Low-dimensional homology groups of mapping class groups: a survey

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Abstract

In this survey paper, we give a complete list of known results on the first and the second homology groups of surface mapping class groups.
Some known results on higher (co)homology are also mentioned.

1. Introduction

Let $\Sigma_{g,r}^n$ be a connected orientable surface of genus g with r boundary components and n punctures. The mapping class group of $\Sigma_{g,r}^n$ may be defined in different ways. For our purpose, it is defined as the group of the isotopy classes of orientation-preserving diffeomorphisms $\Sigma_{g,r}^n \to \Sigma_{g,r}^n$. The diffeomorphisms and the isotopies are assumed to fix each puncture and the points on the boundary. We denote the mapping class group of $\Sigma_{g,r}^n$ by $\Gamma_{g,r}^n$. Here, we see the punctures on the surface as distinguished points. If r and/or n is zero, then we omit it from the notation. We write Σ for the surface $\Sigma_{g,r}^n$ when we do not want to emphasize g, r, n.

The theory of mapping class groups plays a central role in low-dimensional topology. When $r = 0$ and $2g + n \geq 3$, the mapping class group Γ_g^n acts properly discontinuously on the Teichmüller space which is homeomorphic to some Euclidean space and the stabilizer of each point is finite. The quotient of the Teichmüller space by the action of the mapping class group is the moduli space of complex curves.

Recent developments in low-dimensional topology made the algebraic structure of the mapping class group more important. The examples of such developments are the theory of Lefschetz fibrations and the Stein fillability of contact 3-manifolds. Questions about the structure of these can be stated purely as an algebraic problem in the mapping class group, but in this paper we do not address such problems.

The purpose of this survey paper is to give a list of complete known results on the homology groups of the mapping class groups in dimensions one and two. There is no new result in the paper, but there are some new proofs. For example, although the first homology group of the mapping class group in genus one case is known, it seems that it does not appear in the literature. Another example is that we give another proof of the fact that Dehn twists about nonseparating simple closed curves are not enough to generate the mapping class group of a surface of genus one with $r \geq 2$ boundary components (cf. Corollary 5.2 below), as opposed to the higher genus case.

We shall mainly be interested in orientable surfaces. The first and the second homology groups of the mapping class group have been known for more than twenty years. We will give the complete list of the first homologies and we calculate them. An elementary proof of the second homology of the mapping class group was recently given by the author and Stipsicz in [30]. This proof is based on the presentation of the mapping class group and is sketched in Section 6. We then outline some known results for higher dimensional (co)homology. Finally, in the last section, we give the first homology groups of the mapping class groups of nonorientable surfaces.

2. Dehn twists and relations among them

Let Σ be an oriented surface and let a be a simple closed curve on it. We always assume that the curves are unoriented. Cutting the surface Σ along a, twisting one of the sides by 360 degrees to the right and gluing back gives a self-diffeomorphism of the surface Σ (cf. Fig. 1 (a)). Let us denote this diffeomorphism by t_a. In general, a diffeomorphism and its isotopy class will be denoted by the same letter, so that t_a also represents an element of $\Gamma_{g,r}^n$. Accordingly, a simple closed curve and its isotopy class are denoted by the same letter. It can easily be seen that the mapping class t_a depends only on the isotopy class of a. The mapping class t_a is called the (right) Dehn twist about a.

From the definition of a Dehn twist it is easy to see that if $f : \Sigma \to \Sigma$ is a diffeomorphism and a a simple closed curve on Σ, then there is the equality
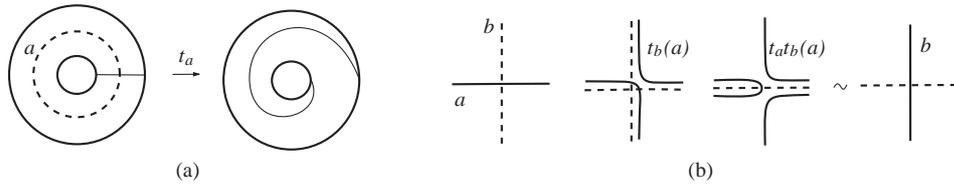
$$f t_a f^{-1} = t_{f(a)}. \tag{1}$$

We note that we use the functional notation for the composition of functions, so that $(fg)(x) = f(g(x))$.

2.1. The braid relations

Suppose that a and b are two disjoint simple closed curves on a surface Σ. Since the support of the Dehn twist t_a can be chosen to be disjoint from b, we have $t_a(b) = b$. Thus by (1), we get

$$t_a t_b = t_a t_b t_a^{-1} t_a = t_{t_a(b)} t_a = t_b t_a. \tag{2}$$

(a)

(b)

Figure 1. The Dehn twist t_a, and the proof of $t_a t_b(a) = b$.

Suppose that two simple closed curves a and b intersect transversely at only one point. It can easily be shown that $t_a t_b(a) = b$ (cf. Fig. 1 (b)). Hence,

$$t_a t_b t_a = t_a t_b t_a t_b^{-1} t_a^{-1} t_a t_b = t_{t_a t_b(a)} t_a t_b = t_b t_a t_b. \tag{3}$$

2.2. The two-holed torus relation

Suppose that a, b, c are three nonseparating simple closed curves on a surface Σ such that a is disjoint from c, and b intersects a and c transversely at one point (cf. Fig. 2 (a)). A regular neighborhood of $a \cup b \cup c$ is a torus with two boundary components, say d and e. Then the Dehn twists about these simple closed curves satisfy the relation

$$(t_a t_b t_c)^4 = t_d t_e. \tag{4}$$

For a proof of this, see [25], Lemma 2.8. We call it the two-holed torus relation. In fact, it follows from the braid relations that the three Dehn twists t_a, t_b, t_c on the left hand side of this relation can be taken in any order.

