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L^p Boundedness of a Class of Singular Integral Operators with Rough Kernels

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Abstract

In this paper, we study the L^{p}mapping properties of singular integral operators with kernels belonging to certain block spaces. These operators have singularities along sets of the form $\{x = \Phi(|y|)y'\}$where Φ satisfies certain growth conditions. Our results improve as well as extend previously known results on singular integrals.

Key Words: Singular integrals, oscillatory integrals, Fourier transform, L^p boundedness, rough kernels, block spaces.

1. Introduction and results

Let $\mathbf{S}^{n-1}$, $n \geq 2$ be the unit sphere in $\mathbf{R}^n$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(\cdot)$. Suppose $\Omega \in L^1(\mathbf{S}^{n-1})$ is a homogeneous function of degree zero on $\mathbf{R}^n$ and satisfies the cancellation condition

$$\int_{\mathbf{S}^{n-1}} \Omega\left(y'\right) d\sigma\left(y'\right) = 0, \tag{1.1}$$

where $y' = \frac{y}{|y|} \in \mathbf{S}^{n-1}$ for any $y \neq 0$.

Define the singular integral operator T by

$$Tf(x) = \text{p.v.} \int_{\mathbf{R}^n} f\left(x-y\right) \frac{\Omega(y')}{|y|^n} h(|y|) dy \tag{1.2}$$

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and the corresponding maximal truncated singular integral T^* by

$$T^* f(x) = \sup_{\varepsilon>0} \left| \int_{|y|>\varepsilon} f(x-y) \frac{\Omega(y')}{|y|^n} h(|y|) dy \right| \tag{1.3}$$

where $h : (0, \infty) \to \mathbf{R}$ is a measurable function, $y' = \frac{y}{|y|}$ for any $y \neq 0$, and $f \in \mathcal{S}(\mathbf{R}^n)$.

The L^p mapping properties of the operators T and T^* were studied extensively by a number of authors (see [3], [5], [8], [13], among others). To improve previously obtained L^p boundedness results for the operators T and T^*, Jiang and Lu introduced the following special class of block spaces $B_q^{\kappa,\upsilon}(\mathbf{S}^{n-1})$:

Definition. *(1) For $x_0' \in \mathbf{S}^{n-1}$ and $0 < \theta_0 \leq 2$, the set*

$$B(x_0', \theta_0) = \{x' \in \mathbf{S}^{n-1} : |x' - x_0'| < \theta_0\}$$

is called a cap on $\mathbf{S}^{n-1}$.

(2) For $1 < q \leq \infty$, a measurable function b is called a $q-$block on $\mathbf{S}^{n-1}$ if b is a function supported on some cap $I = B(x_0', \theta_0)$ with $\|b\|_{L^q} \leq |I|^{-\frac{1}{q'}}$ where $|I| = \sigma(I)$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

(3) $B_q^{\kappa,\upsilon}(\mathbf{S}^{n-1}) = \{\Omega \in L^1(\mathbf{S}^{n-1})$: $\Omega = \sum_{\mu=1}^{\infty} c_\mu b_\mu$ where each c_μ is a complex number; each b_μ is a $q-$block supported on a cap I_μ on $\mathbf{S}^{n-1}$; and $M_q^{\kappa,\upsilon}(\{c_\mu\}) = \sum_{\mu=1}^{\infty} |c_\mu| (1 + \phi_{\kappa,\upsilon}(|I_\mu|)) < \infty\}$, where

$$\phi_{\kappa,\upsilon}(t) = \begin{cases} \int_t^1 u^{-1-\kappa} \log^{\upsilon}(u^{-1})\, du, & \text{if } 0 < t < 1; \\ 0, & \text{if } t \geq 1. \end{cases}$$

Notice that $\phi_{\kappa,\upsilon}(t) \sim t^{-\kappa} \log^{\upsilon}(t^{-1})$ as $t \to 0$ for $\kappa > 0$, $\upsilon \in \mathbf{R}$, and $\phi_{0,\upsilon}(t) \sim \log^{\upsilon+1}(t^{-1})$ as $t \to 0$ for $\upsilon > -1$.

Jiang and Lu proved the following L^2 boundedness theorem which can be found in [11].

Theorem 1. *Let Ω, T and T^* be defined as in (1.1)-(1.3). Assume that $h \in L^\infty(\mathbf{R}^+)$. Then we have*

(i) if $\Omega \in B_q^{0,0}(\mathbf{S}^{n-1})$, then T is a bounded operator on $L^2(\mathbf{R}^n)$

and

(ii) if $\Omega \in B_q^{0,1}(\mathbf{S}^{n-1})$, then T^* is a bounded operator on $L^2(\mathbf{R}^n)$.

