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Diagonal Lift in the Cotangent Bundle and its Applications

S. Akbulut, M. Özdemir, A. A. Salimov

Abstract

The purpose of this paper is to define a diagonal lift $\mathcal{D}g$ of a Riemannian metric g of a manifold M_n to the cotangent bundle $T^*(M_n)$ of M_n , to associate with $\mathcal{D}g$ an Levi-Civita connection of $T^*(M_n)$ in a natural way and to investigate applications of the diagonal lifts.

Key words and phrases: Cotangent bundle, Riemannian metric, diagonal lift, Levi-Civita connection, B-manifold, Killing vector field, geodesic.

1. Introduction

Let M_n be an n -dimensional differentiable manifold of class C^∞ and $T^*(M_n)$ the cotangent bundle over M_n . If x^i are local coordinates in a neighborhood U of a point $x \in M_n$, then a covector p at x which is an element of $T^*(M_n)$ is expressible in the form (x^i, p_i) , where p_i are components of P with respect to the natural frame ∂_i . We may consider $(x^i, p_i) = (x^i, x^{\bar{i}}) = (x^J)$, $i = 1, \dots, n$; $\bar{i} = n + 1, \dots, 2n$; $J = 1, \dots, 2n$ as local coordinates in a neighborhood $\pi^{-1}(U)$ (π is the natural projection $T^*(M_n)$ onto M_n).

Let now M_n be a Riemannian manifold with nondegenerate metric g whose components in a coordinate neighborhood U are g_{ij} and denote by Γ_{ji}^h the Christoffel symbols formed with g_{ji} .

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We denote by $\mathcal{T}_s^r(M_n)$ the module over $F(M_n)$ ($F(M_n)$ is the ring of C^∞ functions in M_n) all tensor fields of class C^∞ and of type (r, s) in M_n . Let $X \in \mathcal{T}_0^1(M_n)$ and $w \in \mathcal{T}_1^0(M_n)$. Then ${}^C X$ (complete lift), ${}^H X$ (horizontal lift) and ${}^V w$ (vertical lift) have, respectively, components [5]

$${}^C X = \begin{pmatrix} X^h \\ -p_m \partial_h X^m \end{pmatrix}, \quad {}^H X = \begin{pmatrix} X^h \\ p_m \Gamma_{hi}^m X^i \end{pmatrix}, \quad {}^V w = \begin{pmatrix} 0 \\ w_h \end{pmatrix} \quad (1)$$

with respect to the coordinates $(x^h, x^{\bar{h}})$ in $T^*(M_n)$, where X^h and w_h are respectively local components of X and w .

In each coordinate neighborhood $U(x^h)$ of M_n , we put

$$X_{(j)} = \frac{\partial}{\partial x^j}, \quad w^{(j)} = dx^j.$$

Taking account of (1), we easily see that the components of ${}^H X_{(j)}$ and ${}^V w^{(j)}$ are respectively given by

$${}^H X_{(j)} = (A_j^H) = \begin{pmatrix} \delta_j^h \\ p_m \Gamma_{jh}^m \end{pmatrix}, \quad {}^V w^{(j)} = (A_{\bar{j}}^H) = \begin{pmatrix} 0 \\ \delta_h^j \end{pmatrix}$$

with respect to the coordinates $(x^h, x^{\bar{h}})$. We call the set $\{{}^H X_{(j)}, {}^V w^{(j)}\}$ the frame adapted to the Riemannian connection Γ in $\pi^{-1}(U) \subset T^*(M_n)$. On putting

$$A_{(j)} = {}^H X_{(j)}, \quad A_{(\bar{j})} = {}^V w^{(j)}$$

we write the adapted frame as $\{A_{(\beta)}\} = \{A_{(j)}, A_{(\bar{j})}\}$.

