

1-1-2002

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### Recommended Citation

MEHRVARZ, A. A. and AZIZI, K. (2002) "n-Commutator Groups," *Turkish Journal of Mathematics*: Vol. 26: No. 2, Article 9. Available at: <https://journals.tubitak.gov.tr/math/vol26/iss2/9>

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## $n$ -Commutator Groups

*A. A. Mehrvarz and K. Azizi*

### Abstract

A sufficient condition such that any element of  $G'$  (the commutator subgroup of  $G$ ) can be represented as a product of  $n$  commutators, was studied in [1]. In this article we study a necessary and sufficient condition such that any element of  $G'$  can be represented as a product of  $n$  commutators, Let  $n$  be the smallest nature number such that any element of finite group  $G$  can be represented as a product of  $n$  commutators. A group  $G$  with this property will be called an  $n$ -commutator group, and  $n$  will be denoted by  $c(G)$ . Then  $\frac{\ln(|G'|)}{\ln(|G:Z(G)|)} \leq 2c(G)$ . In particular, if the all elements of  $G'$  can be represented as a commutator, then  $|G'| \leq |G : Z(G)|^2$ .

**Key Words:** Commutator subgroup, irreducible characters.

### 1. Introduction

Let  $G$  be a finite group and  $G'$  be the commutator subgroup of  $G$ . Also, let  $Irr(G)$  be the set of all complex irreducible characters of  $G$  and  $Lin(G) = \{\chi \in Irr(G) | \chi(1) = 1\}$ ,  $Irr_1(G) = Irr(G) - Lin(G)$ . We suppose that if  $\chi \in Irr(G)$ , then  $T(\chi) = \{g \in G | \chi(g) = 0\}$ .

**Definition 1.** *Let  $n$  be a natural number. Then a finite group  $G$  is called an  $n$ -commutator group if any element of  $G'$  can be represented as a product of  $n$  commutators, and no natural number fewer than  $n$  have this property. We then denote  $n$  by  $c(G)$ .  $\diamond$*

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<sup>2</sup>This research is supported by the Research Institute for Fundamental Sciences, Tabriz, Iran.

## 2. Generalities

**Lemma 1.** *Let  $\chi$  be an irreducible character of  $G$ . Then, for any  $g$  and  $x$  in  $G$ ,*

$$\sum_{t \in G} \chi(g[t, x]) = \frac{|G|}{\chi(1)} \chi(gx) \overline{\chi(x)}. \quad (1)$$

**Proof.** Let  $k_s$  be the class sum for an element  $s \in G$ . Then

$$k_s = \frac{|G : C_G(s)|}{|G|} \sum_{t \in G} s^t.$$

We extend the character  $\chi$  by linearity to  $\mathcal{C}G$  and define a function

$$\omega_\chi : Z(\mathcal{C}G) \longrightarrow \mathcal{C}$$

by

$$\omega_\chi(z) = \frac{\chi(z)}{\chi(1)}$$

for any  $z \in Z(\mathcal{C}G)$ . Then it is clear that  $\omega_\chi$  is a homomorphism of  $Z(\mathcal{C}G)$ . Since

$$\omega_\chi(k_s) = \frac{|G : C_G(s)|}{\chi(1)} \chi(s),$$

it follows that for any  $u, x \in G$ ,

$$\omega_\chi(k_u k_x) = \omega_\chi(k_u) \omega_\chi(k_x) = \frac{|G : C_G(u)| |G : C_G(x)|}{\chi(1)^2} \chi(u) \chi(x). \quad (2)$$