It is clear that Theorem 1 represents an improvement in the special case $p = 2$ over the L^p boundedness theorems obtained by Duoandikoetxea-Rubio de Francia [5], Chen [3] and Namazi [13] regarding the operators T and T^* under the stronger condition $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q > 1$.

Under the same conditions on Ω and h as in Theorem 1, the question of the L^p boundedness of the operators T and T^* (for $p \neq 2$) was left open. The natural question that arises on this result is the following:

Question. *A*re the operators T and T^* bounded on L^p for all $p \neq 2$ under the same conditions on Ω and h as in Theorem 1? Also, does the L^2 (or the L^p) boundedness of T^* still hold if we replace the condition $\Omega \in B_q^{0,1}(\mathbf{S}^{n-1})$ by the weaker condition $\Omega \in B_q^{0,0}(\mathbf{S}^{n-1})$.

In this paper, we shall obtain an answer to the above question as a special case of our L^p estimates for the operators given below by (1.6)-(1.7).

Let $\Phi : (0, \infty) \to \mathbf{R}$ be a smooth function which satisfies the following growth conditions:

$$|\Phi(t)| \leq C_1 t^d, \qquad |\Phi''(t)| \leq C_2 t^{d-2}, \tag{1.4}$$

$$C_3 t^{d-1} \leq |\Phi'(t)| \leq C_4 t^{d-1} \tag{1.5}$$

for some $d \neq 0$ and $t \in (0, \infty)$, where C_1, C_2, C_3, and C_4 are positive constants independent of t.

Define the singular integral operator T_Φ by

$$T_\Phi f(x) = \text{p.v.} \int_{\mathbf{R}^n} f\left(x - \Phi\left(|y|\right) y'\right) \frac{\Omega(y')}{|y|^n} h(|y|) dy \tag{1.6}$$

and the corresponding maximal truncated singular integral T_{Φ}^{*} by

$$T_{\Phi}^{*}f\left(x\right)=\sup_{\varepsilon>0}\left|\int_{|y|>\varepsilon}f\left(x-\Phi\left(|y|\right)y'\right)\frac{\Omega(y')}{|y|^{n}}h(|y|)dy\right| \tag{1.7}$$

where $y'=\frac{y}{|y|}$ and $f\in\mathcal{S}\left(\mathbf{R}^{n}\right).$

Clearly, when $\Phi(t)=t$, we have $T_{\Phi}=T$ and $T_{\Phi}^{*}=T^{*}$.

The class of operators T_{Φ} were first defined and studied by Fan and Pan in [7] and they were able to prove the following result:

Theorem 2. *Let T_{Φ} be given by (1.6). If Ω satisfies (1.1), $\Omega\in H^{1}\left(\mathbf{S}^{n-1}\right)$(the Hardy space on the unit sphere in the sense of Coifman and Weiss [4]) and h satisfies*

$$\sup_{R>0}\left[\frac{1}{R}\int_{0}^{R}\left|h\left(t\right)\right|^{\gamma}dt\right]^{\frac{1}{\gamma}}<\infty \tag{1.8}$$

for some $\gamma>1$, then

$$\left\|T_{\Phi}\left(f\right)\right\|_{L^{p}(\mathbf{R}^{n})}\leq C_{p}\left\|\Omega\right\|_{H^{1}(\mathbf{S}^{n-1})}\left\|f\right\|_{L^{p}(\mathbf{R}^{n})}$$

for all p satisfying $\left|\frac{1}{p}-\frac{1}{2}\right|<\min\left\{\frac{1}{2},\frac{1}{\gamma'}\right\}$ and $f\in L^{p}\left(\mathbf{R}^{n}\right)$.

One observes that when $\gamma\geq 2$, the range of p in Theorem 2 is the entire interval $(1,\infty)$. Also, if we denote by $\Delta_{\gamma}(\mathbf{R}^{+})$ the set of all measurable functions on $\mathbf{R}^{+}$ satisfying (1.8), then it is easy to verify that

$$L^{\infty}(\mathbf{R}^{+})\subset\Delta_{\gamma}(\mathbf{R}^{+})\subset\Delta_{\beta}(\mathbf{R}^{+})$$

for any $\beta<\gamma$ and the inclusions are proper.