It is easily verified that $2n$ local 1-forms

$$\begin{aligned} \tilde{A}^{(i)} &= \left(\tilde{A}^{(i)}_H \right) = \left(\tilde{A}^{(i)}_h, \tilde{A}^{(i)}_{\bar{h}} \right) = (\delta_h^i, 0) = dx^i \quad i = 1, \dots, n, \\ \tilde{A}^{(i)} &= \left(\tilde{A}^{(i)}_H \right) = \left(\tilde{A}^{(i)}_h, \tilde{A}^{(i)}_{\bar{h}} \right) = (-p_m \Gamma_{hi}^m \delta_{hi}) = dp_i - p_m \Gamma_{hi}^m dx^h = \delta p_i \quad \bar{i} = n + 1, \dots, 2n \end{aligned} \quad (2)$$

form a coframe $\{\tilde{A}^\alpha\} = \{\tilde{A}^{(i)}, \tilde{A}^{(\bar{i})}\}$ dual to the adapted frame $\{A_{(\beta)}\}$, i.e. $\tilde{A}^{(\alpha)}_H A_{(\beta)}^H = \delta_\beta^\alpha$.

2. Lift $\mathcal{D}g$ of a Riemannian g to $T^*(M_n)$

On putting locally

$$\mathcal{D}g = g_{ji}\tilde{A}^{(j)} \otimes \tilde{A}^{(i)} + \sum_{j,i=1}^n g^{ji}\tilde{A}^{(\bar{j})} \otimes \tilde{A}^{(\bar{i})} \tag{3}$$

in $T^*(M_n)$, we see that $\mathcal{D}g$ defines a tensor field of type $(0, 2)$ in $T^*(M_n)$, which called the diagonal lift of the tensor field g to $T^*(M_n)$ with respect to Γ . From (2) and (3) we prove that $\mathcal{D}g$ has components of the form

$$({}^{\mathcal{D}}g_{\beta\alpha}) = \left(\begin{array}{c|c} g_{ji} & 0 \\ \hline 0 & g^{ji} \end{array} \right) \tag{4}$$

with respect to the coframe $\{\tilde{A}^{(\alpha)}\}$ (or with respect to the adapted frame $\{A_{(\beta)}\}$) in $T^*(M_n)$ and components

$$\mathcal{D}g = \left(\begin{array}{c|c} g_{ji} + g^{ks}p_m p_\ell \Gamma_{jk}^m \Gamma_{is}^\ell & -g^{is}p_\ell \Gamma_{js}^\ell \\ \hline -g^{js}p_\ell \Gamma_{is}^\ell & g^{ji} \end{array} \right) \tag{5}$$

with respect to the local coordinates $(x^j, x^{\bar{j}})$, where g^{ji} denote contravariant components of g .

From (4) it easily follows that if g is a Riemannian metric in M_n , then $\mathcal{D}g$ is a Riemannian metric in $T^*(M_n)$. The metric $\mathcal{D}g$ is similar to that of the Riemannian extension studied by S. Sasaki in the tangent bundle [4] (for the frame bundle, see [2]).

From (1) and (5) we have

$$\mathcal{D}g({}^H X, {}^H Y) = g(X, Y).$$

We hence have

Theorem 1. Let $X, Y \in \mathcal{T}_0^1(M_n)$. Then the inner product of the horizontal lifts ${}^H X$ and ${}^H Y$ to $T^*(M_n)$ with the metric $\mathcal{D}g$ is equal to the vertical lift of the inner product of X and Y in M_n .

From (1) and (5) we have also

$$\mathcal{D}g({}^V w, {}^V \theta) = {}^V(g(w, \theta)), \quad \forall w, \theta \in \mathcal{T}_1^0(M_n),$$

$$\begin{aligned} \mathcal{D}g(Vw, {}^C X) &= -g^{js}w_j p_\ell (\partial_s X^\ell + \Gamma_{si}^\ell X^i) \\ &= -g^{js}w_j (\iota(\nabla X))_s \\ &= -{}^V(g(w, \iota(\nabla X))), \end{aligned}$$

$$\begin{aligned} \mathcal{D}g({}^C X, {}^C Y) &= g_{ji}x^j y^i + g^{ji}p_k p_\ell (\nabla_j X^k)(\nabla_i Y^\ell) \\ &= g_{ji}x^j y^i + g^{ji}(\iota(\nabla X))_j (\iota(\nabla Y))_i \\ &= {}^V(g(X, Y)) + {}^V(g(\iota(\nabla X), \iota(\nabla Y))), \quad \forall X, Y \in \mathcal{T}_0^1(M_n), \forall w \in \mathcal{T}_1^0(M_n), \end{aligned} \tag{6}$$

where $(\iota(\nabla X))$ is a 1-form with local expression:

$$\iota(\nabla X) = p_\ell \nabla_s X^\ell dx^s.$$

We recall that any element $t \in \mathcal{T}_r^0(T^*(M_n))$ is completely determined by its action on lifts of the type ${}^C X_1, {}^C X_2, \dots, {}^C X_r$, where $X_i, i = 1, \dots, r$ are arbitrary vector fields in M_n [5,p. 237]. Then $\mathcal{D}g$ is completely determined by (6).