Setting  $t_2 t_1^{-1} = t$ , we get

$$\begin{aligned} \omega_\chi(k_u k_x) &= \frac{\chi(k_u k_x)}{\chi(1)} \\ &= \frac{|G : C_G(u)| |G : C_G(x)|}{\chi(1) |G|^2} \sum_{t_1, t_2 \in G} \chi(ux^{t_2 t_1^{-1}}) \\ &= \frac{|G : C_G(u)| |G : C_G(x)|}{\chi(1) |G|} \sum_{t \in G} \chi(ux^t), \end{aligned} \quad (3)$$

which, together with (2), yields

$$\sum_{t \in G} \chi(ux^t) = \frac{|G|}{\chi(1)} \chi(u) \chi(x). \quad (4)$$

Replacing  $x$  by  $x^{-1}$  in (4) and observing that

$$\chi(u(x^{-1})^t) = \chi(u[t, x]x^{-1}) = \chi(x^{-1}u[t, x]),$$

we have

$$\sum_{t \in G} \chi(x^{-1}u[t, x]) = \frac{|G|}{\chi(1)} \chi(u) \overline{\chi(x)}.$$

Taking into account that  $\chi(xg) = \chi(gx)$  and replacing  $x^{-1}u$  by  $g$  in the last equality, we get the required relation.  $\square$

**Lemma 2.** *Let  $\chi$  be an irreducible character of  $G$ . Then any element  $g \in G$  and  $x_1, x_2, \dots, x_n \in G$ ,*

$$\begin{aligned} & \sum_{t_1, t_2, \dots, t_n \in G} \chi(g[t_1, x_1][t_2, x_2] \cdots [t_n, x_n]) \\ &= \left(\frac{|G|}{\chi(1)}\right)^n \chi(gx_1x_2 \cdots x_n) \overline{\chi(x_1)\chi(x_2) \cdots \chi(x_n)}. \end{aligned} \quad (5)$$

**Proof.** For  $n = 1$ , the result is true by Lemma 1. Suppose that the lemma is valid for any  $k < n$ . Then for any  $g, x_1, x_2, \dots, x_{n-1} \in G$  we have

$$\begin{aligned} & \sum_{t_1, t_2, \dots, t_{n-1} \in G} \chi(g[t_1, x_1][t_2, x_2] \cdots [t_{n-1}, x_{n-1}]) \\ &= \left(\frac{|G|}{\chi(1)}\right)^{n-1} \chi(gx_1x_2 \cdots x_{n-1}) \overline{\chi(x_1)\chi(x_2) \cdots \chi(x_{n-1})}. \end{aligned}$$

By Lemma 1,

$$\begin{aligned} & \sum_{t_1, t_2, \dots, t_n \in G} \chi(g[t_1, x_1][t_2, x_2] \cdots [t_n, x_n]) \\ &= \sum_{t_1, t_2, \dots, t_{n-1} \in G} \sum_{t_n} \chi(g[t_1, x_1][t_2, x_2] \cdots [t_{n-1}, x_{n-1}][t_n, x_n]) \\ &= \sum_{t_1, t_2, \dots, t_{n-1} \in G} \frac{|G|}{\chi(1)} \chi(g[t_1, x_1][t_2, x_2] \cdots [t_{n-1}, x_{n-1}]x_n) \overline{\chi(x_n)} \\ &= \frac{|G|}{\chi(1)} \sum_{t_1, t_2, \dots, t_{n-1} \in G} \chi(x_n g[t_1, x_1][t_2, x_2] \cdots [t_{n-1}, x_{n-1}]) \overline{\chi(x_n)}. \end{aligned}$$

By induction,

$$\begin{aligned} & \sum_{t_1, t_2, \dots, t_{n-1} \in G} \chi(x_n g[t_1, x_1][t_2, x_2] \cdots [t_{n-1}, x_{n-1}]) \\ &= \left(\frac{|G|}{\chi(1)}\right)^{n-1} \chi(x_n g x_1 x_2 \cdots x_{n-1}) \overline{\chi(x_1)\chi(x_2) \cdots \chi(x_{n-1})}, \end{aligned}$$

so that

$$\begin{aligned} & \sum_{t_1, t_2, \dots, t_n \in G} \chi(g[t_1, x_1][t_2, x_2] \cdots [t_n, x_n]) \\ &= \left(\frac{|G|}{\chi(1)}\right)^n \chi(gx_1x_2 \cdots x_n) \overline{\chi(x_1)\chi(x_2) \cdots \chi(x_n)}. \end{aligned}$$