Our principal results in this paper are the following:

Theorem 3. *Let T_{Φ} be given by (1.6). Suppose that Ω satisfies (1.1), $\Omega\in B_{q}^{0,0}\left(\mathbf{S}^{n-1}\right)$ for some $q>1$ and $h\in\Delta_{\gamma}(\mathbf{R}^{+})$ for some $\gamma>1$, then*

$$\left\|T_{\Phi}\left(f\right)\right\|_{L^{p}(\mathbf{R}^{n})}\leq C_{p}\left\|f\right\|_{L^{p}(\mathbf{R}^{n})} \tag{1.9}$$

for all p satisfying $\left|\frac{1}{p}-\frac{1}{2}\right| < \min\left\{\frac{1}{2}, \frac{1}{\gamma'}\right\}$ *and* $f \in L^p(\mathbf{R}^n)$.

Theorem 4. *Let* T_Φ^* *be given by (1.7). Suppose that* $h \in L^\infty(\mathbf{R}^+)$, Ω *satisfies (1.1), and* $\Omega \in B_q^{0,0}(\mathbf{S}^{n-1})$ *for some* $q > 1$, *then*

$$\|T_\Phi^*(f)\|_{L^p(\mathbf{R}^n)} \le C_p \|f\|_{L^p(\mathbf{R}^n)} \tag{1.10}$$

for all p, $1 < p < \infty$ *and for all* $f \in L^p(\mathbf{R}^n)$.

It is worth noting that Theorem 1 is a special case of Theorems 3 and 4 if we let $\Phi(t) = t$ and $h \in L^\infty(\mathbf{R}^+)$.

We would like to thank the referee for some helpful comments.

2. Definitions and certain Fourier transforms estimates

We start by recalling some of the necessary properties of block functions $B_q^{\kappa,\upsilon}$. The space $B_q^{\kappa,\upsilon}$ enjoys the following properties which can be found in [11]:

$$\begin{aligned} B_q^{\kappa,\upsilon_2} &\subset B_q^{\kappa,\upsilon_1} \quad (\upsilon_2 > \upsilon_1 > -1 \text{ and } \kappa \ge 0), \\ B_q^{\kappa_2,\upsilon_2} &\subset B_q^{\kappa_1,\upsilon_1} \quad (\upsilon_i > -1,\ i = 1,2, \text{ and } 0 \le \kappa_1 < \kappa_2), \\ B_{q_2}^{\kappa,\upsilon} &\subset B_{q_1}^{\kappa,\upsilon} \quad (1 < q_1 < q_2), \end{aligned} \tag{2.1}$$

and

$$L^q(\mathbf{S}^{n-1}) \subseteq B_q^{\kappa,\upsilon}(\mathbf{S}^{n-1}) \quad (\text{for } \upsilon > -1, \text{ and } \kappa \ge 0).$$

In their investigations of block spaces, Keitoku and Sato showed in [9] that these spaces enjoy the following properties:

Theorem 5. (i) *If* $1 < p \le q \le \infty$, *then for* $\kappa > \frac{1}{p'}$ *we have*

$$B_q^{\kappa,\upsilon}(\mathbf{S}^{n-1}) \subseteq L^p(\mathbf{S}^{n-1}) \text{ for any } \upsilon > -1;$$

(ii)

$$B_q^{\kappa,\upsilon}(\mathbf{S}^{n-1}) = L^q(\mathbf{S}^{n-1}) \text{ if and only if } \kappa \ge \frac{1}{q'} \text{ and } \upsilon \ge 0;$$

and

(ii) for any $v > -1$, we have

$$\bigcup_{q>1} B_q^{0,v}\left(\mathbf{S}^{n-1}\right) \not\subseteq \bigcup_{p>1} L^p\left(\mathbf{S}^{n-1}\right).$$

For a suitable mapping $\Phi : (0, \infty) \to \mathbf{R}$, $\rho \in [2, \infty)$, and a suitable function $\tilde{b}(\cdot)$ on $\mathbf{S}^{n-1}$ we define the measures $\left\{\lambda_{\tilde{b},\Phi,k,\rho}\right\}_{k\in\mathbf{Z}}$ and the maximal operator $\lambda^*_{\tilde{b},\Phi,\rho}$ on $\mathbf{R}^n$ by

$$\int_{\mathbf{R}^n} f\, d\lambda_{\tilde{b},\Phi,k,\rho} = \int_{\rho^k \le |y| < \rho^{k+1}} f\left(\Phi(|y|)y'\right) \frac{\tilde{b}\left(y'\right)}{|y|^n}\, h\left(|y|\right) dy \tag{2.2}$$

and

$$\lambda^*_{\tilde{b},\Phi,\rho} f\left(x\right) = \sup_{k\in\mathbf{Z}} \left|\left|\lambda_{\tilde{b},\Phi,k,\rho}\right| * f(x)\right|. \tag{2.3}$$

