Gamma$. Then $M$ is isomorphic onto the $\Gamma$-ring $D_{n,m}$, where $D_{n,m}$ is the additive abelian group of all rectangular matrices of type $n \times m$ over a division ring $D$ and $\Gamma$ is a nonzero subgroup of the additive abelian group of all rectangular matrices of type $m \times n$ over a division ring $D$. Furthermore $M$ is the $\Gamma$-ring of all $n \times n$ matrices over the field $C_\Gamma$.*

**Proof.** Since $M$ is simple $\Gamma$-ring we have $M = S$ and from Theorem 3.6 we get $M$ has a minimal right ( left ) ideal of $M$. In this case, $M$ is the sum of minimal right ( left ) ideals by Theorem 2.4, that is, $M$ is the sum of minimal right ideals $N_i$, where $N_i = x_i\Gamma N$ ( $N$ is a non-zero minimal right ideal of $M$ ) for some $x_i \in M$. Also, since $M$ has unit ( $1 \in M$ ), $1 \in N_1 + ... + N_n$ for some $n$, we get $M = N_1 + ... + N_n$ and so $M$ is the sum of a finite number of minimal right ideals, each of which is an irredicible right $M$-module. Thus $M$, as a $M$- module, has a composition serises. Thus $M$ has min-r condition and so $M$ is primitive $\Gamma$-ring by Theorem 2.3. In this case, by Theorem 3.6, the commuting ring of $M$ on an irreducible module is $C_\Gamma = Z(M)$, the center of $M$. Thus, this finishes the proof of the theorem by Theorem 2.6. $\square$

**Theorem 3.8.** *Let $M$ be prime $\Gamma$-ring and $C_\Gamma$ the extended centroid of $M$. If $a$ and $b$ are non-zero elements in $S = M\Gamma C_\Gamma$ such that $a\gamma x\beta b = b\beta x\gamma a$ for all $x \in M$ and $\gamma, \beta \in \Gamma$, then $a$ and $b$ are $C_\Gamma$-dependent.*

**Proof.** Firstly, we assume that $a \neq 0$ and $b \neq 0$. Let $U$ be a non- zero ideal of

$M$ such that $a\Gamma U \subseteq M$ and $a\Gamma U \subseteq M$, and set $V = U\Gamma a\Gamma U = \{\sum x_i\gamma_i a\beta_i y_i \mid x_i, y_i \in U, \gamma_i, \beta_i \in \Gamma\}$. We define a mapping $f : V \to M$ defined by $v \mapsto f(v) = f(\sum x_i\gamma_i a\beta_i y_i) = \sum x_i\gamma_i b\beta_i y_i$, for all $x_i, y_i \in U$ and $\gamma_i, \beta_i \in \Gamma$. We suppose that $\sum x_i\gamma_i a\beta_i y_i = 0$. Then,

$$\begin{aligned} 0 &= b\alpha_i m\sigma_i \sum x_i\gamma_i a\beta_i y_i = \sum b\alpha_i(m\sigma_i x_i)\gamma_i a\beta_i y_i \\ &= \sum a\alpha_i(m\sigma_i x_i)\gamma_i b\beta_i y_i = a\alpha_i m\sigma_i \sum x_i\gamma_i b\beta_i y_i \end{aligned}$$

Thus, we get, for all $x_i, y_i \in U$ and $\gamma_i, \beta_i \in \Gamma$

$$a\Gamma M\Gamma(\sum x_i\gamma_i b\beta_i y_i) = 0$$

and so since $a \neq 0$ and $M$ is prime $\Gamma$-ring we get $\sum x_i\gamma_i b\beta_i y_i = 0$. Therefore, $f$ is well defined. Also, specially $f((x\gamma a\beta y)\alpha m) = x\gamma b\beta y\alpha m = f(x\gamma a\beta y)\alpha m$ for all $x, y \in U$ and $m \in M$ and $\gamma, \beta, \alpha \in \Gamma$ and so $f$ is a $M$-module homomorphism. Let $q$ denote the element of $Q$ determined by $f$, that is, $q = Cl(V, f)$. Let $p$ be any element of $Q$ with $p(W) \subseteq M$ for some non-zero ideal $W$ of $M$ by Lemma 2.7. In this case,

