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## Asymptotic Formulas for the Eigenvalues of the Schrodinger Operator

*S. Atılgan, S. Karakılıç, O. A. Veliev*

### Abstract

In this paper, we obtain asymptotic formulas for the eigenvalues of the d-dimensional Schrodinger operator

$$L = -\Delta + q(x)$$

in d-dimensional parallelepiped  $F$  with Dirichlet and Neumann boundary conditions.

Let  $\Omega = \{m_1 w_1 + m_2 w_2 + \dots + m_d w_d : m_i \in Z, i = 1, 2, \dots, d\}$  be a lattice in  $R^d$  with the reduced orthonormal basis

$$w_1 = (a_1, 0, \dots, 0), w_2 = (0, a_2, 0, \dots, 0), \dots, w_d = (0, \dots, 0, a_d)$$

and  $\Gamma = \{m_1 \gamma_1 + m_2 \gamma_2 + \dots + m_d \gamma_d : m_i \in Z, i = 1, 2, \dots, d\}$  be the dual lattice of  $\Omega$ , where the vectors  $\{\gamma_i\}_{i=1}^d$  are biorthogonal to the vectors  $\{w_i\}_{i=1}^d$ . Denote by  $F \equiv [0, a_1] \times [0, a_2] \times \dots \times [0, a_d]$  the fundamental domain  $R^d/\Omega$  of the lattice  $\Omega$ .

We consider the Schrodinger operators  $L_D(q(x))$  and  $L_N(q(x))$ , defined by the differential expression

$$Lu = -\Delta u + q(x)u \tag{1}$$

in  $L_2(F)$  with the Dirichlet boundary condition

$$u|_{\partial F} = 0 \tag{2}$$

and the Neumann boundary condition

$$\frac{\partial u}{\partial n} \Big|_{\partial F} = 0, \tag{3}$$

respectively.

Here  $\partial F$  denotes the boundary of  $F$ ,  $x = (x_1, x_2, \dots, x_d) \in R^d$ ,  $d \geq 2$ ,  $\Delta$  is the Laplace operator in  $R^d$ ,  $\frac{\partial}{\partial n}$  denotes differentiation along outward normal  $n$  and  $q(x)$  is a real valued, periodic (with respect to lattice  $\Omega$ ) function of  $W_2^l(F)$ , where  $l \geq \frac{(d+2)(d-1)}{2} + d + 1$ .

First asymptotic formula for the eigenvalue of Schrodinger operator in parallelepiped with quasiperiodic boundary condition is obtained in papers [6], [7], [8]. The other asymptotic formulas for quasiperiodic boundary conditions in two and three dimensional cases are obtained in [4], [5], [1], [2]. The asymptotic formula for Dirichlet boundary condition in two dimension is obtained in [3].

We use the method of papers [7], [8] to find the asymptotic formula for the eigenvalues of  $L_D(q(x))$  and  $L_N(q(x))$  in arbitrary dimension.

We denote the eigenfunctions and the eigenvalues of the operator  $L_D(q(x))$  by  $\Phi_n$  and  $\mu_n$ , respectively and denote the eigenfunctions and the eigenvalues of the operator  $L_N(q(x))$  by  $\Psi_n$  and  $\Lambda_n$ , respectively.

The eigenvalues of the operators  $L_D(0)$  and  $L_N(0)$  are  $|\gamma|^2$  for  $\gamma \in \frac{\Gamma}{2}$ . The normalized eigenfunctions of the operators  $L_D(0)$  and  $L_N(0)$ , corresponding to the eigenvalue  $|\gamma|^2$  are  $\sum_{\alpha \in A_\gamma} (\text{sign} \prod_{i=1}^d \alpha_i) e^{i\langle \alpha, x \rangle}$  and  $\sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x \rangle}$ , respectively, where  $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_d) \in \frac{\Gamma}{2}$  and

$$A_\gamma = \{ \alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in R^d : |\alpha_i| = |\gamma_i|, i = 1, 2, \dots, d \}.$$