3. Levi-Civita Connection of $\mathcal{D}g$

The components of the adapted frame $\{A_{(\beta)}\}$ are given by

$$(A_\beta{}^H) = (A_j{}^H, A_{\bar{j}}{}^H) = \left(\begin{array}{c|c} \delta_j^h & 0 \\ \hline p_m \Gamma_{jh}^m & \delta_h^j \end{array} \right). \tag{7}$$

The indices $\alpha, \beta, \dots = 1, \dots, 2n$ indicate the indices with respect to the adapted frame. The inverse of the matrix (7) is given by

$$(\tilde{A}^\beta{}_H) = \begin{pmatrix} \tilde{A}^j{}_H \\ \tilde{A}^{\bar{j}}{}_H \end{pmatrix}, \tag{7'}$$

$\tilde{A}^j{}_H$ and $\tilde{A}^{\bar{j}}{}_H$ being defined by

$$\tilde{A}^j{}_H = (\delta_h^j, 0), \quad \tilde{A}^{\bar{j}}{}_H = (-p_m \Gamma_{hj}^m, \delta_j^h).$$

We now consider local vector fields A_β defined in $\pi^{-1}(U)$ by

$$A_\beta = A_\beta^H \partial_H,$$

which will give for various types of indices

$$A_j = \partial_j + \sum_{h=1}^n p_m \Gamma_{jh}^m \partial_{\bar{h}}, \quad A_{\bar{j}} = \partial_{\bar{j}}. \quad (8)$$

If $\Omega_{\gamma\beta}^\alpha$ denote the non-holonomic object with respect to the vector fields A_β , then we have

$$[A_\gamma, A_\beta] = \Omega_{\gamma\beta}^\alpha A_\alpha.$$

According to (7), (7') and (8), the components of the non-holonomic object are given by

$$\Omega_{\gamma\beta}^\alpha = (A_\gamma A_\beta^C - A_\beta A_\gamma^C) A^\alpha_C.$$

The only non-vanishing components of $\Omega_{\gamma\beta}^\alpha$ are

$$\begin{aligned} \Omega_{ji}^{\bar{h}} &= p_m R_{jih}^m, \\ \Omega_{ij}^{\bar{h}} &= -p_m R_{jih}^m, \\ \Omega_j^{\bar{i}} &= -\Gamma_{jh}^i, \\ \Omega_{ij}^{\bar{h}} &= \Gamma_{jh}^i, \end{aligned} \quad (9)$$

where R_{jih}^m are components of the curvature tensor of Γ with metric g_{ij} .

Components of the Riemannian connection determined by the metric ${}^{\mathcal{D}}g$ are given by

$${}^{\mathcal{D}}\Gamma_{\gamma\beta}^\alpha = \frac{1}{2} {}^{\mathcal{D}}g^{\alpha\epsilon} (A_\gamma {}^{\mathcal{D}}g_{\epsilon\beta} + A_\beta {}^{\mathcal{D}}g_{\gamma\epsilon} - A_\epsilon {}^{\mathcal{D}}g_{\gamma\beta}) + \frac{1}{2} (\Omega_{\gamma\beta}^\alpha + \Omega^\alpha_{\gamma\beta} + \Omega^\alpha_{\beta\gamma}), \quad (10)$$

where $\Omega^\alpha_{\gamma\beta} = {}^{\mathcal{D}}g^{\alpha\epsilon} {}^{\mathcal{D}}g_{\delta\beta} \Omega_{\epsilon\gamma}^\delta$, ${}^{\mathcal{D}}g^{\alpha\epsilon}$ are the contravariant components of the metric ${}^{\mathcal{D}}g$ with respect to the adapted frame:

$$({}^{\mathcal{D}}g^{\beta\alpha}) = \left(\begin{array}{c|c} g^{ji} & 0 \\ \hline 0 & g_{ji} \end{array} \right) \quad (11)$$