Theorem 6. *Let $\Phi : (0, \infty) \to \mathbf{R}$ be a function, $\left\{\lambda_{\tilde{b},\Phi,k,\rho}\right\}_{k\in\mathbf{Z}}$ be given as in (2.2), and let $\tilde{b}(\cdot)$ be a function on $\mathbf{S}^{n-1}$ satisfying the following conditions:*
(i) $\int_{\mathbf{S}^{n-1}} \tilde{b}\left(u\right) d\sigma\left(u\right) = 0$; (ii) $\left\|\tilde{b}\right\|_{L^q(\mathbf{S}^{n-1})} \le |I|^{-\frac{1}{q'}}$ for some $q > 1$ and for some cap I on $\mathbf{S}^{n-1}$; (iii) $\left\|\tilde{b}\right\|_{L^1(\mathbf{S}^{n-1})} \le 1$.

If Φ satisfies the conditions (1.4)-(1.5) for some $d \neq 0$ and h satisfies (1.8) for some $\gamma, 1 < \gamma \le 2$, then there exist constants C, $\alpha > 0$ such that for all $k \in \mathbf{Z}$

$$\left|\hat{\lambda}_{\tilde{b},\Phi,k,\rho}(\xi)\right| \le C \log\left(|I|^{-1}\right) \left|\rho^{kd}\, \xi\right|^{\pm\frac{\alpha}{\gamma' \log|I|}} \quad \text{if} \quad \rho = 2^{\log(|I|^{-1})} \text{ and } |I| < e^{-1}, \tag{2.4}$$

whereas

$$\left|\hat{\lambda}_{\tilde{b},\Phi,k,\rho}(\xi)\right| \le C \left|\rho^{kd}\, \xi\right|^{\pm\frac{\alpha}{\gamma'}} \quad \text{if } \rho = 2 \text{ and } |I| \ge e^{-1} \tag{2.5}$$

where $t^{\pm\alpha} = \inf\{t^{\alpha}, t^{-\alpha}\}$. The constant C is independent of k, $\tilde{b}$, ξ, and $\Phi\left(\cdot\right)$.

Proof: We shall prove our estimates only for the case $d > 0$, because the proof for the case $d < 0$ is essentially the same. We shall first assume that $|I| < e^{-1}$ and

$\rho = 2^{\log(|I|^{-1})}$. By Hölder's inequality we have

$$\left|\hat{\lambda}_{\tilde{b},\Phi,k,\rho}(\xi)\right| \leq \left(\int_{\rho^k}^{\rho^{k+1}} |h(t)|^{\gamma} \frac{dt}{t}\right)^{\frac{1}{\gamma}} \left(\int_1^{\rho} |\Gamma_{k,\xi}(t)|^{\gamma'} \frac{dt}{t}\right)^{\frac{1}{\gamma'}}$$

where

$$\Gamma_{k,\xi}(t) = \int_{\mathbf{S}^{n-1}} e^{-i\Phi(\rho^k t)|\xi|(\xi'\cdot y')} \tilde{b}(y') d\sigma(y'),$$

and $\xi' = \frac{\xi}{|\xi|}$. Since

$$\int_{\rho^k}^{\rho^{k+1}} |h(t)|^{\gamma} \frac{dt}{t} \leq \sum_{s=1}^{[\log|I|^{-1}]+1} \int_{\rho^k 2^{s-1}}^{\rho^k 2^s} |h(t)|^{\gamma} \frac{dt}{t},$$

and $|\Gamma_{k,\xi}(t)| \leq 1$ we obtain

$$\left|\hat{\lambda}_{\tilde{b},\Phi,k,\rho}(\xi)\right| \leq C(\log|I|^{-1})^{\frac{1}{\gamma}} \left(\int_1^{\rho} |\Gamma_{k,\xi}(t)|^2 \frac{dt}{t}\right)^{\frac{1}{\gamma'}}$$

where $[\cdot]$ denotes the greatest integer function.