The potential  $q(x)$  in the expression (1) can be written in the form

$$q(x) = \sum_{\gamma \in \frac{\Gamma}{2}} q_\gamma \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x \rangle}, \tag{4}$$

where  $q_\gamma = \int_F q(x) \overline{\sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x \rangle}} dx$  for  $\gamma \in \frac{\Gamma}{2}$  (without loss of generality we can assume  $q_0 = \int_F q(x) dx = 0$ .) are the Fourier coefficients of the potential  $q(x)$  with respect to the basis  $\{ \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x \rangle} : \gamma \in \frac{\Gamma}{2} \}$ . Since  $q(x) \in W_2^l(F)$ , one can write

$$q(x) = \sum_{\gamma \in \Gamma(\rho^\alpha)} q_\gamma \sum_{\beta \in A_\gamma} e^{i\langle \beta, x \rangle} + O(\rho^{-p\alpha}), \quad (5)$$

where  $p = l - d$ ,  $\Gamma(\rho^\alpha) = \{\gamma \in \frac{\Gamma}{2} : 0 < |\gamma| < \rho^\alpha\}$ ,  $\alpha = 1/(d+2)$  and  $\rho$  is a large parameter.

Let us introduce the following notations:

$$M \equiv \sum_{\gamma \in \frac{\Gamma}{2}} |q_\gamma| \quad (6)$$

$$V_b(\rho^\alpha) \equiv \{x \in R^d : ||x|^2 - |x + b|^2| < \rho^\alpha\}$$

$$U(\rho^\alpha, p) \equiv R^d \setminus \bigcup_{b \in \Gamma(p\rho^\alpha)} V_b(\rho^\alpha).$$

The domain  $U(\rho^\alpha, p)$  is said to be non-resonance domain and the eigenvalues  $|\gamma|^2$  are called non-resonance eigenvalues, if  $\gamma \in U(\rho^\alpha, p)$ . The domains  $V_b(\rho^\alpha)$  for all  $b \in \Gamma(p\rho^\alpha)$  are called resonance domains and the eigenvalues  $|\gamma|^2$  are called resonance eigenvalues, if  $\gamma \in V_b(\rho^\alpha)$ . Note that the number of non-resonance eigenvalues is essentially greater than the number of resonance eigenvalues. Namely, if  $N_n(\rho)$  and  $N_r(\rho)$  denote the number of  $\gamma \in U(\rho^\alpha, p) \cap (R(2\rho) \setminus R(\rho))$  and  $\gamma \in \bigcup_{b \in \Gamma(p\rho^\alpha)} V_b(\rho^\alpha) \cap (R(2\rho) \setminus R(\rho))$ , respectively, then

$$\frac{N_r(\rho)}{N_n(\rho)} = O(\rho^{(d+1)\alpha-1}) = o(1) \quad (7)$$

for  $(d+1)\alpha < 1$  where  $R_\rho = \{x \in R^d : |x| \leq \rho\}$  (see remark 1).

In this paper, we obtain asymptotic formulas for non-resonance eigenvalues by using the following well-known formulas:

$$(\Lambda_n - |\gamma|^2)(\Psi_n, \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x \rangle}) = (\Psi_n, q(x) \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x \rangle}) \quad (8)$$

$$(\mu_n - |\gamma|^2)(\Phi_n, \sum_{\alpha \in A_\gamma} (\text{sign} \prod_{i=1}^n \alpha_i) e^{i\langle \alpha, x \rangle}) = (\Phi_n, q(x) \sum_{\alpha \in A_\gamma} (\text{sign} \prod_{i=1}^d \alpha_i) e^{i\langle \alpha, x \rangle}) \quad (9)$$

where  $(\cdot, \cdot)$  is the inner product in  $L_2(F)$ .

Note that (8) can be obtained from

$$-\Delta\Psi_n(x) + q(x)\Psi_n(x) = \Lambda_n\Psi_n(x) \tag{10}$$

by multiplying both sides of this equation by  $\sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x \rangle}$ .

The Formula (9) can be obtained in the same way.

We say that  $|\gamma|^2$  is of the order of  $\rho^2$  and write  $|\gamma|^2 \sim \rho^2$ , if  $c_1\rho^2 < |\gamma|^2 < c_2\rho^2$ , where by  $c_i, i = 1, 2, \dots$  we denote the positive, independent on  $\rho$  constants whose exact values are not important.