□

**Lemma 3.** *Let  $\chi$  be an irreducible character of  $G$ . Then*

a) *For any natural number  $n$  and  $g \in G$*

$$\sum_{g_1 g_2 \cdots g_n = g} \chi(g_1)\chi(g_2) \cdots \chi(g_n) = \left(\frac{|G|}{\chi(1)}\right)^{n-1} \chi(g).$$

b) *For any  $g \in G$ ,*

$$\sum_{\substack{t_i, x_i \in G \\ i \in \{1, 2, \dots, n\}}} \chi(g[t_1, x_1][t_2, x_2] \cdots [t_n, x_n]) = \left(\frac{|G|}{\chi(1)}\right)^{2n} \chi(g). \quad (6)$$

**Proof.** a) The element

$$e_\chi = \frac{\chi(1)}{|G|} \sum_{g \in G} \overline{\chi(g)} g$$

is an idempotent of the algebra  $Z(\mathcal{O}G)$ . Since  $e_\chi^n = e_\chi$ , it follows that

$$\begin{aligned} & \frac{\chi(1)}{|G|} \sum_{g \in G} \overline{\chi(g)} g \\ &= \prod_{i=1}^n \left( \frac{\chi(1)}{|G|} \sum_{g_i \in G} \overline{\chi(g_i)} g_i \right) \\ &= \left(\frac{\chi(1)}{|G|}\right)^n \sum_{g_i \in G} \overline{\chi(g_1) \cdots \chi(g_n)} g_1 \cdots g_n \\ &= \left(\frac{\chi(1)}{|G|}\right)^n \sum_{g \in G} \left( \sum_{g_1 g_2 \cdots g_n = g} \overline{\chi(g_1)\chi(g_2) \cdots \chi(g_n)} \right) g. \end{aligned}$$

Comparing the coefficients of  $g$  in the first and the last expressions, we get the required result.

b) Summing up equations (5) over  $x_1, x_2, \dots, x_n \in G$  we get

$$\begin{aligned} & \sum_{t_i, x_i \in G} \chi(g[t_1, x_1][t_2, x_2] \cdots [t_n, x_n]) = \\ & \left(\frac{|G|}{\chi(1)}\right)^n \sum_{x_1, x_2, \dots, x_n \in G} \chi(gx_1x_2 \cdots x_n)\chi(x_1^{-1})\chi(x_2^{-1}) \cdots \chi(x_n^{-1}) = \\ & \left(\sum_{x_1, x_2, \dots, x_n \in G} \chi(gx_1x_2 \cdots x_n)\chi(x_n^{-1})\chi(x_{n-1}^{-1}) \cdots \chi(x_1^{-1})\right)\left(\frac{|G|}{\chi(1)}\right)^n. \end{aligned} \quad (7)$$

Put

$$u_1 = gx_1 \cdots x_n, \quad u_2 = x_n^{-1}, \quad \dots, \quad u_{n+1} = x_1^{-1}.$$

Then  $u_1 \cdots u_{n+1} = g$ , and the last expression in (7) can be rewritten as

$$\left(\frac{|G|}{\chi(1)}\right)^n \sum_{u_1 \cdots u_{n+1} = g} \chi(u_1)\chi(u_2) \cdots \chi(u_{n+1}),$$

and hence, by part (a), it is equal to

$$\left(\frac{|G|}{\chi(1)}\right)^{2n} \chi(g),$$

as required. □

**Theorem 1.** *Let  $G$  be a finite group. Then  $G$  is an  $n$ -commutator group if and only if*

$$|G|^{2n-1} \sum_{\chi \in Irr(G)} \frac{\chi(g)}{\chi(1)^{2n-1}} \quad (8)$$

*is a natural number for all  $g \in G'$ , where  $n$  is the smallest natural number with this property.*