(a)   (b)
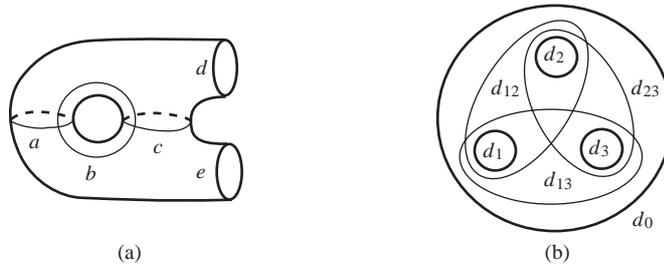
Figure 2. The circles of the two-holed torus and the lantern relations.

2.3. The lantern relation

Consider a sphere X with four holes embedded in a surface Σ in such a way that the boundary circles of X are the simple closed curves d_0, d_1, d_2 and d_3 on Σ (cf. Fig. 2 (b)). There are three circles d_{12}, d_{13} and d_{23} on X such that there is the relation

$$t_{d_0} t_{d_1} t_{d_2} t_{d_3} = t_{d_{12}} t_{d_{13}} t_{d_{23}}. \tag{5}$$

This relation is called the lantern relation. It was first discovered by Dehn [10] and rediscovered and made popular by Johnson [24].

The two-holed torus and the lantern relations can be proved easily: Choose a set of arcs dividing the supporting subsurface into a disc and show that the actions on this set of arcs of the diffeomorphisms on the two sides are equal up to homotopy.

3. Generating the mapping class group

The search of the algebraic structures of the mapping class group was initiated by the work of Dehn [10]. He proved that that the mapping class group of a closed orientable surface is generated by finitely many (Dehn) twists about nonseparating simple closed curves.

In [31, 33], Lickorish reproduced this result; he proved that the mapping class group Γ_g can be generated by $3g - 1$ Dehn twists, all of which are about nonseparating simple closed curves. In [19], Humphries reduced this number to $2g + 1$: The mapping class group Γ_g of a closed orientable surface Σ_g of genus g is generated by Dehn twists about $2g + 1$ simple closed curves $a_0, a_1, \ldots, a_{2g}$ of Fig. 3. In the figure, we glue a disc to the boundary component of the surface to get the closed surface Σ_g. Humphries also showed that the number $2g + 1$ is minimal; the mapping class group of a closed orientable surface of genus $g \geq 2$ cannot be generated by $2g$ (or less) Dehn twists. For any generating set the situation is different of course; Γ_g is generated by two elements. This result is due to Wajnryb [45]. This is the least number of generators, because Γ_g is not commutative.
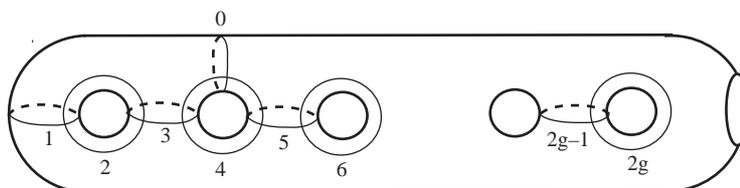
Figure 3. The label n represents the circle a_n.

Let Σ_g^n be an orientable surface of genus g with n punctures. Let $n \geq 1$ and let us fix a puncture x. By forgetting the puncture x, every diffeomorphism $\Sigma_g^n \to \Sigma_g^n$ induces a diffeomorphism $\Sigma_g^{n-1} \to \Sigma_g^{n-1}$. This gives an epimorphism $\Gamma_g^n \to \Gamma_g^{n-1}$ whose kernel is isomorphic to the fundamental group of Σ_g^{n-1} at the base point x (cf. [2]). Therefore, we have a short exact sequence

$$1 \to \pi_1(\Sigma_g^{n-1}) \to \Gamma_g^n \to \Gamma_g^{n-1} \to 1. \tag{6}$$

Now let $\Sigma_{g,r}^n$ be an orientable surface of genus g with r boundary components and n punctures. Assume that $r \geq 1$. Let P be one of the boundary components. By gluing a disc D with one puncture along P, we get a surface $\Sigma_{g,r-1}^{n+1}$ of genus g with $r - 1$ boundary components and $n + 1$ punctures. A diffeomorphism $\Sigma_{g,r}^n \to \Sigma_{g,r}^n$ extends to a diffeomorphism $\Sigma_{g,r-1}^{n+1} \to \Sigma_{g,r-1}^{n+1}$ by defining the extension to be the identity on D. This way we get an epimorphism $\Gamma_{g,r}^n \to \Gamma_{g,r-1}^{n+1}$. Note that a Dehn twist on $\Sigma_{g,r}^n$ along a simple closed curve parallel to P gives a diffeomorphism $\Sigma_{g,r-1}^{n+1} \to \Sigma_{g,r-1}^{n+1}$ isotopic to the identity.

Essentially, this is the only vanishing mapping class under the map $\Gamma_{g,r}^n \to \Gamma_{g,r-1}^{n+1}$. More precisely, we have the short exact sequence

$$1 \to \mathbb{Z} \to \Gamma_{g,r}^n \to \Gamma_{g,r-1}^{n+1} \to 1, \tag{7}$$

where $\mathbb{Z}$ is the subgroup of $\Gamma_{g,r}^n$ generated by the Dehn twist along a simple closed curve parallel to P.

It follows from the description of the homomorphisms in the short exact sequences (6) and (7), and the fact that the mapping class group Γ_g is generated by Dehn twists about finitely many nonseparating simple closed curves, the group $\Gamma_{g,r}^n$ is generated by Dehn twists along finitely many nonseparating simple closed curves and the Dehn twist along a simple closed curve parallel to each boundary component.