**Lemma 1** *Let  $|\gamma|^2$  be the eigenvalue of the operators  $L_D(0)$  and  $L_N(0)$  of the order of  $\rho^2$ . Then there are  $n_1$  and  $n_2$  such that  $|\Lambda_{n_1} - |\gamma|^2| < 2M$ ,  $|\mu_{n_2} - |\gamma|^2| < 2M, |(\Psi_{n_1}, \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x \rangle})| > c_3\rho^{\frac{-(d-1)}{2}}$  and  $|(\Phi_{n_2}, \sum_{\alpha \in A_\gamma} (\text{sign} \prod_{i=1}^d \alpha_i) e^{i\langle \alpha, x \rangle})| > c_4\rho^{\frac{-(d-1)}{2}}$ , where  $M$  is the number defined in (6).*

proof: It is well known that the set of eigenfunctions  $\Psi_n$  of the self-adjoint operator  $L_N(q(x))$  is an orthonormal basis in  $L_2(F)$ . Using (8) and (6) we get

$$\sum_{n: |\Lambda_n - |\gamma|^2| > 2M} |(\Psi_n(x), \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x \rangle})|^2 \leq \frac{1}{4}.$$

Hence by the Parsevals equality, we have

$$\sum_{n: |\Lambda_n - |\gamma|^2| \leq 2M} |(\Psi_n(x), \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x \rangle})|^2 > \frac{3}{4}. \tag{11}$$

On the other hand, it is well known that if  $a \sim \rho$  then the number of  $\gamma \in \frac{\mathbb{F}}{2}$  satisfying  $||\gamma| - a| < 1$  is less than  $c_5\rho^{d-1}$ . Therefore the number of eigenvalues of  $L_N(0)$  lying in  $(a^2 - \rho, a^2 + \rho)$  is less than  $c_6\rho^{d-1}$ . Since, by general perturbation theory, the  $n$ -th eigenvalue of  $L_N(q(x))$  lies in  $M$ -neighborhood of the  $n$ -th eigenvalue of  $L_N(0)$ , the number of the eigenvalues  $\Lambda_n$  of the operator  $L_N(q(x))$  in the interval  $I = [|\gamma|^2 - 2M, |\gamma|^2 + 2M]$  is less than  $c_7\rho^{d-1}$ . By this fact and the inequality (11), there

exists  $n_1 \in I$  such that

$$|(\Psi_{n_1}(x), \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x \rangle})| > c_3 \rho^{-\frac{(d-1)}{2}}$$

Similarly, by using (9) for  $\Phi_n(x)$ , we get

$$|(\Phi_{n_2}, \sum_{\alpha \in A_\gamma} (\text{sign} \prod_{i=1}^d \alpha_i) e^{i\langle \alpha, x \rangle})| > c_4 \rho^{-\frac{(d-1)}{2}}$$

The lemma is proved.  $\square$

**Lemma 2** *Let  $\gamma \in U(\rho^\alpha, p)$ , i.e.  $|\gamma|^2$  be the non-resonance eigenvalue of  $L_D(0)$  and  $L_N(0)$  and  $\Lambda_n$  and  $\mu_n$  be the eigenvalues of  $L_N(q(x))$  and  $L_D(q(x))$ , respectively, lying in the interval  $I = [|\gamma|^2 - 2M, |\gamma|^2 + 2M]$ , then  $|\Lambda_n - |\gamma + b|^2| > \frac{1}{2}\rho^\alpha$  and  $|\mu_n - |\gamma + b|^2| > \frac{1}{2}\rho^\alpha$  for all  $b \in \Gamma(m\rho_\alpha)$ .*

proof: If  $\gamma \in U(\rho^\alpha, p)$ , then for all  $b \in \Gamma(m\rho_\alpha)$  we have the inequality

$$||\gamma|^2 - |\gamma + b|^2| \geq \rho^\alpha$$

which, together with the fact that  $\Lambda_n \in I$ , implies

$$|\Lambda_n - |\gamma + b|^2| = |\Lambda_n - |\gamma + b|^2 \mp |\gamma|^2| \geq ||\gamma|^2 - |\gamma + b|^2| - |\Lambda_n - |\gamma|^2| \geq |\rho^\alpha - 2M|,$$

where  $\rho^\alpha$  is sufficiently large so the result follows. Similarly  $|\mu_n - |\gamma + b|^2| > \frac{1}{2}\rho^\alpha$ .  $\square$