**Proof.** Let  $\rho = \sum_{\chi} \chi(1)\chi$  be the regular character of  $G$ . Multiplying both sides of (6) by  $\chi(1)$  and summing over all  $\chi \in Irr(G)$ , we get

$$\sum_{\substack{t_i, x_i \in G, \\ i \in \{1, 2, \dots, n\}}} \rho(g[t_1, x_1][t_2, x_2] \cdots [t_n, x_n]) = |G|^{2n} \sum_{\chi \in Irr(G)} \frac{\chi(g)}{\chi(1)^{2n-1}}. \quad (9)$$

Suppose that  $G$  is an  $n$ -commutator group. We now deduce from the first equality in (9) that if  $g \in G'$ , then since  $g$  can be represented as a product of  $n$  commutators, we have

$$|G|^{2n-1} \sum_{\chi \in Irr(G)} \frac{\chi(g)}{\chi(1)^{2n-1}} = f_n(g),$$

where  $f_n(g)$  is the number of representations of  $g$  as a product of  $n$  commutators. Since,  $f_n(g) \geq 1$  for any  $g \in G'$ ,

$$|G|^{2n-1} \sum_{\chi \in Irr(G)} \frac{\chi(g)}{\chi(1)^{2n-1}}$$

is a natural number for any  $g \in G'$ . If  $|G|^{2n-1} \sum_{\chi \in Irr(G)} \frac{\chi(g)}{\chi(1)^{2n-1}}$  is a natural number for all  $g \in G'$ , where  $n$  is the smallest natural number with this property, then we deduce from the equality (9) that if  $g \in G'$ , then  $g$  can be represented as a product of  $n$  commutators, and  $n$  is the smallest natural number with this property. Thus,  $G$  is an  $n$ -commutator group.  $\square$

**Remark 1:** Let  $G$  be a finite group. Then by using the character table of  $G$  one can say that whether  $G$  is an  $n$ -commutator group or not, an observation that follows immediately from Theorem 1.

Gallagher proved in [1] that:

**Theorem 2.** (Gallagher) Let  $\{x_1, \dots, x_n\}$  be a complete system of representatives of the sets  $T(\chi)(\chi \in Irr_1(G))$ . Then any element of  $G'$  can be represented as

$$[g_1, x_1][g_2, x_2] \cdots [g_n, x_n],$$

where  $g_i \in G, i \in \{1, 2, \dots, n\}$ .

**Corollary.** For any finite group  $G$ ,  $c(G) \leq |Irr_1(G)|$ .

**Proof.** Proof is obvious by Theorem 2.  $\square$

**Proposition.** Let  $G$  be a finite group. Then  $\frac{\ln(|G'|)}{\ln(|G:Z(G)|)} \leq 2c(G)$ . In particular, if  $c(G) = 1$ , then  $|G'| \leq |G : Z(G)|^2$ .

**Proof.** If  $T$  is a transversal for  $Z(G)$  in  $G$ , an easy calculation shows that every commutator in  $G$  actually has the form  $[s, t]$  for elements  $s, t \in T$ . Thus, by definition of  $c(G)$  we have  $|G'| \leq (|T|)^{2c(G)} = (|G : Z(G)|)^{2c(G)}$ . Thus,  $\frac{\ln(|G'|)}{\ln(|G:Z(G)|)} \leq 2c(G)$ .  $\square$

**Question:** Let  $n$  be a natural number. Does there exist a finite group  $G$  such that  $c(G) = n$ ?

**Remark 2:** Generalize this for some class of simple groups see attached paper. From Theorem 1 we have  $c(A_5) = 1$ . But  $A_5$  is not solvable. Thus, it is not true that if  $c(G) = 1$ , then  $G$  is solvable.

**Conjecture:** Let  $G$  be a finite solvable group. Then  $c(G) \leq$  derived length of  $G$ .

### References

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Received 28.11.2001