Writing $|\Gamma_{k,\xi}(t)|^2$ as

$$|\Gamma_{k,\xi}(t)|^2 = \int_{\mathbf{S}^{n-1}} \int_{\mathbf{S}^{n-1}} \tilde{b}(x') \overline{\tilde{b}(y')} e^{-i\Phi(\rho^k t)|\xi|\xi'\cdot(y'-x')} d\sigma(x') d\sigma(y')$$

and using integration by parts and the conditions (1.4)-(1.5), we obtain

$$\left|\int_1^{\rho} e^{-i\Phi(\rho^k t)|\xi|\xi'\cdot(y'-x')} \frac{dt}{t}\right| \leq C\rho \left|\rho^{kd}\xi\right|^{-1} |\xi'\cdot(y'-x')|^{-1}$$

which when combined with the trivial estimate

$$\left|\int_1^{\rho} e^{-i\Phi(\rho^k t)|\xi|\xi'\cdot(y'-x')} \frac{dt}{t}\right| \leq C\log\left(|I|^{-1}\right)$$

gives

$$\left|\int_1^\rho e^{-i\Phi(\rho^k t)|\xi|\xi'\cdot(y'-x')}\frac{dt}{t}\right| \leq C\log\left(|I|^{-1}\right)\rho^{\alpha}\left|\rho^{kd}\xi\right|^{-\alpha}|\xi'\cdot(y'-x')|^{-\alpha}$$

where $0<\alpha<1$. Thus,

$$\left|\hat{\lambda}_{\tilde{b},\Phi,k,\rho}(\xi)\right| \leq C\log\left(|I|^{-1}\right)\rho^{\alpha}\left|\rho^{kd}\xi\right|^{-\frac{\alpha}{\gamma'}}\times$$

$$\left(\int_{\mathbf{S}^{n-1}}\int_{\mathbf{S}^{n-1}}\left|\tilde{b}(x')\tilde{b}(y')\right||\xi'\cdot(y'-x')|^{-\alpha}\ d\sigma(x')\,d\sigma(y')\right)^{\frac{1}{\gamma'}}$$

$$\leq C\log\left(|I|^{-1}\right)\rho^{\alpha}\left|\rho^{kd}\xi\right|^{-\frac{\alpha}{\gamma'}}\left\|\tilde{b}\right\|_{L^q(\mathbf{S}^{n-1})}^{\frac{2}{\gamma'}}\left(\int_{\mathbf{S}^{n-1}}\int_{\mathbf{S}^{n-1}}|x_1-y_1|^{-\alpha q'}\,d\sigma(x')\,d\sigma(y')\right)^{\frac{1}{\gamma q'}}.$$

We choose α so that $0<\alpha q'<1$ and by the conditions (ii)-(iii), we get

$$\left|\hat{\lambda}_{\tilde{b},\Phi,k,\rho}(\xi)\right| \leq C\min\left\{\log\left(|I|^{-1}\right),\ \log\left(|I|^{-1}\right)\rho^{\alpha}\left|\rho^{kd}\xi\right|^{-\frac{\alpha}{\gamma'}}|I|^{-\frac{2}{q'\gamma'}}\right\}.$$

Thus $\left|\hat{\lambda}_{\tilde{b},\Phi,k,\rho}(\xi)\right|$ is majorized by $C\log\left(|I|^{-1}\right)\left|\rho^{kd}\,\xi\right|^{\frac{\alpha}{\gamma'\log|I|}}$ which yields the first estimate in (2.4) with a plus sign in the exponent.

Next, by the conditions (i) and (iii) we have

$$\begin{aligned}\left|\hat{\lambda}_{\tilde{b},\Phi,k,\rho}(\xi)\right| &\leq \int_{\mathbf{S}^{n-1}}\int_1^\rho\left|e^{-i(\Phi(\rho^k t)|\xi|\xi'\cdot y')}-1\right|\left|h(\rho^k t)\right|\tilde{b}(y')\frac{dt}{t}d\sigma(y') \\ &\leq \rho^d\left|\rho^{dk}\xi\right|\int_1^\rho\left|h(\rho^k t)\right|\frac{dt}{t} \\ &\leq C\log\left(|I|^{-1}\right)\rho^d\left|\rho^{kd}\xi\right|\end{aligned}$$

which when combined with the trivial estimate $\left|\hat{\lambda}_{\tilde{b},\Phi,k,\rho}(\xi)\right| \leq C\log\left(|I|^{-1}\right)$, yields the second desired estimate in (2.4).

Finally, the proof of (2.5) is the same as the proof of (2.4). We shall omit the details. This ends the proof of our theorem.