**Theorem 1** *Let  $\gamma \in U(\rho^\alpha, p)$ ,  $|\gamma| \sim \rho$ ; i.e.,  $|\gamma|^2$  be non-resonance eigenvalue of the operators  $L_D(0)$  and  $L_N(0)$ . Then there exists an eigenvalue  $\Lambda_n$  of the operator  $L_N(q(x))$  and an eigenvalue  $\mu_n$  of the operator  $L_D(q(x))$  satisfying the following formulas :*

$$\Lambda_n = |\gamma|^2 + O(\rho^{-\alpha}) \tag{12}$$

$$\mu_n = |\gamma|^2 + O(\rho^{-\alpha}). \tag{13}$$

proof: First, we prove the theorem for  $L_N(q(x))$ , i.e., we prove (12).

By Lemma 1, there is an index  $n$  such that  $|\Lambda_n - |\gamma|^2| \leq 2M$  and

$|(\Psi_n(x), \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x \rangle})| > c_3 \rho^{-\frac{(d-1)}{2}}$ . We prove that this eigenvalue satisfies the Formula (12). Substituting the decomposition (5) of the potential  $q(x)$  in the Formula (8) we have:

$$(\Lambda_n - |\gamma|^2)(\Psi_n, \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x \rangle}) = \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} q_{\gamma_1}(\Psi_n, \sum_{\beta_1 \in A_{\gamma_1}} e^{i\langle \beta_1, x \rangle} \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x \rangle}) + O(\rho^{-p\alpha}).$$

Using the formula

$$\sum_{\beta_1 \in A_{\gamma_1}} e^{i\langle \beta_1, x \rangle} \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x \rangle} = \sum_{\beta_1 \in A_{\gamma_1}} \sum_{\alpha \in A_{\gamma+\beta_1}} e^{i\langle \alpha, x \rangle}, \quad (14)$$

which can be easily proved by direct calculation, we get

$$(\Lambda_n - |\gamma|^2)(\Psi_n, \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x \rangle}) = \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} \sum_{\beta_1 \in A_{\gamma_1}} q_{\gamma_1}(\Psi_n, \sum_{\alpha \in A_{\gamma+\beta_1}} e^{i\langle \alpha, x \rangle}) + O(\rho^{-p\alpha}).$$

Since  $\gamma + \beta_1 \in \frac{\Gamma}{2}$ , i.e.;  $|\gamma + \beta_1|^2$  is an eigenvalue of the operator  $L_N(0)$  with the corresponding eigenfunction  $\sum_{\alpha \in A_{\gamma+\beta_1}} e^{i\langle \alpha, x \rangle}$ , we can use the Formula (8). Therefore using (8) in the last equation we obtain

$$\begin{aligned} (\Lambda_n - |\gamma|^2)(\Psi_n, \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x \rangle}) &= \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} \sum_{\beta_1 \in A_{\gamma_1}} q_{\gamma_1} \frac{(\Psi_n, q(x) \sum_{\alpha \in A_{\gamma+\beta_1}} e^{i\langle \alpha, x \rangle}) + O(\rho^{-p\alpha})}{\Lambda_n - |\gamma + \beta_1|^2} \\ (\Lambda_n - |\gamma|^2)(\Psi_n, \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x \rangle}) &= \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} \sum_{\beta_1 \in A_{\gamma_1}} q_{\gamma_1} \frac{(\Psi_n, q(x) \sum_{\alpha \in A_{\gamma+\beta_1}} e^{i\langle \alpha, x \rangle})}{\Lambda_n - |\gamma + \beta_1|^2} \\ &\quad + O(\rho^{-p\alpha}) \end{aligned} \quad (15)$$

