Braids and symplectic four-manifolds with abelian fundamental group

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Abstract

We explain how a version of Floer homology can be used as an invariant of
symplectic manifolds with $b_1 > 0$. As a concrete example, we look at four-
manifolds produced from braids by a surgery construction. The outcome shows
that the invariant is nontrivial; however, it is an open question whether it is
stronger than the known ones.

On a symplectic manifold with nonzero first Betti number, there is a distinction between
symplectic and Hamiltonian vector fields. This is usually perceived as adding to the
difficulty of understanding such manifolds, and indeed it raises many questions, some of
which are still open (for instance, the flux conjecture [5]). But the additional complexity
also means that more symplectic invariants become available. The aim of this note is to
draw attention to one of these, which we call "non-Hamiltonian Floer homology". The
concept is by no means new, but it has been underrated a bit, partially because the first
computations were in cases where it reduces to a version of ordinary homology [6].

To demonstrate its usefulness, we consider a construction that associates to any d-stranded
braid (to be precise, framed spherical transitive braid) a symplectic four-manifold with
fundamental group $\mathbb{Z} \times \mathbb{Z}/d$. This is a variation of earlier constructions due to Smith
[13, 12], McMullen-Taubes [9] and Fintushel-Stern [2, 3]. Braids can be represented
as diffeomorphisms of a punctured two-sphere. Moreover, these representatives can be
chosen to be symplectic, and then there is a symplectic Floer homology group measuring
their fixed point theory. We will prove that the non-Hamiltonian Floer homology of
the associated four-manifold recovers this Floer homology of the braid (more precisely,
recovers its total dimension).

At present, it is an open question whether the same information could be obtained
from Gromov-Witten theory, or even from more classical topological invariants. Non-
Hamiltonian Floer homology is not invariant under deformations of the symplectic class,
which seems to indicate that it cannot be expressed in terms of Gromov-Witten invari-
ants alone. This still does not quite answer the question, so the importance of our results

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remains somewhat dubious; which is one reason why this is only an announcement, containing no proofs.

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1. Floer homology

Let (M, ω) be a closed symplectic manifold. To any symplectic automorphism ϕ of M one can associate its Floer homology group $HF_*(\phi)$, which is a finite-dimensional $\mathbb{Z}/2$-graded vector space over the "universal Novikov field" Λ. We will use Floer homology only for manifolds of dimension ≤ 4, where the definition becomes considerably easier (as a consequence of "weak monotonicity"). Here are some basic properties:

(i) $HF_*(\mathrm{id}) \cong H_*(M; \Lambda)$ canonically,
(ii) $HF_*(\phi^{-1})$ is the vector space dual of $HF_*(\phi)$,
(iii) for any ϕ, ψ there is a canonical isomorphism $HF_*(\phi) \cong HF_*(\psi\phi\psi^{-1})$,
(iv) $HF_*(\phi)$ is invariant under Hamiltonian isotopies of ϕ.

We emphasize that $HF_*(\phi)$ is not in general invariant under symplectic isotopies (an exception is the case when $\mathrm{id} - \phi^* : H^1(M; \mathbb{R}) \to H^1(M; \mathbb{R})$ is an isomorphism, because then any symplectic isotopy of ϕ can be replaced by a Hamiltonian isotopy followed by conjugation).

Floer homology can be formally described as Morse theory for the action functional on the twisted free loop space $\mathcal{L}(M, \phi) = \{u \in C^\infty(\mathbb{R}, M) : u(t) = \phi(u(t+1))\}$. More concretely, it is the homology of a chain complex whose generators are the fixed points of ϕ (in the generic situation where these are nondegenerate). The grading is given by the Lefschetz index $sign(det(1 - D\phi))$. This implies that the Euler characteristic of Floer cohomology is the Lefschetz fixed point number:

$$dim\, HF_0(\phi) - dim\, HF_1(\phi) = L(\phi). \tag{1}$$

For more information about the construction of Floer homology, see [1], [11]. There is a considerable amount of additional structure on these groups, which we will not mention at all here.

An example: Floer homology for braids. On the two-sphere S^2, choose a set $\Delta = \{z_1, \ldots, z_d\}$ of $d \geq 2$ marked points. Around each of these points choose distinguished local coordinates, which means an oriented embedding $\iota : D_\epsilon \times \{1, \ldots, d\} \to S^2$ with $\iota(0, k) = z_k$, where D_ϵ is the closed disc of radius $\epsilon > 0$. Let $\mathcal{G}_d$ be the group of diffeomorphisms $\phi : S^2 \to S^2$ which preserve Δ and are compatible with the local coordinates, in

the sense that $\phi(\iota(z,k)) = \iota(z,\sigma(k))$ for some permutation $\sigma \in S_d$. The framed spherical braid group is defined to be $\pi_0(\mathcal{G}_d)$. An element β of this group will be called a transitive braid if the induced permutation of Δ is $\sigma = (12\ldots d)$.