Thus, according to (8), (9), (10) and (11), the components ${}^{\mathcal{D}}\Gamma_{\gamma\beta}^{\alpha}$ with respect to adapted frame are given by

$$\begin{aligned} {}^{\mathcal{D}}\Gamma_{ji}^h &= \Gamma_{ji}^h, & {}^{\mathcal{D}}\Gamma_{j\bar{i}}^h &= \frac{1}{2}p_m R_j^{him}, \\ {}^{\mathcal{D}}\Gamma_{ji}^{\bar{h}} &= \frac{1}{2}p_m R_i^{hjm}, & \Gamma_{j\bar{i}}^{\bar{h}} &= 0, \\ {}^{\mathcal{D}}\Gamma_{ji}^{\bar{h}} &= \frac{1}{2}p_m R_{jih}{}^m, & {}^{\mathcal{D}}\Gamma_{j\bar{i}}^{\bar{h}} &= -\Gamma_{jh}^i, & {}^{\mathcal{D}}\Gamma_{j\bar{i}}^{\bar{h}} &= 0, \\ \Gamma_{j\bar{i}}^{\bar{h}} &= 0, \end{aligned} \tag{12}$$

where $R_j^{him} = g^{hl}g^{ki}R_{ljk}{}^m$.

The covariant derivative of the diagonal lift ${}^{\mathcal{D}}\varphi (\varphi \in T_1^1(M_n))$ has components

$${}^{\mathcal{D}}\nabla_{\delta} {}^{\mathcal{D}}\varphi_{\beta}^{\alpha} = A_{\delta} {}^{\mathcal{D}}\varphi_{\beta}^{\alpha} + {}^{\mathcal{D}}\Gamma_{\delta\epsilon}^{\alpha} {}^{\mathcal{D}}\varphi_{\beta}^{\epsilon} - {}^{\mathcal{D}}\Gamma_{\delta\beta}^{\epsilon} {}^{\mathcal{D}}\varphi_{\epsilon}^{\alpha}$$

with respect to the adapted frame, where the components of ${}^{\mathcal{D}}\varphi$ are given by [5, p. 291]

$${}^{\mathcal{D}}\varphi_{\beta}^{\alpha} = \left(\begin{array}{c|c} \varphi_j^i & 0 \\ \hline 0 & -\varphi_i^j \end{array} \right) \tag{13}$$

with respect to the adapted frame.

Let us consider a $2n$ -dimensional Riemannian manifold M_{2n} with the almost complex structure φ . If tensor of Riemann metric g_{ij} satisfies

$$g_{mj}\varphi_i^m = g_{im}\varphi_j^m,$$

then we call this Riemann metric a pure metric in an almost complex manifold M_{2n} and call an almost \mathcal{B} -manifold an almost complex space with a pure metric.

Now, if a pure metric satisfies

$$\nabla_j F_i^h = 0 \quad \text{or} \quad \phi_k g_{ij} = 0,$$

where ϕ is the Tachibana operator, then we call this manifold a \mathcal{B} -manifold (see [1], [3]) (∇_k denotes the covariant differentiation with respect to the Christoffel symbols formed with g_{ij}). Taking account of (8) and (12), we find that ${}^{\mathcal{D}}\nabla {}^{\mathcal{D}}\varphi$ has components given

by

$$\begin{aligned} \mathcal{D}\nabla_k \mathcal{D}\varphi_j^i &= \nabla_k \varphi_j^i, \\ \mathcal{D}\nabla_k \mathcal{D}\varphi_j^{\bar{i}} &= -\nabla_k \varphi_i^j, \\ \mathcal{D}\nabla_{\bar{k}} \mathcal{D}\varphi_j^i &= \frac{1}{2}p_\ell (R_m^{i\ k\ell} \varphi_j^m - R_j^{m\ k\ell} \varphi_m^i), \\ \mathcal{D}\nabla_k \mathcal{D}\varphi_j^i &= -\frac{1}{2}p_\ell (R_k^{i\ m\ell} \varphi_m^j + R_k^{m\ j\ell} \varphi_m^i), \\ \mathcal{D}\nabla_k \mathcal{D}\varphi_j^{\bar{i}} &= \frac{1}{2}p_\ell (R_{kmi}{}^\ell \varphi_j^m + R_{kjm}{}^\ell \varphi_i^m). \end{aligned}$$

all the others being zero, with respect to the adapted frame.