Suppose that the genus of the surface $\Sigma_{g,r}^n$ is at least 2. The four-holed sphere X of the lantern relation can be embedded in the surface $\Sigma_{g,r}^n$ in such a way that one of the boundary components of X is a given boundary component of $\Sigma_{g,r}^n$ and all other six curves of the lantern relation are nonseparating on $\Sigma_{g,r}^n$. We conclude from this that

Theorem 3.1. *If $g \geq 2$ then the mapping class group $\Gamma_{g,r}^n$ is generated by Dehn twists about finitely many nonseparating simple closed curves.*

We note that this theorem does not hold for $g = 1$ and $r \geq 2$. See Corollary 5.2 in Section 5 below. Dehn twists about boundary parallel simple closed curves are needed in this case. In the case that $g = 0$, there is no nonseparating simple closed curve.

4. Presenting the mapping class groups Γ_g and $\Gamma_{g,1}$

The mapping class groups are finitely presented. The presentation of Γ_2 was first obtained by Birman and Hilden [5]. For $g \geq 3$, this fact was first proved by McCool [35] using combinatorial group theory without giving an actual presentation. A geometric proof of this was given by Hatcher and Thurston [18], again without an explicit presentation. Their proof used the connectedness and the simple connectedness of a certain complex formed by so called cut systems. Harer [14] modified the Hatcher-Thurston complex of cut systems in order to calculate the second homology groups of mapping class groups of orientable surfaces of genus $g \geq 5$. Using this modified complex, simple presentations of the mapping class groups $\Gamma_{g,1}$ and Γ_g were finally obtained by Wajnryb [44]. Minor errors in [44] were corrected in [6]. The proof of Hatcher and Thurston is very complicated. In [46], Wajnryb gave an elementary proof of the presentations of $\Gamma_{g,1}$ and Γ_g. This proof does not use the results of Hatcher-Thurston and Harer. It turns out that all the relations needed to present the mapping class groups are those given in Section 2, which were obtained by Dehn [10].

We now give the Wajnryb presentations of $\Gamma_{g,1}$ and Γ_g. So suppose that $n = 0$ and $r \leq 1$. As a model for $\Sigma_{g,r}$, consider the surface in Fig. 3. On the surface $\Sigma_{g,r}$, consider the simple closed curves $a_0, a_1, \ldots, a_{2g}$ illustrated in Fig. 3.

Let F be the nonabelian free group freely generated by $x_0, x_1, \ldots, x_{2g}$. For $x, y \in F$, let $[x, y]$ denote the commutator $xyx^{-1}y^{-1}$. In the group F, we define some words as

follows. Let
$$A_{ij} = [x_i, x_j]$$
if the curve a_i is disjoint from the curve a_j in Fig. 3, and let
$$B_0 = x_0 x_4 x_0 x_4^{-1} x_0^{-1} x_4^{-1},$$
$$B_i = x_i x_{i+1} x_i x_{i+1}^{-1} x_i^{-1} x_{i+1}^{-1}$$
for $i = 1, 2, \ldots, 2g - 1$. Let us also define the words
$$C = (x_1 x_2 x_3)^4 x_0^{-1} (x_4 x_3 x_2 x_1^2 x_2 x_3 x_4) x_0^{-1} (x_4 x_3 x_2 x_1^2 x_2 x_3 x_4)^{-1}$$
and
$$D = x_1 x_3 x_5 w x_0 w^{-1} x_0^{-1} t_2^{-1} x_0^{-1} t_2 (t_2 t_1)^{-1} x_0^{-1} (t_2 t_1),$$
where
$$t_1 = x_2 x_1 x_3 x_2, \ t_2 = x_4 x_3 x_5 x_4,$$
and
$$w = x_6 x_5 x_4 x_3 x_2 (t_2 x_6 x_5)^{-1} x_0 (t_2 x_6 x_5)(x_4 x_3 x_2 x_1)^{-1}.$$
In the group F, we define one more element E to be
$$E = [x_{2g+1}, x_{2g} x_{2g-1} \cdots x_3 x_2 x_1^2 x_2 x_3 \cdots x_{2g-1} x_{2g}],$$
where
$$
\begin{aligned}
x_{2g+1} &= (u_{g-1} u_{g-2} \cdots u_1) x_1 (u_{g-1} u_{g-2} \cdots u_1)^{-1}, \\
u_1 &= (x_1 x_2 x_3 x_4)^{-1} v_1 x_4 x_3 x_2, \\
u_i &= (x_{2i-1} x_{2i} x_{2i+1} x_{2i+2})^{-1} v_i x_{2i+2} x_{2i+1} x_{2i} \text{ for } i = 2, \ldots, g - 1, \\
v_1 &= (x_4 x_3 x_2 x_1^2 x_2 x_3 x_4) x_0 (x_4 x_3 x_2 x_1^2 x_2 x_3 x_4)^{-1}, \\
v_i &= (w_i w_{i-1})^{-1} v_{i-1} (w_i w_{i-1}) \text{ for } i = 2, \ldots, g - 1, \\
w_i &= x_{2i} x_{2i+1} x_{2i-1} x_{2i} \text{ for } i = 1, 2, \ldots, g - 1.
\end{aligned}
$$

We would like to note that if we define a homomorphism from F to Γ_g or $\Gamma_{g,1}$ by $x_i \mapsto t_{a_i}$, then the relation C maps to a two-holed torus relation and D maps to a lantern relation such that all seven simple closed curves in the relation are nonseparating. A_{ij} and B_i map to the braid relations.

Let us denote by R_1 the normal subgroup of F normally generated by the elements $A_{ij}, B_0, B_1, \ldots, B_{2g-1}, C$ and D, and let R_0 denote the normal subgroup of F normally generated by R_1 and E. The Wajnryb presentation of the mapping class groups Γ_g and $\Gamma_{g,1}$ can be summarized as the next theorem.