3. General lemmas and proofs of our results

For a given sequence $\{\mu_k : k \in \mathbf{Z}\}$ of non negative Borel measures on $\mathbf{R}^n$ we define the maximal function μ^* by

$$\mu^*(f) = \sup_{k\in\mathbf{Z}} |\mu_k * f|.$$

By following a similar argument as in the proof of Lemma 3.1 in [2] we get the following lemma which is an extension of a result of Duoandikoetxea and Rubio de Francia in [5]:

Lemma 7. *Let $\{\mu_k : k \in \mathbf{Z}\}$ be a sequence of non negative Borel measures on $\mathbf{R}^n$. Let L: $\mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation. Suppose that for all $k \in \mathbf{Z}$, $\xi \in \mathbf{R}^n$, for some $a \geq 2$, $\alpha, C > 0$ and for some constant $B > 1$ we have*

(i) $\|\mu_k\| \leq B$;
(ii) $|\hat{\mu}_k(\xi)| \leq CB(a^{kB}|L(\xi)|)^{-\frac{\alpha}{B}}$;
(iii) $|\hat{\mu}_k(\xi)-1| \leq CB(a^{kB}|L(\xi)|)^{\frac{\alpha}{B}}$.

Then the inequality

$$\|\mu^*(f)\|_p \leq B\|f\|_p \tag{3.1}$$

holds for all $1 < p \leq \infty$ and f in $L^p(\mathbf{R}^n)$. The constant C_p is independent of B.

By a quick investigation of the proof of the lemma given in ([5], page 544) we have the following:

Lemma 8. *Let $\{\lambda_k : k \in \mathbf{Z}\}$ be a sequence of Borel measures in $\mathbf{R}^n$ and let $\lambda^*(f) = \sup_{k\in\mathbf{Z}} ||\lambda_k| * f|$. Assume that*

$$\|\lambda^*(f)\|_q \leq B\|f\|_q \text{ for some } q > 1 \text{ and } B > 0. \tag{3.2}$$

Then, for arbitrary functions $\{g_k\}$ *on* $\mathbf{R}^n$ *and* $\left|\frac{1}{p_0}-\frac{1}{2}\right|=\frac{1}{2q}$, *we have*

$$\left\|\left(\sum_{k\in\mathbf{Z}}\left|\lambda_k * g_k\right|^2\right)^{\frac{1}{2}}\right\|_{p_0} \leq \sqrt{B\sup_{k\in\mathbf{Z}}\|\lambda_k\|}\left\|\left(\sum_{k\in\mathbf{Z}}|g_k|^2\right)^{\frac{1}{2}}\right\|_{p_0}. \tag{3.3}$$

We need the following result from [1] which is an extension of a result of Duoandikoetxea and Rubio de Francia in [5] (see also [6]).

Lemma 9. *Let* $\{\sigma_k : k\in\mathbf{Z}\}$ *be a sequence of Borel measures on* $\mathbf{R}^n$. *Suppose that for all* $k\in\mathbf{Z}$, $\xi\in\mathbf{R}^n$, *for some* $a\geq 2$, α, $C>0$ *and for some* $B>1$ *we have*

(i) $|\hat{\sigma}_k(\xi)|\leq CB(a^{kB}\,|L(\xi)|)^{\pm\frac{\alpha}{B}}$;

(ii) For some $p_0\in(2,\infty)$

$$\left\|\left(\sum_{k\in\mathbf{Z}}\left|\sigma_k * g_k\right|^2\right)^{\frac{1}{2}}\right\|_{p_0} \leq CB\left\|\left(\sum_{k\in\mathbf{Z}}|g_k|^2\right)^{\frac{1}{2}}\right\|_{p_0} \tag{3.4}$$

for arbitrary functions $\{g_k\}$ *on* $\mathbf{R}^n$. *Then for* $p_0' < p < p_0$ *there exists a positive constant* C_p *which is independent of* B *such that*

$$\left\|\sum_{k\in\mathbf{Z}}\sigma_k * f\right\|_{L^p(\mathbf{R}^n)} \leq C_pB\,\|f\|_{L^p(\mathbf{R}^n)} \tag{3.5}$$

holds for all f *in* $L^p(\mathbf{R}^n)$.

We are now ready to present the proofs of our main results.

Proofs of main results.

Without loss of generality, we may assume that $1<\gamma\leq 2$ and p satisfies $\left|\frac{1}{p}-\frac{1}{2}\right|<\frac{1}{\gamma'}$. By assumption, the function Ω can be written as $\Omega=\sum_{\mu=1}^{\infty}c_\mu b_\mu$ where $c_\mu\in\mathbf{C}$, b_μ is a

q-block supported on a cap I_μ on $\mathbf{S}^{n-1}$ and

$$M_q^{0,0}\left(\{c_\mu\}\right)=\sum_{\mu=1}^{\infty}|c_\mu|\left(1+\log\left|I_\mu\right|^{-1}\right)<\infty. \tag{3.6}$$

To each block function $b_\mu(\cdot)$, let $\tilde{b}_\mu(\cdot)$ be a function defined by

$$\tilde{b}_\mu(x) = b_\mu(x) - \int_{\mathbf{S}^{n-1}} b_\mu(u)d\sigma(u). \tag{3.7}$$