Here, we use the fact (see Lemma 2) that the denominator of the fraction in (15) satisfies

$$|\Lambda_n - |\gamma + \beta_1|^2| > \frac{1}{2} \rho^\alpha$$

since  $\beta_1 \in \Gamma(\rho^\alpha)$ . Again, substituting the decomposition of  $q(x)$  in Equation (15) and using the last inequality, we get

$$\begin{aligned} & (\Lambda_n - |\gamma|^2)(\Psi_n, \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x \rangle}) \\ = & \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} \sum_{\beta_1 \in A_{\gamma_1}} q_{\gamma_1} \frac{(\Psi_n, \sum_{\gamma_2 \in \Gamma(\rho^\alpha)} q_{\gamma_2} \sum_{\alpha \in A_{\gamma_2}} e^{i\langle \alpha, x \rangle} \sum_{\alpha \in A_{\gamma+\beta_1}} e^{i\langle \alpha, x \rangle}) + O(\rho^{-p\alpha})}{\Lambda_n - |\gamma + \beta_1|^2} \\ & + O(\rho^{-p\alpha}). \end{aligned}$$

Now using the Equation (14), we have

$$\begin{aligned} (\Lambda_n - |\gamma|^2)(\Psi_n, \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x \rangle}) = & \sum_{\gamma_1, \gamma_2 \in \Gamma(\rho^\alpha)} \sum_{\beta_1 \in A_{\gamma_1}, \beta_2 \in A_{\gamma_2}} q_{\gamma_1} q_{\gamma_2} \frac{(\Psi_n, \sum_{\alpha \in A_{\gamma+\beta_1+\beta_2}} e^{i\langle \alpha, x \rangle})}{\Lambda_n - |\gamma + \beta_1|^2} \\ & + O(\rho^{-p\alpha}) \end{aligned}$$

If the terms with coefficient  $(\Psi_n, \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x \rangle})$  are isolated, we obtain

$$\begin{aligned} (\Lambda_n - |\gamma|^2)(\Psi_n, \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x \rangle}) = & \sum_{\gamma_1, \gamma_2 \in \Gamma(\rho^\alpha)} \sum_{\substack{\beta_2 = -\beta_1 \\ \beta_1 \in A_{\gamma_1} \\ \beta_2 \in A_{\gamma_2}}} q_{\gamma_1} q_{\gamma_2} \frac{(\Psi_n, \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x \rangle})}{\Lambda_n - |\gamma + \beta_1|^2} \\ & + \sum_{\gamma_1, \gamma_2 \in \Gamma(\rho^\alpha)} \sum_{\substack{\beta_2 \neq -\beta_1 \\ \beta_1 \in A_{\gamma_1} \\ \beta_2 \in A_{\gamma_2}}} q_{\gamma_1} q_{\gamma_2} \frac{(\Psi_n, \sum_{\alpha \in A_{\gamma+\beta_1+\beta_2}} e^{i\langle \alpha, x \rangle})}{\Lambda_n - |\gamma + \beta_1|^2} + O(\rho^{-p\alpha}) \quad (16) \end{aligned}$$

By the same method as above, iterating  $p$  times the formula (16) and isolating each time the terms with multiplicand  $(\Psi_n, \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x \rangle})$ , we get

$$(\Lambda_n - |\gamma|^2)(\Psi_n, \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x \rangle}) = \left( \sum_{i=1}^p S_i \right) (\Psi_n, \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x \rangle}) + C_p + O(\rho^{-p\alpha}), \quad (17)$$



where

$$S_m(\Lambda_n) = \sum_{\gamma_1, \dots, \gamma_{m+1} \in \Gamma(\rho^\alpha)} \sum_{\substack{\beta_{m+1} = -(\beta_1 + \dots + \beta_m) \\ \beta_1 \in A_{\gamma_1}, \dots, \beta_{m+1} \in A_{\gamma_{m+1}}}} \frac{q_{\gamma_1} \dots q_{\gamma_{m+1}}}{(\Lambda_n - |\gamma + \beta_1|^2) \dots (\Lambda_n - |\gamma + \beta_1 + \dots + \beta_m|^2)} \quad (18)$$

$$C_p = \sum_{\gamma_1, \dots, \gamma_{p+1} \in \Gamma(\rho^\alpha)} \sum_{\substack{\beta_{p+1} \neq -(\beta_1 + \dots + \beta_p) \\ \beta_1 \in A_{\gamma_1}, \dots, \beta_{p+1} \in A_{\gamma_{p+1}}}} \frac{q_{\gamma_1} \dots q_{\gamma_{p+1}} (\Psi_n, \sum_{\alpha \in A_{\gamma + \beta_1 + \dots + \beta_{p+1}}} e^{i\langle \alpha, x \rangle})}{(\Lambda_n - |\gamma + \beta_1|^2) \dots (\Lambda_n - |\gamma + \beta_1 + \dots + \beta_p|^2)} \quad (19)$$