Theorem 4.1 ([46], Theorems 1′ and 2). *Let $g \geq 2$. Then there are two short exact sequences*

$$1 \longrightarrow R_1 \longrightarrow F \xrightarrow{\phi_1} \Gamma_{g,1} \longrightarrow 1 \tag{8}$$

and

$$1 \longrightarrow R_0 \longrightarrow F \xrightarrow{\phi_0} \Gamma_g \longrightarrow 1, \tag{9}$$

where $\phi_i(x_j)$ is the Dehn twist t_{a_j} about the curve a_j.

Remark 4.1. If $g = 2$ then the relation D is not supported in the surface. In this case, one should omit the element D from the definition of R_0 and R_1.

Notice that the presentation of $\Gamma_{2,1}$ in Theorem 2 in [46] is slightly different but equivalent to the presentation above.

A finite presentation of the mapping class group $\Gamma_{g,r}$ is obtained by Gervais in [13].
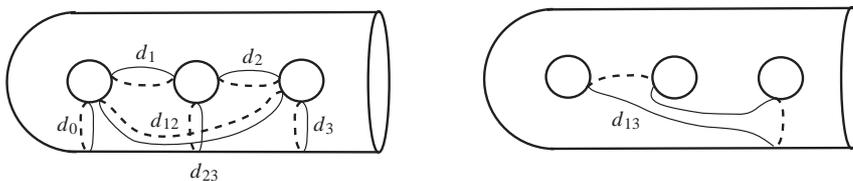
5. The first homology

Recall that for a discrete group G, the first homology group $H_1(G;\mathbb{Z})$ of G with integral coefficients is isomorphic to the derived quotient $G/[G,G]$, where $[G,G]$ is the subgroup of G generated by all commutators $[x,y]$ for $x,y \in G$. Here, $[x,y] = xyx^{-1}y^{-1}$.

From the presentation of the mapping class group Γ_g given in Theorem 4.1, the group $H_1(\Gamma_g;\mathbb{Z})$ can be computed easily; it is isomorphic to $\mathbb{Z}_{10}$ if $g = 2$ and 0 if $g \geq 3$. The fact that $H_1(\Gamma_2;\mathbb{Z})$ is isomorphic to $\mathbb{Z}_{10}$ was first proved by Mumford [40] and that $H_1(\Gamma_g;\mathbb{Z}) = 0$ for $g \geq 3$ by Powell [42]. We prove this result for $g \geq 3$ without appealing to the presentation. We also determine the first homology groups for arbitrary r and n, which is well known.

If a and b are two nonseparating simple closed curves on a surface $\Sigma_{g,r}^n$, then by the classification of surfaces there is a diffeomorphism $f : \Sigma_{g,r}^n \rightarrow \Sigma_{g,r}^n$ such that $f(a) = b$. Thus, by (1), we have that $t_b = ft_af^{-1}$. This can also be written as $t_b = [f,t_a]t_a$. Therefore, t_a and t_b represent the same class τ in $H_1(\Gamma_{g,r}^n;\mathbb{Z})$. Since the mapping class group $\Gamma_{g,r}^n$ is generated by Dehn twists about nonseparating simple closed curves for $g \geq 2$, it follows that the group $H_1(\Gamma_{g,r}^n;\mathbb{Z})$ is cyclic and is generated by τ.

Suppose that $g \geq 3$. The four-holed sphere X of the lantern relation can be embedded in $\Sigma_{g,r}^n$ such that all seven curves involved in the lantern relation become nonseparating on $\Sigma_{g,r}^n$ (cf. Fig. 4). This gives us the relation $4\tau = 3\tau$ in $H_1(\Gamma_{g,r}^n;\mathbb{Z})$. Hence, $H_1(\Gamma_{g,r}^n;\mathbb{Z})$ is trivial.

Suppose now that $g = 2$. The two-holed torus of the two-holed torus relation can be embedded in $\Sigma_{2,r}^n$ so that all five curves in the relation becomes nonseparating on $\Sigma_{2,r}^n$. This gives us $12\tau = 2\tau$, i.e. $10\tau = 0$. On the other hand, since there is an epimorphism $\Gamma_{2,r}^n \rightarrow \Gamma_2$, it follows that $H_1(\Gamma_{2,r}^n) = \mathbb{Z}_{10}$.

Figure 4. An embedding of the lantern with all curves nonseparating.

Although we usually deal with the surfaces of genus at least two, we would like to mention the first homology groups in the genus one case as well.

Consider a torus $\Sigma_{1,r}^n$ with n punctures and r boundary components, $P_1, P_2, \ldots, P_r$. For each $i = 1, 2, \ldots, r$, let ∂_i be a simple closed curve parallel to P_i. The mapping class group Γ_1 is generated by the Dehn twists about two (automatically nonseparating) simple closed curves intersecting transversely at one point. It can be proved by the exact sequences (6) and (7) that the mapping class group $\Sigma_{1,r}^n$ is generated by Dehn twists about finitely many nonseparating simple closed curves and r Dehn twists about $\partial_1, \partial_2, \ldots, \partial_r$. By the use of the lantern relation, it can be shown that we may omit any one of ∂_i, say ∂_n. It will follow from the following computation of the first homology of $\Gamma_{1,r}^n$ that in fact no more ∂_i may be omitted.

By the argument in the case of higher genus, any two Dehn twists about nonseparating simple closed curves are conjugate. Hence, they represent the same class τ in $H_1(\Gamma_{1,r}^n; \mathbb{Z})$.