To associate a Floer group to a transitive braid, choose a symplectic form on S^2 such that ι is symplectic with respect to the standard form on D_ϵ. By Moser's lemma on volume forms, β has a symplectic representative ϕ. Consider $S' = S^2 \setminus \iota(int(D_{\epsilon/2}) \times \{1,\ldots,d\})$ and its symplectic automorphism $\phi' = \phi \,|\, S'$. One defines

$$HF_*(\beta) \overset{def}{=} HF_*(\phi'). \tag{2}$$

We are overstepping the bounds slightly, since we did not originally introduce Floer homology for manifolds with boundary. However, this is not really a problem: by transitivity of β, there are no fixed points on the boundary, and an easy maximum principle argument shows that the "connecting orbits" of Floer theory never touch the boundary. (An alternative way of defining $HF_*(\beta)$ is to glue in a torus to each boundary component, which yields a closed genus d surface. One can extend ϕ' to a symplectic map of this surface which permutes the tori transitively, and then define $HF_*(\beta)$ to be the Floer homology of this extension. The techniques of [10] ensure that both approaches yield the same result.) Again using Moser's lemma, we know that the representative ϕ is unique up to symplectic isotopies within $\mathcal{G}_d$. All such isotopies are Hamiltonian, even though the Hamiltonian functions involved do not necessarily vanish near $\partial S'$. As a consequence (2) is independent of the choice of ϕ. This is evident if one defines $HF_*(\beta)$ in terms of the closed genus d surface, since the induced isotopy on that surface is Hamiltonian. If one wants to stay on S', the proof requires another application of the maximum principle to solutions of the "continuation" equation.

It is not entirely clear how much information $HF_*(\beta)$ contains. Conjugate braids have the same Floer homology, by (iii) above. A "neck-stretching" argument in the spirit of [10], using the transitivity of β, shows that Floer homology does not really see the framings: if $\tau \in \mathcal{G}_d$ is a Dehn twist along a small loop encircling just one point of Δ, then $HF_*(\beta) \cong HF_*(\beta \circ [\tau])$ for all β. On the positive side, Nielsen theory provides a nontrivial lower bound. Recall that for each connected component $l \in \pi_0(\mathcal{L}(S', \phi'))$ of the twisted free loop space there is a Nielsen number $N_l(\phi') \in \mathbb{Z}$, which counts (with the usual sign) the fixed points whose associated constant paths lie in that component. Floer homology admits a corresponding splitting into direct summands, whose Euler characteristics are the $N_l(\phi')$; this is a refinement of (1). As a consequence

$$dim\, HF_*(\beta) \geq \sum_l |N_l(\phi')|. \tag{3}$$

This is a general feature of Floer homology, not at all limited to braids, but it is particularly relevant in this case due to the richness of $\pi_1(S')$. For all I know, (3) might be an equality for all transitive β (note however, that there are plenty of counterexamples among more general surface diffeomorphisms, starting with Poincaré's geometric theorem).

Non-Hamiltonian Floer homology. Let M be a closed symplectic manifold. To any $a \in H^1(M; \mathbb{R})$ one can associate a Hamiltonian isotopy class of automorphisms: it consists of maps ϕ_a which can be obtained from the identity by a symplectic isotopy with Calabi class a (see [8, Chapter 10] for a thorough discussion; what we call the Calabi class is the "flux" in their terminology, and moreover our sign convention is opposite to theirs). The "non-Hamiltonian Floer homology" $HF_*(\phi_a)$ is an invariant of (M, a). From the properties stated above one sees that for $a = 0$ it is ordinary homology; that the groups associated to a and $-a$ are dual; and that $HF_*(\phi_a) \cong HF_*(\phi_{a+\gamma})$ for all γ in the flux subgroup $\Gamma \subset H^1(M; \mathbb{R})$ (again, this is defined in [8]).

For monotone symplectic manifolds (of arbitrary dimension) $HF_*(\phi_a)$ can be determined completely. This has been done by Lê and Ono in [6]; their result, stated in a slightly different form, is

THEOREM 1 (Lê-Ono). *Assume that $[\omega] = \lambda\, c_1(M)$ with some $\lambda \neq 0$ (if λ is negative, they also assume that the minimal Chern number N satisfies $2N \geq \dim M - 4$). Then $HF_*(\phi_a) \cong H_*(M; \underline{\Lambda}_a)$, where $\underline{\Lambda}_a \to M$ is the flat bundle of invertible Λ-modules canonically associated to a.*

The assumption on N can probably be dropped, in view of the technical developments which have occurred in the meantime. On the other hand, the monotonicity condition is essential, as we will see now.