For all  $m = 1, 2, \dots, p, \gamma_m \in \Gamma(\rho^\alpha)$  and  $\beta_m \in A_{\gamma_m} \Rightarrow |\gamma_m| = |\beta_m| < \rho^\alpha$  and  $|\beta_1 + \beta_2 + \dots + \beta_m| < p\rho^\alpha$ , hence we can use Lemma 2 and the Equation (6). Then we have

$$\sum_{m=1}^p S_m(\Lambda_n) = O(\rho^{-\alpha}), \quad C_p = O(\rho^{-p\alpha}). \quad (20)$$

Taking into account that for  $\Lambda_n$ , we only used the condition  $\Lambda_n \in I$ , we have

$$\sum_{m=1}^p S_m(a) = O(\rho^{-\alpha}), \quad \forall a \in I \quad (21)$$

If we substitute (20) into (17), we get

$$(\Lambda_n - |\gamma|^2)(\Psi_n, \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x \rangle}) = O(\rho^{-\alpha})(\Psi_n, \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x \rangle}) + O(\rho^{-p\alpha}) \quad (22)$$

dividing both sides of the Equation (22) by  $(\Psi_n, \sum_{\alpha \in A_\gamma} e^{i\langle \alpha, x \rangle})$ , using Lemma 1 and the obvious inequality  $p\alpha > \frac{d-1}{2} + \alpha$  (see definition of  $p$  and  $\alpha$ ), we get the proof for  $L_N(q(x))$ .

By the same way, we can prove the theorem for  $L_D(q(x))$ , i.e, for the non-resonance eigenvalue  $|\gamma|^2$  of  $L_D(0)$  ( $\gamma \in U(\rho^\alpha, l)$ ), there is an eigenvalue  $\mu_n$  of  $L_D(q(x))$  such that

the Formula (13) is satisfied. Indeed, to prove this, instead of (8), we use the Formula (9) with the same decomposition (5) of  $q(x)$  and we get

$$\begin{aligned} & (\mu_n - |\gamma|^2)(\Phi_n, \sum_{\alpha \in A_\gamma} (\text{sign} \prod_{i=1}^d \alpha_i) e^{i\langle \alpha, x \rangle}) \\ = & \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} q_{\gamma_1}(\Phi_n, \sum_{\beta_1 \in A_{\gamma_1}} e^{i\langle \beta_1, x \rangle} \sum_{\alpha \in A_\gamma} (\text{sign} \prod_{i=1}^d \alpha_i) e^{i\langle \alpha, x \rangle}) + O(\rho^{-p\alpha}) \end{aligned}$$

and instead of (14), using the following formula

$$\sum_{\beta_1 \in A_{\gamma_1}} e^{i\langle \beta_1, x \rangle} \sum_{\alpha \in A_\gamma} (\text{sign} \prod_{i=1}^d \alpha_i) e^{i\langle \alpha, x \rangle} = \sum_{\beta_1 \in A_{\gamma_1}} \sum_{\alpha \in A_{\gamma+\beta_1}} (\text{sign} \prod_{i=1}^d \alpha_i) e^{i\langle \alpha, x \rangle}$$

we get

$$\begin{aligned} & (\mu_n - |\gamma|^2)(\Phi_n, \sum_{\alpha \in A_\gamma} (\text{sign} \prod_{i=1}^d \alpha_i) e^{i\langle \alpha, x \rangle}) \\ = & \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} \sum_{\beta_1 \in A_{\gamma_1}} q_{\gamma_1}(\Phi_n, \sum_{\alpha \in A_{\gamma+\beta_1}} (\text{sign} \prod_{i=1}^d \alpha_i) e^{i\langle \alpha, x \rangle}) + O(\rho^{-p\alpha}) \end{aligned}$$