Assume first that $r = 0$. The group Γ_1 is isomorphic to $SL(2, \mathbb{Z})$. Hence, $H_1(\Gamma_1; \mathbb{Z})$ is isomorphic to $\mathbb{Z}_{12}$ and is generated by τ. Since Γ_1^n is generated by Dehn twists about nonseparating simple closed curves, the homology group $H_1(\Gamma_1^n; \mathbb{Z})$ is cyclic and generated by τ. It was shown in Theorem 3.4 in [28] that $12\tau = 0$. On the other hand, the surjective homomorphism $\Gamma_1^n \to \Gamma_1$ obtained by forgetting the punctures induces a surjective homomorphism between the first homology groups, mapping τ to the generator of $H_1(\Gamma_1; \mathbb{Z})$. It follows that $H_1(\Gamma_1^n; \mathbb{Z})$ is isomorphic to $\mathbb{Z}_{12}$.

The group Γ_1^1 is also isomorphic to $SL(2, \mathbb{Z})$. Let a and b be two simple closed curve on $\Sigma_{1,1}$ intersecting each other transversely at one point. By examining the short exact sequence

$$1 \to \mathbb{Z} \to \Gamma_{1,1} \to \Gamma_1^1 \to 1$$

it can be shown easily that $\Gamma_{1,1}$ has a presentation with generator t_a, t_b and with a unique relation $t_a t_b t_a = t_b t_a t_b$. That is, $\Gamma_{1,1}$ is isomorphic to the braid group on three strings. Hence, by abelianizing this presentation, we see that $H_1(\Gamma_{1,1}; \mathbb{Z})$ is isomorphic to $\mathbb{Z}$.

Assume now that $r \geq 1$. Since $\Gamma_{1,r}^n$ is generated by Dehn twists about nonseparating simple closed curves and the curves $\partial_1, \partial_2, \ldots, \partial_{r-1}$, the group $H_1(\Gamma_{1,r}^n; \mathbb{Z})$ is generated by $\tau, \delta_1, \delta_2, \ldots, \delta_{r-1}$, where δ_i is the class in $H_1(\Gamma_{1,r}^n; \mathbb{Z})$ of the Dehn twist about ∂_i.

We prove that $\tau, \delta_1, \delta_2, \ldots, \delta_{r-1}$ are linearly independent. Let $n_0\tau + n_1\delta_1 + n_2\delta_2 + \ldots + n_{r-1}\delta_{r-1} = 0$ with $n_i \in \mathbb{Z}$. Gluing a disc to each P_i for $i \leq r - 1$ and forgetting the punctures gives rise to an epimorphism $H_1(\Gamma_{1,r}^n; \mathbb{Z}) \to H_1(\Gamma_{1,1}; \mathbb{Z})$. Under this map, τ is mapped to the generator of $H_1(\Gamma_{1,1}; \mathbb{Z})$ and all δ_i to zero. Hence, $n_0 = 0$. Similarly, gluing a disc to each boundary component but P_i and forgetting the punctures induces an epimorphism $H_1(\Gamma_{1,r}^n; \mathbb{Z}) \to H_1(\Gamma_{1,1}; \mathbb{Z})$ mapping δ_i to 12τ and each δ_j, $j \neq i$, to 0, where τ is the generator of $H_1(\Gamma_{1,1}; \mathbb{Z})$. This shows that $n_i = 0$ for each $i = 1, 2, \ldots, r - 1$. We conclude that $H_1(\Gamma_{1,r}^n; \mathbb{Z})$ is isomorphic to $\mathbb{Z}^r$.

We collect the results of this section in the next theorem.

Theorem 5.1. *Let $g \geq 1$. The first homology group $H_1(\Gamma_{g,r}^n; \mathbb{Z})$ of the mapping class group is isomorphic to $\mathbb{Z}_{12}$ if $(g,r) = (1,0)$, $\mathbb{Z}^r$ if $g = 1, r \geq 1$, $\mathbb{Z}_{10}$ if $g = 2$ and 0 if $g \geq 3$.*

Corollary 5.2. *Let $r \geq 2$. The mapping class group $\Gamma_{1,r}^n$ cannot be generated by Dehn twists about nonseparating simple closed curves.*

We note that this corollary was proved by Gervais in [12] by a different argument.

6. The second homology

The second homology group of the mapping class group $\Gamma_{g,r}^n$ for $g \geq 5$ was first computed by Harer in [14]. His proof relies on the simple connectedness of a complex obtained by modifying the Hatcher-Thurston complex [18]. But this proof is extremely complicated to understand. The computation of $H_2(\Gamma_g; \mathbb{Z})$ in [14] was incorrect. It was corrected later. See, for example, [15] or [38].

In [41], Pitsch gave a simple proof of $H_2(\Gamma_{g,1}; \mathbb{Z}) = \mathbb{Z}$ for $g \geq 4$. His method used the presentation of the mapping class group $\Gamma_{g,1}$ and the following theorem of Hopf (cf. [7]): Given a short exact sequence of groups

$$1 \rightarrow R \rightarrow F \rightarrow G \rightarrow 1,$$

where F is free, then

$$H_2(G; \mathbb{Z}) = \frac{R \cap [F,F]}{[R,F]}. \tag{10}$$

In [30], Stipsicz and the author extended Pitsch's proof to $H_2(\Gamma_g; \mathbb{Z}) = \mathbb{Z}$ for $g \geq 4$. Then the homology stabilization theorem of Harer [15] and a use of the Hochschild-Serre spectral sequence for group extensions enabled us to give a new proof of Harer's theorem on the second homology of mapping class groups, by extending it to the $g = 4$ case.

We now outline the proof of $H_2(\Gamma_g; \mathbb{Z}) = \mathbb{Z}$ for $g \geq 4$.