Given an arbitrary symplectic automorphism ψ of M, one can form the symplectic mapping torus E, which is the quotient of $S^1 \times \mathbb{R} \times M$ by the $\mathbb{Z}$-action $(s, t, x) \mapsto (s, t-1, \psi(x))$, with the obvious product symplectic structure. Informally speaking, the purpose of mapping tori is to make maps isotopic to the identity. In our context, this means that the symplectic automorphism ϕ of E given by $(s, t, x) \mapsto (s, t, \psi(x))$ is the time one map of the symplectic vector field $\partial/\partial t$. In terms of the notation introduced above, $\phi = \phi_a$ with $a = [ds] \in H^1(M; \mathbb{R})$.

PROPOSITION 2. $HF_*(\phi) \cong HF_*(\psi) \otimes_\Lambda H_*(T^2; \Lambda)$.

On a naive level, this "suspension" isomorphism reflects the fact that each fixed point of ψ gives rise to a T^2 of fixed points of ϕ. Note that mapping tori are never monotone: and indeed, in contrast to Theorem 1, non-Hamiltonian Floer homology does not reduce to a form of ordinary homology. Another obvious remark is that non-Hamiltonian Floer homology depends not only on a, but also on the cohomology class of the symplectic form. Indeed, if one rescales a by some small amount and keeps the symplectic form on E, then all fixed points of ϕ_a disappear; and the same happens if one keeps a fixed and changes the symplectic form to $\delta(ds \wedge dt) + \omega$, for $\delta \notin \mathbb{Z}$.

For the purposes of four-dimensional symplectic topology, Proposition 2 is not very satisfactory, since the mapping tori of surface diffeomorphisms have highly nontrivial fundamental groups, so that one can often distinguish between them on topological grounds.

Path components. When explaining (3), we had already mentioned the decomposition of $HF_*(\phi)$ into direct summands corresponding to components of the twisted free loop space. For $\phi = \phi_a$ which is homotopic to the identity, this is just the ordinary free loop space, so summands are enumerated by conjugacy classes in $\pi_1(M)$. For our purpose, it is sufficient to have a coarser splitting, which distinguishes only homology classes. We denote this by

$$HF_*(\phi_a) \cong \bigoplus_{c \in H_1(M;\mathbb{Z})} HF_*(\phi_a; c). \tag{4}$$

The duality between $HF_*(\phi_a)$ and $HF_*(\phi_{-a})$ relates the components belonging to c and $-c$. Some examples: in Theorem 1 only the summand corresponding to $c = 0$ is nonzero. In fact, the proof of that result is essentially by deformation to $a = 0$, where one can arrange that all fixed points lie in the trivial connected component. In Proposition 2 there can be several nontrivial summands, but all of them are for $c \neq 0$. The reason is that the isotopy from the identity to $\phi(s, t, x) = (s, t, \psi(x))$ winds once around the base T^2 in t-direction.

2. A surgery construction

Let β be a transitive framed spherical braid on d strands. We choose a symplectic representative ϕ as in the previous section. Let $M_1 = (S^1 \times \mathbb{R} \times S^2)/\mathbb{Z}$ be the symplectic mapping torus of ϕ. The "graph of the braid" is a canonical symplectic torus $H_1 \subset M_1$: it is the image of the embedding $T^2 \to M_1$ which sends (s, t), for $k - 1/d \leq t \leq k/d$, to $(s, d \cdot t - k + 1, z_k)$. Because we are working with framed braids, the normal bundle of H_1 has a canonical trivialization. Moreover, $vol(H_1) = d$. Next take $M_2 = T^4$ with the disjointly embedded tori

$$H_1' = S^1 \times S^1 \times \{\xi_1\} \times \{\xi_2\},$$
$$H_3' = \{\xi_3\} \times S^1 \times S^1 \times \{\xi_4\},$$
$$H_4' = \{\xi_5\} \times S^1 \times \{\xi_6\} \times S^1$$

where the ξ_r are all different. We identify each of these tori with T^2 by using coordinates in the given order. Their normal bundles have preferred (translation-invariant) trivializations. Equip M_2 with some constant symplectic form which makes all the H_k' symplectic, and such that $vol(H_1') = d$. The remaining parts M_3, M_4 will be elliptically fibered K3 surfaces, with embedded tori $H_3 \subset M_3$, $H_4 \subset M_4$ which are the fibres. We identify them with T^2 in an arbitrary way, and use the standard trivialization of their normal bundles (given by the fibration). The symplectic forms should be normalized in such a way that

$vol(H_3) = vol(H'_3)$, $vol(H_4) = vol(H'_4)$. Now glue together all these pieces by Gompf-style sums, pairing the tori H_k with H'_k for $k = 1, 3, 4$. The outcome (keeping the choices of M_2, M_3, M_4 fixed) depends up to symplectic isomorphism only on the conjugacy class of the framed braid β. We denote it by M^β.