By the similar considerations, we can iterate the above formula  $p$  times and by isolating the coefficient of  $(\Phi_n, \sum_{\alpha \in A_\gamma} (\text{sign} \prod_{i=1}^d \alpha_i) e^{i\langle \alpha, x \rangle})$ , we obtain the equation

$$\begin{aligned} (\mu_n - |\gamma|^2)(\Phi_n, \sum_{\alpha \in A_\gamma} (\text{sign} \prod_{i=1}^d \alpha_i) e^{i\langle \alpha, x \rangle}) &= (\sum_{m=1}^p S_m)(\Phi_n, \sum_{\alpha \in A_\gamma} (\text{sign} \prod_{i=1}^d \alpha_i) e^{i\langle \alpha, x \rangle}) \quad (23) \\ &+ C_p + O(\rho^{-p\alpha}), \end{aligned}$$

instead of (17), where  $S_m$  is the same as Equation (18) and

$$C_p = \sum_{\gamma_1, \dots, \gamma_{p+1} \in \Gamma(\rho^\alpha)} \sum_{\substack{\beta_{p+1} \neq -(\beta_1 + \dots + \beta_p) \\ \beta_1 \in A_{\gamma_1}, \dots, \beta_{p+1} \in A_{\gamma_{p+1}}}} \frac{q_{\gamma_1} \dots q_{\gamma_{p+1}} (\Phi_n, \sum_{\alpha \in A_{\gamma_1 + \beta_1 + \dots + \beta_{p+1}}} (\text{sign} \prod_{i=1}^d \alpha_i) e^{i\langle \alpha, x \rangle})}{(\mu_n - |\gamma + \beta_1|^2) \dots (\mu_n - |\gamma + \beta_1 + \dots + \beta_p|^2)}.$$

Hence, by similar calculations, we get the proof.  $\square$

**Theorem 2** *Let  $\gamma \in U(\rho^\alpha, p)$ ,  $|\gamma| \sim \rho$  then there is an eigenvalue  $\Lambda_n$  of the operator  $L_N(q(x))$  and an eigenvalue  $\mu_n$  of the operator  $L_D(q(x))$  satisfying the formulas*

$$\Lambda_n = |\gamma|^2 + F_{k-1} + O(\rho^{-k\alpha}), \tag{24}$$

and

$$\mu_n = |\gamma|^2 + F_{k-1} + O(\rho^{-k\alpha}), \tag{25}$$

for all  $k = 1, 2, \dots, p - z$  where

$$F_0 = 0, F_1 = \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} \sum_{\beta_1 \in A_{\gamma_1}} \frac{|q_{\gamma_1}|^2}{|\gamma|^2 - |\gamma - \beta_1|^2},$$

$$F_s = \sum_{i=1}^s S_i(|\gamma|^2 + F_{s-1}), s = 2, 3, \dots, p$$

and  $z = [\frac{d-1}{2\alpha}] + 1$ . ( $[\frac{d-1}{2\alpha}]$  is the integer part of  $\frac{d-1}{2\alpha}$ .)

proof: We prove that for the eigenvalues  $\Lambda_n$  and  $\mu_n$  satisfying the Formulas (12) and (13) the Formulas (24) and (25) hold, respectively. Let us prove it by mathematical induction on  $k$  :

for  $k = 1$  ; by Theorem 1,  $\Lambda_n$  and  $\mu_n$  satisfy the equations

$$\Lambda_n = |\gamma|^2 + F_0 + O(\rho^{-\alpha}),$$

$$\mu_n = |\gamma|^2 + F_0 + O(\rho^{-\alpha}),$$

where  $F_0 = 0$ ,

for  $k = j$  ; assume that it is true, i.e

$$\Lambda_n = |\gamma| + F_{j-1} + O(\rho^{-j\alpha}). \quad (26)$$

$$\mu_n = |\gamma| + F_{j-1} + O(\rho^{-j\alpha}). \quad (27)$$

For  $k = j + 1$ , we must prove that

$$\Lambda_n = |\gamma|^2 + F_j + O(\rho^{-(j+1)\alpha}), \quad (28)$$

$$\mu_n = |\gamma|^2 + F_j + O(\rho^{-(j+1)\alpha}). \quad (29)$$

To prove this we put Expression (26) into  $S_m(\Lambda_n)$  and (27) into  $S_m(\mu_n)$  and divide both sides of (17) by  $(\Psi_n(x), \sum_{\alpha \in A_\gamma} e^{i(\alpha, x)})$  and (23) by