Consider the short exact sequence (9). Recall that F in (9) is the free group generated freely by $x_0, x_1, \ldots, x_{2g}$ and R_0 is the normal subgroup of F normally generated by the elements $A_{ij}, B_0, B_1, \ldots, B_{2g-1}, C, D$ and E. By Hopf's theorem, we have

$$H_2(\Gamma_g; \mathbb{Z}) = \frac{R_0 \cap [F,F]}{[R_0,F]}.$$

Hence, every element in $H_2(\Gamma_g; \mathbb{Z})$ has a representative of the form

$$A B_0^{n_0} B_1^{n_1} \cdots B_{2g-1}^{n_{2g-1}} C^{n_C} D^{n_D} E^{n_E}, \tag{11}$$

where A is a product of A_{ij}.

Note that each A_{ij} and E represent elements of $H_2(\Gamma_g; \mathbb{Z})$ since they are contained in $R_0 \cap [F,F]$.

It was shown in [41] by the use of the lantern relation that each A_{ij} represents the trivial class in $H_2(\Gamma_{g,1}; \mathbb{Z}) = (R_1 \cap [F,F])/[R_1,F]$. The same proof applies to show that the class

of each A_{ij} in $H_2(\Gamma_g; \mathbb{Z})$ is zero. The main reason for this is that since $g \geq 4$, for any nonseparating simple closed curve a on Σ_g, the four-holed sphere of the lantern relation can be embedded in $\Sigma_g - a$ such that all seven curves of the relation are nonseparating on $\Sigma_g - a$.

The main improvement in [30] after [41] is to show that E represents the zero element in $H_2(\Gamma_g; \mathbb{Z})$. The proof of this uses the braid relations and the two-holed torus relation. Therefore, we may delete A and E in (11).

Note that an element of F is in the derived subgroup $[F, F]$ of F if and only if the sum of the exponents of each generator x_i is zero. Since the expression (11) must be in $[F, F]$, by looking at the sum of the exponents of the generators $x_{2g}, x_{2g-1}, \ldots, x_6$, one can see easily that $n_{2g-1}, n_{2g-2}, \ldots, n_5$ must be zero. Then, by looking at the sums of the exponents of the other generators, it can be concluded that there must be an integer k such that $n_0 = -18k, n_1 = 6k, n_2 = 2k, n_3 = 8k, n_4 = -10k, n_C = k$ and $n_D = -10k$. This says that $H_2(\Gamma_g; \mathbb{Z})$ is cyclic and is generated by the class of the element

$$B_0^{-18} B_1^6 B_2^2 B_3^8 B_4^{-10} C D^{-10}. \tag{12}$$

On the other hand, for every $g \geq 3$, the existence of a genus-g surface bundle with nonzero signature guarantees that $H^2(\Gamma_g; \mathbb{Z})$ contains an infinite cyclic subgroup (cf. [36]); the signature cocycle is of infinite order. The universal coefficient theorem implies that $H_2(\Gamma_g; \mathbb{Z})$ contains an element of infinite order. This shows that $H_2(\Gamma_g; \mathbb{Z}) = \mathbb{Z}$ for $g \geq 4$.

By omitting E from the above proof, the same argument also proves that $H_2(\Gamma_{g,1}; \mathbb{Z}) = \mathbb{Z}$ for $g \geq 4$.

A special case of Harer's homology stability theorem in [15] says that for $g \geq 4$ and $r \geq 1$ the inclusion mapping $\Sigma_{g,r} \to \Sigma_{g,r+1}$ obtained by gluing a disc with two boundary components to $\Sigma_{g,r}$ along one of the boundary components induces an isomorphism $H_2(\Gamma_{g,r}; \mathbb{Z}) \to H_2(\Gamma_{g,r+1}; \mathbb{Z})$. Also, an application of the Hochschild-Serre spectral sequence to (7) shows that $H_2(\Gamma_{g,r}^n; \mathbb{Z}) = H_2(\Gamma_{g,r+1}^{n-1}; \mathbb{Z}) \oplus \mathbb{Z}$ for $g \geq 3$.

We can summarize the results mentioned above as follows. The details of the proof may be found in [30].

Theorem 6.1. *If $g \geq 4$ then $H_2(\Gamma_{g,r}^n; \mathbb{Z})$ is isomorphic to $\mathbb{Z}^{n+1}$.*

The same method above also proves that $H_2(\Gamma_2; \mathbb{Z}) = H_2(\Gamma_{2,1}; \mathbb{Z})$ is isomorphic to either 0 or $\mathbb{Z}_2$, and the groups $H_2(\Gamma_3; \mathbb{Z})$ and $H_2(\Gamma_{3,1}; \mathbb{Z})$ are isomorphic to either $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}_2$. By the work of Benson-Cohen [1], $H_2(\Gamma_2; \mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. It follows now from the universal coefficient theorem that $H_2(\Gamma_2; \mathbb{Z})$ is not trivial, hence $\mathbb{Z}_2$. To the best knowledge of the author, the computations of $H_2(\Gamma_{2,r}^n; \mathbb{Z})$ in the remaining cases and $H_2(\Gamma_{3,r}^n; \mathbb{Z})$ are still open.

7. Higher (co)homologies

Here we will mention a few known results on the (co)homology group of the mapping class group. In Section 6, we appealed to a special case of the homology stability theorem of Harer. The original theorem asserts that in a given dimension the homology group

of the mapping class group of a surface of with boundary components does not depend on the genus if the genus of the surface is sufficiently high. This result was improved by Ivanov in [20, 21]. In [21], Ivanov also proved a stabilization theorem for the homology with twisted coefficients of the mapping class groups of closed surfaces.

The third homology group of $\Gamma_{g,r}$ with rational coefficients was computed by Harer in [16]. It turns out that $H_3(\Gamma_{g,r}; \mathbb{Q}) = 0$ for $g \geq 6$.