From (11) and (13) we easily find that $\mathcal{D}g$ is pure with respect to the structure $\mathcal{D}\varphi$. Thus we have

Theorem 2. The cotangent bundle of \mathcal{B} -manifold is \mathcal{B} -manifold with respect to the metric $\mathcal{D}g$ and the structure $\mathcal{D}\varphi$ if and only if the Riemannian manifold is locally flat.

4. Killing vector fields

A vector field $X \in T_0^1(M_n)$ is said to be an infinitesimal isometry or a Killing vector field of a Riemannian manifold with metric g , if $\mathcal{L}_X g = 0$ [5, p78]. In terms of components g_{ji} of g , X is a Killing vector field if and only if

$$\mathcal{L}_X g_{ji} = X^\alpha \nabla_\alpha g_{ji} + g_{\alpha i} \nabla_j X^\alpha + g_{j\alpha} \nabla_i X^\alpha = \nabla_j X_i + \nabla_i X_j = 0,$$

X^α being components of X , where ∇ is the Riemannian connection of the metric g .

Let \tilde{X} be a covector field in $T^*(M_n)$ and

$$\left(\tilde{X}_\alpha\right) = \left(\tilde{X}_h, \tilde{X}_{\bar{h}}\right)$$

its components with respect to the adapted frame. Then the covariant derivative $\mathcal{D}\nabla\tilde{X}$ has components

$$\mathcal{D}\nabla_\beta \tilde{X}_\alpha = A_{\beta\alpha} \tilde{X}_\alpha + \mathcal{D}\Gamma_{\beta\alpha}^\gamma \tilde{X}_\gamma, \tag{14}$$

${}^{\mathcal{D}}\Gamma_{\beta\alpha}$ being given by (12), with respect to the adapted frame. From (1) we see that components of ${}^C X$, ${}^H X$ and ${}^V w$

$${}^C X^\alpha = A^\alpha_A {}^C X^A, \quad {}^H X^\alpha = A^\alpha_A {}^H X^A, \quad {}^V w^\alpha = A^\alpha_A {}^V w^A$$

with respect to the adapted frame ${}^C X$, ${}^H X$ and ${}^V w$ are given respectively by

$$({}^C X^\alpha) = \begin{pmatrix} X^h \\ -p_m \nabla_h X^m \end{pmatrix}, \quad ({}^H X^\alpha) = \begin{pmatrix} X^h \\ 0 \end{pmatrix}, \quad ({}^V w^\alpha) = \begin{pmatrix} 0 \\ w_h \end{pmatrix}$$

by virtue of (7').

The associated covector fields of the complete, horizontal and vertical lifts to $T^*(M_n)$ with the metric ${}^{\mathcal{D}}g$ are given respectively by

$$\begin{aligned} ({}^C X_\beta) &= ({}^{\mathcal{D}}g_{\beta\alpha} {}^C X^\alpha) = (X_j, -g^{ji} p_m \nabla_i X^m), \\ ({}^H X_\beta) &= ({}^{\mathcal{D}}g_{\beta\alpha} {}^H X^\alpha) = (X_j, 0), \\ ({}^V w_\beta) &= ({}^{\mathcal{D}}g_{\beta\alpha} {}^V w^\alpha) = (0, w^j) \end{aligned} \quad (15)$$

with respect to the adapted frame, where $X_j = g_{ji} X^i$, $w^j = g^{ji} w_i$.