It is easy to verify that $\tilde{b}_\mu$ enjoys the following properties:

$$\int_{\mathbf{S}^{n-1}} \tilde{b}_\mu(u)\, d\sigma(u) = 0, \tag{3.8}$$

$$\left\|\tilde{b}_\mu\right\|_{L^q(\mathbf{S}^{n-1})} \leq 2\left|I_\mu\right|^{-\frac{1}{q'}} \tag{3.9}$$

and

$$\left\|\tilde{b}_\mu\right\|_{L^1(\mathbf{S}^{n-1})} \leq 2. \tag{3.10}$$

Using the assumption that Ω has the mean zero property (1.1), and the definition of $\tilde{b}_\mu$, we deduce that Ω can be written as

$$\Omega = \sum_{\mu=1}^{\infty} c_\mu \tilde{b}_\mu \tag{3.11}$$

which implies

$$\|T_\Phi f\|_p \leq \sum_{\mu=1}^{\infty} \left|c_\mu\right| \left\|T_{\Phi,\tilde{b}_\mu} f\right\|_p \tag{3.12}$$

where

$$T_{\Phi,\tilde{b}_\mu} f(x) = \text{p.v.} \int_{\mathbf{R}^n} f\left(x - \Phi\left(|y|\right) y'\right) \frac{\tilde{b}_\mu\left(y'\right)}{|y|^n} h\left(|y|\right) dy.$$

By Theorem 6, and Lemma 7 we immediately get

$$\left\|\lambda^*_{\tilde{b}_\mu,\Phi,\rho_\mu}(f)\right\|_{L^p(\mathbf{R}^n)} \leq C_p(\log\left|I_\mu\right|^{-1}) \|f\|_{L^p(\mathbf{R}^n)} \quad \text{if } \rho_\mu = 2^{\log(\left|I_\mu\right|^{-1})} \text{ and } \left|I_\mu\right| < e^{-1} \tag{3.13}$$

and

$$\left\|\lambda^{*}_{\tilde{b}_{\mu},\Phi,\rho_{\mu}}(f)\right\|_{L^{p}(\mathbf{R}^{n})} \leq C_{p}\left\|f\right\|_{L^{p}(\mathbf{R}^{n})} \text{ if } \rho_{\mu}=2 \text{ and } \left|I_{\mu}\right| \geq e^{-1}. \tag{3.14}$$

By following a similar argument as in the proof of Theorem 7.5 in [6], and (3.15)-(3.16), there exists a constant C_p which is independent of $\tilde{b}_{\mu}$ such that

$$\left\|\left(\sum_{k\in\mathbf{Z}}\left|\lambda_{\tilde{b}_{\mu},\Phi,k,\rho_{\mu}}*g_{k}\right|^{2}\right)^{\frac{1}{2}}\right\|_{p} \leq C_{p}(\log\left|I_{\mu}\right|^{-1})\left\|\left(\sum_{k\in\mathbf{Z}}\left|g_{k}\right|^{2}\right)^{\frac{1}{2}}\right\|_{p} \tag{3.15}$$

if $\rho_{\mu}=2^{\log(\left|I_{\mu}\right|^{-1})}$ and $\left|I_{\mu}\right|<e^{-1}$, whereas

$$\left\|\left(\sum_{k\in\mathbf{Z}}\left|\lambda_{\tilde{b}_{\mu},\Phi,k,\rho}*g_{k}\right|^{2}\right)^{\frac{1}{2}}\right\|_{p} \leq C_{p}\left\|\left(\sum_{k\in\mathbf{Z}}\left|g_{k}\right|^{2}\right)^{\frac{1}{2}}\right\|_{p} \text{ if } \rho_{\mu}=2 \text{ and } \left|I_{\mu}\right| \geq e^{-1} \tag{3.16}$$

for any p satisfying $\left|\frac{1}{p}-\frac{1}{2}\right|<\frac{1}{\gamma'}$ and for any $f\in L^{p}(\mathbf{R}^{n})$.