$$\begin{aligned} & (f1_{M\alpha}p)(\sum w_i\acute{\gamma_i}m_i\alpha n_i\acute{\beta_i}x_i\gamma_i a\beta_i y_i) \\ = {} & f(\sum p(w_i)\acute{\gamma_i}m_i\alpha n_i\acute{\beta_i}x_i\gamma_i a\beta_i y_i) \\ = {} & \sum p(w_i)\acute{\gamma_i}m_i\alpha n_i\acute{\beta_i}x_i\gamma_i b\beta_i y_i \\ = {} & p(\sum w_i\acute{\gamma_i}m_i\alpha n_i\acute{\beta_i}x_i\gamma_i b\beta_i y_i) \\ = {} & p(1_{M\alpha}f(\sum w_i\acute{\gamma_i}m_i\alpha n_i\acute{\beta_i}x_i\gamma_i a\beta_i y_i)) \\ = {} & (p1_{M\alpha}f)(\sum w_i\acute{\gamma_i}m_i\alpha n_i\acute{\beta_i}x_i\gamma_i a\beta_i y_i) \end{aligned}$$

and so $q\alpha p = Cl(W\Gamma M\alpha M\Gamma V,\ f1_{M\alpha}P) = Cl(W\Gamma M\alpha M\Gamma V,\ P1_{M\alpha}f) = p\alpha q$. Thus, we get $q \in C_\Gamma$. For $\gamma,\ \beta,\ \alpha \in \Gamma$,

$$\begin{aligned} q\gamma(x\alpha a\beta y) &= Cl(V, f)Cl(M\gamma M,\ 1_{M\gamma})Cl(\acute{V},\ x\alpha a\beta y) \\ &= Cl(\acute{V}\Gamma M\gamma M\Gamma V,\ f1_{M\gamma}(x\alpha a\beta y)) \\ &= Cl(\acute{V}\Gamma M\gamma M\Gamma V,\ x\alpha a\beta y) \\ &= x\alpha b\beta y, \end{aligned}$$

Hence we have $(x\gamma q\alpha a - x\alpha b)\beta y = 0$ for all $x, y \in U$ and $\gamma, \beta, \alpha \in \Gamma$. Therefore, since $M$ is prime $\Gamma$ -ring we get $x\gamma q\alpha a - x\alpha b = 0$ for all $x, y \in U$ and $\gamma, \alpha \in \Gamma$. Now writing $\alpha + \gamma$ for in the previous equation we get, $x\gamma(q\gamma a - b) = 0$ for all $\gamma, \in \Gamma$ and $x \in U$. Thus, since $M$ is prime $\Gamma$-ring, we get, $q\gamma a = b$ for all $\gamma \in \Gamma$ and so this completes the proof. □

**Theorem 3.9.** *Let $M$ be prime $\Gamma$-ring, $Q$ quotient $\Gamma$-ring of $M$ and $C_\Gamma$ the extended centroid of $M$. If $q$ is non-zero element in $Q$ such that $q\gamma_1 x\gamma_2 q\beta_1 y\beta_2 q = q\beta_1 y\beta_2 q\gamma_1 x\gamma_2 q$ for all $x, y \in M$, $\gamma_1, \gamma_2, \beta_1, \beta_2 \in \Gamma$ then $S$ is a primitive $\Gamma$-ring with minimal right (left) ideal such that $e\Gamma S$, where $e$ is idempotent and $C_\Gamma \Gamma e$ is the commuting ring of $S$ on $e\Gamma S$.*

**Proof.** If $q \in M$, then the proof finishes from Theorem 3.6. If $q \in Q$ then one can pick $a \in M$ such that $\acute{q} = q\alpha a$ is a non-zero element of $M$ by Lemma 2.7. Also, $\acute{q}$ satisfies $\acute{q}\gamma_1 x\gamma_2\acute{q}\beta_1 y\beta_2\acute{q} = \acute{q}\beta_1 y\beta_2\acute{q}\gamma_1 x\gamma_2\acute{q}$ for all $x, y \in M$, $\gamma_1, \gamma_2, \beta_1, \beta_2 \in \Gamma$ and so this completes the proof. □

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Received 06.10.2000