We now compute the Lie derivatives of the metric ${}^{\mathcal{D}}g$ with respect to ${}^C X$, ${}^H X$ and ${}^V w$, by means of (14) and (15). The Lie derivatives of ${}^{\mathcal{D}}g$ with respect to ${}^C X$, ${}^H X$ and ${}^V w$ have respectively components

$$\begin{aligned} (\mathcal{L}_{{}^C X} {}^{\mathcal{D}}g_{\beta\alpha}) &= ({}^{\mathcal{D}}\nabla_\beta {}^C X_\alpha + {}^{\mathcal{D}}\nabla_\alpha {}^C X_\beta) \\ &= \left(\begin{array}{c|c} \nabla_j X_i + \nabla_i X_j & -p_m g^{ki} g^{tm} (\nabla_j \nabla_k X_t + R_{\ell j k t} X^\ell) \\ \hline -p_m g^{kj} g^{tm} (\nabla_i \nabla_k X_t + R_{\ell i k t} X^\ell) & - (g^{is} \nabla_s X^j + g^{js} \nabla_s X^i) \end{array} \right), \end{aligned} \quad (16)$$

$$(\mathcal{L}_{{}^H X} {}^{\mathcal{D}}g_{\beta\alpha}) = ({}^{\mathcal{D}}\nabla_\beta {}^H X_\alpha + {}^{\mathcal{D}}\nabla_\alpha {}^H X_\beta) = \left(\begin{array}{c|c} \nabla_j X_i + \nabla_i X_j & -p_m g^{ki} R_{\ell j k}{}^m X^\ell \\ \hline -p_m g^{kj} R_{\ell i k}{}^m X^\ell & 0 \end{array} \right),$$

$$(\mathcal{L}_{{}^V w} {}^{\mathcal{D}}g_{\beta\alpha}) = ({}^{\mathcal{D}}\nabla_\beta {}^V w_\alpha + {}^{\mathcal{D}}\nabla_\alpha {}^V w_\beta) = \left(\begin{array}{c|c} 0 & g^{is} \nabla_j w_s \\ \hline g^{js} \nabla_i w_s & 0 \end{array} \right)$$

with respect to the adapted frame in $T^*(M_n)$.

Since we have

$$\nabla_i \nabla_k X_t + R_{\ell i k t} X^\ell = 0, \quad \mathcal{L}_X g^{ji} = -(g^{js} \nabla_s X^i + g^{is} \nabla_s X^j) = 0$$

as a consequence of $\mathcal{L}_X g_{ji} = \nabla_j X_i + \nabla_i X_j = 0$ (see [6, p.17]), we conclude by means of (16) that the complete lift ${}^C X$ is a Killing vector field in $T^*(M_n)$ if and only if X is a Killing vector field in M_n .

We next have

$$R_{\ell i k}{}^m X^\ell = 0$$

as a consequence of the vanishing of the second covariant derivative of X . Conversely, the conditions $\mathcal{L}_X g_{ji} = \nabla_j X_i + \nabla_i X_j = 0$ and $R_{\ell i k}{}^m X^\ell = 0$ imply that the second covariant derivative of X vanishes. Summing up these results, we have

Theorem 3. Necessary and sufficient conditions in order that the

- a) complete ${}^C X \in \mathcal{T}_0^1(T^*(M_n))$,
- b) horizontal ${}^H X \in \mathcal{T}_0^1(T^*(M_n))$ and
- c) vertical ${}^V w \in \mathcal{T}_0^1(T^*(M_n))$

lifts to $T^*(M_n)$ with the metric $\mathcal{D}g$, of a vector field X and covector field w in M_n be a Killing vector field in $T^*(M_n)$ are that,

- a) X is a Killing vector field in M_n ,
- b) X is a Killing vector field with vanishing second covariant derivative in M_n and
- c) w is parallel in M_n .

5. Geodesics in $T^*(M_n)$ with metric $\mathcal{D}g$

Let C be a curve in M_n expressed locally by $x^h = x^h(t)$ and $w_h(t)$ be a covector field along C . Then, in the cotangent bundle $T^*(M_n)$, we define a curve \tilde{C} by

$$x^h = x^h(t), \quad x^{\tilde{h}} \stackrel{\text{def}}{=} p_h = w_h(t) \tag{17}$$

If the curve C satisfies at all the points the relation

$$\frac{\delta w_n}{dt} = \frac{dw_n}{dt} - \Gamma_{j h}^i \frac{dx^j}{dt} w_i = 0, \tag{18}$$

then the curve \tilde{C} is said to be a horizontal lift of the curve C in M_n . Thus, if the initial condition $w_h = w_h^0$ for $t = t_0$ is given, there exists a unique horizontal lift expressed by (17).