The idea can be summarized as follows. By deforming ϕ to the identity inside $Diff(S^2)$, one can identify M_1 with $T^2 \times S^2$. From this point of view, the two-torus $H_1 \subset T^2 \times S^2$ is knotted in a way which is determined by the braid (as observed in [3] and [13], one can use the fundamental group of the complement to verify that many different knot types occur). At this point, one could choose to directly glue in a K3 surface to H_1. This is an appealing possibility, somewhat similar to [3, Section 5], but since the resulting fundamental group is $\mathbb{Z}/d$, non-Hamiltonian Floer homology cannot be applied in a meaningful way. The role of the intermediate piece M_2, which we have borrowed from [9] and [12], is to let a slightly larger part of $\pi_1(M_1 \setminus H_1)$ survive.

Topological aspects. As has been already mentioned, $\pi_1(M^\beta) \cong \mathbb{Z} \times \mathbb{Z}/d$ for all β. More explicitly, let $z_0 \in S^2 \setminus \Delta$ be a fixed point of ϕ, and consider the loops $l_1 = S^1 \times \{0\} \times \{z_0\}$, $l_2 = \{0\} \times S^1 \times \{z_0\}$ in $M_1 \setminus H_1 \subset M^\beta$, oriented in the obvious way. The isomorphism can be chosen such that $[l_1], [l_2]$ correspond to $(1, 0)$ and $(0, 1)$ in $\mathbb{Z} \oplus \mathbb{Z}/d$, respectively. To verify this, it is useful to carry out the gluing in a particular order. Consider the manifold M'_2 obtained by putting together M_2, M_3, M_4 in the way described above; this still contains the torus H'_1. It is a familiar fact that $\pi_1(M_3 \setminus H_3) = \pi_1(M_4 \setminus H_4) = 1$, and as a consequence

$$
\begin{aligned}
\pi_1(M'_2 \setminus H'_1) &\cong \pi_1(M_2 \setminus H'_1)/\langle \pi_1(H'_3), \pi_1(H'_4) \rangle \\
&\cong \langle d_1, d_2, d_3, d_4 \ : \ [d_1, d_2] = 1 \rangle / \langle d_2, d_3, d_4 \rangle \\
&= \langle d_1 \rangle \cong \mathbb{Z}.
\end{aligned}
$$

Note that d_1 is the first longitude of H'_1; the other longitude (which would be d_2) and the meridian (which would be $[d_3, d_4]$) have got killed. As for the remaining piece, the fundamental group of $M_1 \setminus H_1$ is quite large, but it is generated by l_1 (which is the first longitude of H_1, and commutes with all other elements) together with l_2 and various conjugates of the meridian of H_1. Joining together this with $M'_2 \setminus H'_1$ kills the meridian and identifies l_1 with d_1, from which one sees that the fundamental group becomes abelian.

The characteristic numbers are $c_2(M^\beta) = 48$, $c_1(M^\beta)^2 = 0$. In fact $-d \cdot c_1(M^\beta)$ can be represented by the disjoint union of $6d - 2$ embedded symplectic tori, each of which has trivial normal bundle: $2d - 2$ parallel copies of H_1, and $2d$ copies of H_3 and H_4 each (see [12] for how to do this kind of computation). The next step would be to compute the homology of the universal cover, as a $\pi_1(M^\beta)$-module, and the intersection form on it (presumably, that goes a long way towards determining the homeomorphism type of M^β). We have not done this, but informal considerations suggest that it might turn out to be the same for all β.

Floer homology. Let $a_1 \in H^1(M^\beta; \mathbb{R})$ be the unique class with $\langle a_1, [l_1] \rangle = 1$. The next result determines certain summands in the splitting (4) of the non-Hamiltonian Floer group $HF_*(\phi_{a_1})$.

THEOREM 3. $HF_*(\phi_{a_1}; [l_2]) \cong HF_*(\beta) \otimes_\Lambda H_*(T^2; \Lambda)$. *Moreover* $HF_*(\phi_{a_1}; c) = 0$ *for any other nonzero torsion class* $c \in H_1(M^\beta; \mathbb{Z})$.