By Theorem 6, (3.15)-(3.16) and Lemma 9 we have

$$\left\|T_{\Phi,\tilde{b}_{\mu}}f\right\|_{p}=\left\|\sum_{k\in\mathbf{Z}}\lambda_{\tilde{b}_{\mu},\Phi,k,\rho_{\mu}}*f\right\|_{p} \leq C_{p}(\log\left|I_{\mu}\right|^{-1})\left\|f\right\|_{p} \tag{3.17}$$

if $\rho_{\mu}=2^{\log(\left|I_{\mu}\right|^{-1})}$ and $\left|I_{\mu}\right|<e^{-1}$, whereas

$$\left\|T_{\Phi,\tilde{b}_{\mu}}f\right\|_{p}=\left\|\sum_{k\in\mathbf{Z}}\lambda_{\tilde{b}_{\mu},\Phi,k,\rho_{\mu}}*f\right\|_{p} \leq C_{p}\left\|f\right\|_{p} \text{ if } \rho_{\mu}=2 \text{ and } \left|I_{\mu}\right| \geq e^{-1} \tag{3.18}$$

for every $f\in L^{p}(\mathbf{R}^{n})$ and for p satisfies $\left|\frac{1}{p}-\frac{1}{2}\right|<\frac{1}{\gamma'}$. Therefore, by (3.6), (3.12) and (3.17)-(3.18) we get (1.9) which ends the proof of Theorem 3.

Finally, Theorem 4 follows by (3.6), (3.11), (3.13)-(3.14) and by the same argument as in [1]. We omit the details.

4. Further results

Define the maximal operator

$$\mathcal{M}_{\Phi}f(x) = \sup_{r>0} \frac{1}{r^n} \int\limits_{|y|\le r} |f(x - \Phi(|y|)\, y')|\, |\Omega(y')|\, |h(|y|)|\, dy \tag{4.1}$$

where Ω and Φ are given as in Section 1 and $y' = \frac{y}{|y|} \in \mathbf{S}^{n-1}$. Also, we define the oscillatory singular integral operator S_{λ} by

$$S_{\lambda} f(x) = \text{p.v.} \int\limits_{\mathbf{R}^n} e^{i\lambda\Phi(|y|)y'} \frac{\Omega(y')}{|y|^n} h(|y|) f(x-y) dy$$

where $\lambda \in \mathbf{R}$.

Let $A = \{\mu \in \mathbf{N} : |I_{\mu}| \ge e^{-1}\}$ and $B = \{\mu \in \mathbf{N} : |I_{\mu}| < e^{-1}\}$. For $\mu \in \mathbf{N}$, we set

$$\rho_{\mu} = \begin{cases} 2 & , \text{ if } \mu \in A \\ 2^{\log(|I_{\mu}|^{-1})} & , \text{ if } \mu \in B. \end{cases}$$

Then by noticing that

$$\mathcal{M}_{\Phi}f(x) \le 4 \sum_{\mu=1}^{\infty} |c_{\mu}|\, \lambda^{*}_{\tilde{b}_{\mu},\Phi,k,2} (|f|)(x)$$

$$\le 4 \sum_{\mu \in A}^{\infty} |c_{\mu}|\, \lambda^{*}_{\tilde{b}_{\mu},\Phi,k,\rho_{\mu}} (|f|)(x) + 8 \sum_{\mu \in B}^{\infty} |c_{\mu}|\, \lambda^{*}_{\tilde{b}_{\mu},\Phi,k,\rho_{\mu}} (|f|)(x) \tag{4.2}$$

and using (3.15)-(3.16) we get the following:

Theorem 10. *Let Ω be a homogeneous function on $\mathbf{R}^n$ which satisfies (1.1), $\Omega \in B_q^{0,0}(\mathbf{S}^{n-1})$ and $h \in \Delta_{\gamma}(\mathbf{R}^{+})$ for some γ, $q > 1$. Assume that Φ satisfies the conditions (1.4)-(1.5). Then for all $\gamma' < p \le \infty$, there exists a constant C_p such that*

$$\|\mathcal{M}_{\Phi}(f)\|_{L^p(\mathbf{R}^n)} \le C_p \|f\|_{L^p(\mathbf{R}^n)} \tag{4.3}$$

for all $f \in L^p(\mathbf{R}^n)$.

As a consequence of Theorem 3 we also obtain the following uniform L^p boundedness result of the oscillatory singular integral operator S_λ. In fact, we have the following:

Theorem 11. *Let Ω be a homogeneous function on $\mathbf{R}^n$ which satisfies (1.1), $\Omega \in B_q^{0,0}\left(\mathbf{S}^{n-1}\right)$ and $h \in \Delta_\gamma(\mathbf{R}^+)$ for some γ, $q > 1$. Suppose that Φ satisfies the conditions (1.4)-(1.5). Then the operator S_λ is bounded from $L^p\left(\mathbf{R}^n\right)$ to itself for all p satisfying $\left|\frac{1}{p} - \frac{1}{2}\right| < \min\left\{\frac{1}{2}, \frac{1}{\gamma'}\right\}$. The bound for the operator norm is independent of λ.*

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