Let $\mathbb{Q}[z_2, z_4, z_6, \dots]$ denote the polynomial algebra of generators z_{2n} in dimensions $2n$ for each positive integer n. Then there are classes $y_2, y_4, y_6, \dots$ with $y_{2n} \in H^{2n}(\Gamma_g; \mathbb{Q})$ such that the homomorphism of algebras

$$\mathbb{Q}[z_2, z_4, z_6, \dots] \to H^*(\Gamma_g; \mathbb{Q})$$

given by $z_{2n} \mapsto y_{2n}$ is an injection in dimensions less than $g/3$. This result was proved by Miller [37].

The entire mod-2 cohomology of Γ_2 is also known. Benson and Cohen [1] computed the Poincaré series for mod-2 cohomology to be

$$(1 + t^2 + 2t^3 + t^4 + t^5)/(1 - t)(1 - t^4) = 1 + t + 2t^2 + 4t^3 + 6t^4 + 7t^5 + \cdots.$$

8. Nonorientable surfaces

In this last section, we outline the known results about the generators and the homology groups of the mapping class groups of nonorientable surfaces. So let S_g^n denote a nonorientable surface of genus g with n punctures. Recall that the genus of a closed nonorientable surface is defined as the number of real projective planes in a connected sum decomposition. Let us define the mapping class group Γ_g^n as in the orientable case; diffeomorphisms and isotopies are required to fix each puncture.

The mapping class group Γ_1 of the real projective plane is trivial and the group Γ_2 is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ (cf. [32]). Lickorish [32, 34] and Chillingworth [8] proved that if $g \geq 3$ then Γ_g is generated by a finite set consisting of Dehn twists about two-sided nonseparating simple closed curves and a crosscap slide (or Y-homeomorphism). See also [4]. Using this result the author [25] computed $H_1(\Gamma_g; \mathbb{Z})$. This result was extended to the punctured cases in [27]. We note that the group Γ_g^n of this section is called the pure mapping class group in [27] and denoted by $\mathcal{PM}_{g,n}$. The first homology group of Γ_g^n with integer coefficients is as follows.

Theorem 8.1 ([27]). *Let $g \geq 7$. Then the first homology group $H_1(\Gamma_g^n; \mathbb{Z})$ of Γ_g^n is isomorphic to $\mathbb{Z}_2^{n+1}$.*

If we define $\mathcal{M}_g^n$ as the group of the diffeomorphisms $S_g^n \to S_g^n$ modulo the diffeomorphisms which are isotopic to the identity by an isotopy fixing each puncture, then more is known. Let $\mathbb{N}_+$ and $\mathbb{N}$ denote the set of positive integers and the set of nonnegative integers, respectively. Define a function $k : \mathbb{N}_+ \times \mathbb{N} \to \mathbb{N}$ by declaring $k(1, 0) = 0$, $k(4, 0) = 3$, $k(g, 0) = 2$ if $g = 2, 3, 5, 6$, $k(g, 0) = 1$ if $g \geq 7$, $k(g, 1) = k(g, 0) + 1$ and $k(g, n) = k(g, 0) + 2$ if $n \geq 2$.

Theorem 8.2 ([27]). *The first homology group $H_1(\mathcal{M}_g^n; \mathbb{Z})$ of the mapping class group $\mathcal{M}_g^n$ of a nonorientable surface of genus g with n punctures is isomorphic to the direct sum of $k(g,n)$ copies of $\mathbb{Z}_2$.*

It is easy to see that the groups Γ_g^n and $\mathcal{M}_g^n$ fit into a short exact sequence

$$1 \longrightarrow \Gamma_g^n \longrightarrow \mathcal{M}_g^n \longrightarrow Sym(n) \longrightarrow 1,$$

where $Sym(n)$ is the symmetric group on n letters.

No higher homology groups of Γ_g^n or $\mathcal{M}_g^n$ are known.

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References

[1] D. Benson and F. Cohen, *The mod-2 cohomology of the mapping class group for a surface of genus 2*, Mem. Amer. Math. Soc. **443** (1991), 93–104.

[2] J. S. Birman, *Braids, links and mapping class groups*, Annals of Mathematics Studies, No. 82. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1974.

[3] J. S. Birman, *The algebraic structure of surface mapping class groups*, Discrete groups and automorphic functions (Proc. Conf., Cambridge, 1975), 163–198. Academic Press, London, 1977.

[4] J. S. Birman, D. R. J. Chillingworth, *On the homeotopy group of a non-orientable surface*, Math. Proc. Camb. Phil. Soc. **71** (1972), 437-448.

[5] J. S. Birman, H. M. Hilden, *On the mapping class groups of closed surfaces as covering spaces*, Advances in the theory of Riemann surfaces (Proc. Conf., Stony Brook, N.Y., 1969), 81–115. Ann. of Math. Studies, No. 66. Princeton Univ. Press, Princeton, N.J., 1971.

[6] J. S. Birman, B. Wajnryb, *Errata: Presentations of the mapping class group,* Israel J. Math. **88** (1994), 425–427.

[7] K. S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics 87, Springer-Verlag, New York-Berlin, 1982.

[8] D. R. J. Chillingworth, *A finite set of generators for the homeotopy group of a non-orientable surface*, Math. Proc. Camb. Phil. Soc. **65** (1969), 409-430.

[9] F. R. Cohen, *On the mapping class groups for punctured spheres, the hyperelliptic mapping class groups, $SO(3)$, and $spin^c(3)$*, American Journal of Mathematics **115** (1993), 389-434.

[10] M. Dehn, *Die gruppe der abdildungsklassen*, Acta Math. **69** (1938), 135-206.