We now consider differential equations of the geodesics of the cotangent bundle $T^*(M_n)$ with the metric $\mathcal{D}g$. If t is the arc length of a curve $x^A = x^A(t)$ in $T^*(M_n)$, equations of geodesic in $T^*(M_n)$ have the usual form

$$\frac{\delta^2 x^A}{dt^2} = \frac{d^2 x^A}{dt^2} + \mathcal{D} \Gamma_C^A{}^B \frac{dx^C}{dt} \frac{dx^B}{dt} = 0 \quad (19)$$

with respect to the induced coordinates $(x^i, x^{\bar{i}}) = (x^i, p_i)$ in $T^*(M_n)$.

We find it more convenient to refer equations (19) to the adapted frame $\{A_{(i)}, A_{(\bar{i})}\}$. Using (2), we now write

$$\theta^h = A^{(h)}{}_A dx^A = dx^h,$$

$$\theta^{\bar{h}} = A^{(\bar{h})}{}_A dx^A = \delta p_h,$$

and put

$$\frac{\theta^h}{dt} = A^{(h)}{}_A \frac{dx^A}{dt} = \frac{dx^h}{dt},$$

$$\frac{\theta^{\bar{h}}}{dt} = A^{(\bar{h})}{}_A \frac{dx^A}{dt} = \frac{\delta p_h}{dt} = \frac{\delta p_h}{dt} - \Gamma_{j\ h}^i \frac{dx^j}{dt} p_i$$

along a curve $x^A = x^A(t)$, i.e., $x^h = x^h(t)$, $p_h = p_h(t)$ in $T^*(M_n)$.

If we therefore write down the form equivalent to (19), namely,

$$\frac{d}{dt} \left(\frac{\theta^\alpha}{dt} \right) + \mathcal{D} \Gamma_\delta{}^\alpha{}_\beta \left(\frac{\theta^\gamma}{dt} \right) \left(\frac{\theta^\beta}{dt} \right) = 0$$

with respect to the adapted frame and take account of (12), then we have

$$\begin{cases} \frac{\delta^2 x^h}{dt^2} + p_m R^h{}_{j\ im} \frac{dx^j}{dt} \frac{\delta p_i}{dt} = 0, \\ \frac{\delta^2 p_h}{dt^2} + \frac{1}{2} p_m R_{j\ ih}{}^m \frac{dx^j}{dt} \frac{dx^i}{dt} = 0 \end{cases} \quad (20)$$

Since we have

$$R_{j i h}{}^m \frac{dx^j}{dt} \frac{dx^i}{dt} = 0$$

as a consequence of $R_{(j i) h}{}^m = 0$, we conclude by means of (20) that a curve $x^i = x^i(t)$, $p_h = p_h(t)$ in $T^*(M_n)$ with the metric $\mathcal{D}g$ is a geodesic in $T^*(M_n)$, if and only if

$$\begin{cases} \frac{\delta^2 x^h}{dt^2} + p_m R_{j i h}{}^m \frac{dx^j}{dt} \frac{\delta p_i}{dt} = 0, & \text{(a)} \\ \frac{\delta^2 p_h}{dt^2} = 0 & \text{(b)} \end{cases} \quad (21)$$

If a curve satisfying (21) lies on a fibre given by $x^h = \text{const}$, then (20, (b)) reduces to

$$\frac{d^2 p_h}{dt^2} = 0$$

so that $p_h = a_h t + b_h$, a_h and b_h being constant. Thus we have

Theorem 4. If geodesic $x^h = x^h(t)$, $p_h = p_h(t)$ lies in a fibre of $T^*(M_n)$ with the metric $\mathcal{D}g$, the a geodesic is expressed by linear equations $x^h = c^h$, $p_h = a_h t + b_h$, where a_h , b_h and c^h are constant.

From (18) and (21), we have

Theorem 5. The horizontal lift of a geodesic in M_n is always geodesic in $T^*(M_n)$ with the metric $\mathcal{D}g$.

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