Consider the direct sum of the groups $HF_*(\phi_a; c)$ where a ranges over the two generators of $H^1(M^\beta; \mathbb{Z}) \subset H^1(M^\beta; \mathbb{R})$, and c over all nonzero torsion elements of $H_1(M^\beta; \mathbb{Z})$. Using the duality between $HF_*(\phi_a; c)$ and $HF_*(\phi_{-a}; -c)$ one computes that the total dimension of the direct sum is $8 \, dim(HF_*(\beta))$. On the other hand, the direct sum is defined without reference to any particular basis of homology. As a consequence:

COROLLARY 4. *The total dimension of* $HF_*(\beta)$ *is a symplectic invariant of* M^β.

As in our discussion of mapping tori, $HF_*(\phi_a; c)$ becomes zero if one changes the symplectic class slightly, by changing the area of the T^2 factor in M_1, and rescaling the symplectic forms of the other M_k accordingly. However, one can compensate for this by rescaling a, and recover the Floer homology groups in this way. Of course, it is not clear what happens under "large" deformations of the symplectic class.

The proof of Theorem 3 consists of two steps. The first is a variant of Proposition 2 adapted to surfaces with boundary. The second step is a "Mayer-Vietoris" argument in which one considers the behaviour of Floer homology under Gompf sums. In general this is a hard problem, as a look at the formulae for Gromov-Witten invariants shows [4, 7], but the particular case needed here is fairly simple. The reason is essentially topological: the components c which we are interested in lie in the image of $H_1(M_1 \setminus H_1; \mathbb{Z}) \to H_1(M^\beta; \mathbb{Z})$, but not in that of $H_1(M_2' \setminus H_1'; \mathbb{Z}) \to H_1(M^\beta; \mathbb{Z})$. Therefore only the fixed points lying in $M_1 \setminus H_1$ are relevant, which are precisely those coming from the braid.

Finally, we would like to point out that non-Hamiltonian Floer homology also merits some attention in higher dimensions. For instance, take a symplectic manifold M with an automorphism ϕ that is differentiably, but not symplectically, isotopic to the identity (in dimension ≥ 4, plenty of such exist). Then the symplectic mapping torus E is diffeomorphic to $T^2 \times M$ but with a potentially nonstandard symplectic structure, which one could try to detect using Proposition 2. There are also examples of "fragile" symplectic automorphisms, which become symplectically isotopy to the identity after a slight change of the symplectic class [11]. In that case, one can hope to show that E is symplectically deformation equivalent, but not symplectomorphic, to $T^2 \times M$.

References

1. S. Dostoglou and D. Salamon, *Self dual instantons and holomorphic curves*, Annals of Math. **139** (1994), 581–640.
2. R. Fintushel and R. Stern, *Knots, links, and 4-manifolds*, Invent. Math. **134** (1998), 363–400.
3. _____, *Symplectic surfaces in a fixed homology class*, J. Differential Geom. **52** (1999), 203–222.
4. E.-N. Ionel and T. Parker, *Gromov-Witten invariants of symplectic sums*, Math. Res. Lett. **5** (1998), 563–576.
5. F. Lalonde, D. McDuff, and L. Polterovich, *On the flux conjectures*, Geometry, topology, and dynamics (Montreal, PQ, 1995) (F. Lalonde, ed.), Amer. Math. Soc., 1998, pp. 69–85.
6. Le Hong Van and K. Ono, *Cup-length estimate for symplectic fixed points*, Contact and symplectic geometry (C. B. Thomas, ed.), Cambridge Univ. Press, 1996, pp. 268–295.
7. A.-M. Li and Y. Ruan, *Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds*, Invent. Math. **145** (2001), 151–218.
8. D. McDuff and D. Salamon, *Introduction to symplectic topology*, Oxford University Press, 1995.
9. C. McMullen and C. Taubes, *4-manifolds with inequivalent symplectic forms and 3-manifolds with inequivalent fibrations*, Math. Res. Lett. **6** (1999), 681–696.
10. P. Seidel, *The symplectic Floer homology of a Dehn twist*, Math. Research Lett. **3** (1996), 829–834.
11. _____, *Floer homology and the symplectic isotopy problem*, Ph.D. thesis, Oxford University, 1997.
12. I. Smith, *On moduli spaces of symplectic forms*, Math. Res. Lett. **7** (2000), 779–788.
13. _____, *Symplectic submanifolds from surface fibrations*, Pacific J. Math. **198** (2001), 197–205.

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