[11] H. Endo, M. Korkmaz, D. Kotschick, B. Ozbagci, A. Stipsicz, *Commutators, Lefschetz fibrations and signatures of surface bundles*, Topology, to appear.

[12] S. Gervais, *Presentation and central extensions of mapping class groups*, Trans. Amer. Math. Soc. **348** (1996), 3097–3132.

[13] S. Gervais, *A finite presentation of the mapping class group of a punctured surface*, Topology **40** (2001), 703–725.

[14] J. L. Harer, *The second homology group of the mapping class group of an orientable surface*, Inv. Math. **72** (1982), 221-239.

[15] J. L. Harer, *Stability of the homology of the mapping class groups of orientable surfaces,* Annals of Mathematics **121** (1985), 215-249.

[16] J. L. Harer, *The third homology group of the moduli space of curves,* Duke Mathematical Journal **63** (1991), 25-55.

[17] J. L. Harer, *The cohomology of the moduli space of curves*, Theory of Moduli, 138-221, Lecture Notes in Math., 1337, Springer-Verlag, Berlin 1988.

[18] A. Hatcher, W. Thurston, *A presentation for the mapping class group of a closed orientable surface*, Topology **19** (1980), 221–237.

[19] S. Humphries, *Generators for the mapping class group*, Topology of low-dimensional manifolds (Proc. Second Sussex Conf., Chelwood Gate, 1977), 44–47, Lecture Notes in Math., 722, Springer-Verlag, Berlin, 1979.

[20] N. V. Ivanov, *Stabilization of the homology of Teichmüller modular groups*, (Russian) Algebra i Analiz **1** (1989), 110–126; translation in Leningrad Math. J. **1** (1990), 675–691.

[21] N. V. Ivanov, *On the homology stability for Teichmüller modular groups: closed surfaces and twisted coefficients*, Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991), 149–194, Contemp. Math., 150, Amer. Math. Soc., Providence, RI, 1993.

[22] N. V. Ivanov, *Mapping class groups*, Handbook of Geometric Topology, Elsevier, to appear.

[23] N. V. Ivanov, J. D. McCarthy, *On injective homomorphisms between Teichmüller modular groups. I*, Invent. Math. **135** (1999), 425–486.

[24] D. L. Johnson, *Homeomorphisms of a surface which act trivially on homology*, Proc. Amer. Math. Soc. **75** (1979), 119-125.

[25] M. Korkmaz, *First homology group of mapping class groups of nonorientable surfaces*, Math. Proc. Camb. Phil. Soc. **123** (1998), 487-499.

[26] M. Korkmaz, *Noncomplex smooth 4-manifolds with Lefschetz fibrations*, International Mathematics Research Notices **2001**, 115-128.

[27] M. Korkmaz, *Mapping class groups of nonorientable surfaces*, Geometriae Dedicata, to appear.

[28] M. Korkmaz, J. D. McCarthy, *Surface mapping class groups are ultrahopfian*, Math. Proc. Camb. Phil. Soc. **129** (2000), 35-53.

[29] M. Korkmaz, B. Ozbagci, *Minimal number of singular fibers in a Lefschetz fibration*, Proc. Amer. Math. Soc. **129** (2001), 1545-1549.

[30] M. Korkmaz, A. I. Stipsicz, *The second homology groups of mapping class groups of orientable surfaces*, Math. Proc. Camb. Phil. Soc., to appear.

[31] W. B. R. Lickorish, *A representation of orientable combinatorial 3-manifolds*, Annals of Mathematics **76** (1962), 531-540.

[32] W. B. R. Lickorish, *Homeomorphisms of non-orientable 2-manifolds*, Math. Proc. Camb. Phil. Soc. **59** (1963), 307-317.

[33] W. B. R. Lickorish, *A finite set of generators for the homeotopy group of a 2-manifold*, Math. Proc. Camb. Phil. Soc. **60** (1964), 769-778.

[34] W. B. R. Lickorish, *On the homeomorphisms of a non-orientable surface*, Math. Proc. Camb. Phil. Soc. **61** (1965), 61-64.

[35] J. McCool, *Some finitely presented subgroups of the automorphism group of a free group*, J. Algebra **35** (1975), 205–213.

[36] W. Meyer, *Die Signatur von Flächenbündlen*, Math. Ann. **201** (1973), 239–264.

[37] E. Y. Miller, *The homology of the mapping class group*, J. Differential Geometry **24** (1986), 1-14.

[38] S. Morita, *Families of Jacobian manifolds and characteristic classes of surface bundles. I*, Ann. Inst. Fourier (Grenoble) **39** (1989), 777–810.

[39] S. Morita, *Structure of the mapping class group of surfaces: a survey and a prospect*, Geometry and Topology Monographs, V. 2. (Proceedings of the Kirbyfest, 1999), 349–406.

[40] D. Mumford, *Abelian quotients of the Teichmüller modular group*, J. d'Anal. Math. **18** (1967), 227-244.

[41] W. Pitsch, *Un calcul élémentaire de $H_2(\mathcal{M}_{g,1}, \mathbf{Z})$ pour $g \geq 4$*, C. R. Acad. Sci. Paris, t. **329**, Série I (1999), 667-670.

[42] J. Powell, *Two theorems on the mapping class group of a surface*, Proc. Amer. Math. Soc. **68** (1978), 347-350.

[43] A. I. Stipsicz, *Surface bundles with nonvanishing signatures*, preprint.

[44] B. Wajnryb, *A simple presentation of the mapping class group of an orientable surface*, Israel J. Math. **45** (1983), 157–174.

[45] B. Wajnryb, *Mapping class group of a surface is generated by two elements*, Topology **35** (1996), 377–383.

[46] B. Wajnryb, *An elementary approach to the mapping class group of a surface*, Geometry and Topology **3** (1999), 405-466.

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