$(\Phi_n(x), \sum_{\alpha \in A_\gamma} (\text{sign} \prod_{i=1}^d \alpha_i) e^{i(\alpha, x)})$  , we get

$$\Lambda_n = |\gamma|^2 + \sum_{m=1}^p S_m(|\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha}) + O(\rho^{-(p-z)\alpha})) \quad (30)$$

$$\mu_n = |\gamma|^2 + \sum_{m=1}^p S_m(|\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha}) + O(\rho^{-(p-z)\alpha})) \quad (31)$$

adding and subtracting the term  $\sum_{m=1}^p S_m(|\gamma|^2 + F_{j-1})$  in (30) and (31), we have

$$\begin{aligned} \Lambda_n = |\gamma|^2 + [ \sum_{m=1}^p S_m(|\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha})) - S_m(|\gamma|^2 + F_{j-1}) ] \\ + \sum_{m=1}^p S_m(|\gamma|^2 + F_{j-1}) + O(\rho^{-(p-z)\alpha}) \end{aligned} \quad (32)$$

$$\begin{aligned} \mu_n = |\gamma|^2 + & \left[ \sum_{m=1}^p S_m(|\gamma|^2 + F_{j-1} + O(\rho^{-j\alpha})) - S_m(|\gamma|^2 + F_{j-1}) \right] \\ & + \sum_{m=1}^p S_m(|\gamma|^2 + F_{j-1}) + O(\rho^{-(p-z)\alpha}) \end{aligned} \tag{33}$$

$\sum_{m=1}^p S_m(|\gamma|^2 + F_{j-1}) = F_j$ , so we need only to show that the expressions in the square brackets in (32) and (33) are equal to  $O(\rho^{-(j+1)\alpha})$ . First we prove that  $F_j = O(\rho^{-\alpha})$  for all  $j = 0, 1, 2, \dots, p$  by induction. By Theorem 1  $F_0 = 0$ . Suppose  $F_{j-1} = O(\rho^{-\alpha})$  then by (21)  $F_j = S_m(|\gamma|^2 + F_{j-1}) = O(\rho^{-\alpha})$ .

Using this and Lemma 2, we have

$$||\gamma|^2 + F_{j-1} + O(\rho^{-(j)\alpha}) - |\gamma + \beta_1 + \dots + \beta_m|^2| > \frac{1}{3}\rho^\alpha$$

$$||\gamma|^2 + F_{j-1} - |\gamma + \beta_1 + \dots + \beta_m|^2| > \frac{1}{3}\rho^\alpha, m = 1, 2, \dots, p$$

Hence, by direct calculations and using the above inequalities, it can be easily checked that the expressions in the square brackets are equal to  $O(\rho^{-(j+1)\alpha})\square$ .

**Remark 1** *It is clear that  $V_b(\rho^\alpha) \cap (R(2\rho) \setminus R(\rho))$  is the part of  $(R(2\rho) \setminus R(\rho))$  which is contained between two parallel hyperplanes  $\{x : |x|^2 - |x + b|^2 = -\rho^\alpha\}$  and  $\{x : |x|^2 - |x + b|^2 = \rho^\alpha\}$ . The distance of this hyperplanes from the origin is  $\frac{\rho^\alpha}{|b|}$ . Therefore  $\mu(V_b(\rho^\alpha) \cap (R(2\rho) \setminus R(\rho))) = O(\rho^{d-1+\alpha})$ . Since the number of the vectors  $\gamma$  in  $\Gamma(p\rho^\alpha)$  is equal to  $\rho^{d\alpha}$  and  $\mu(R(2\rho) \setminus R(\rho)) \sim \rho^d$ , we have  $\mu(\bigcup_{b \in \Gamma(p\rho^\alpha)} V_b(\rho^\alpha) \cap (R(2\rho) \setminus R(\rho))) = O(\rho^{d-1+(d+1)\alpha})$  and  $\mu(U(\rho^\alpha, p) \cap (R(2\rho) \setminus R(\rho))) = \mu((R(2\rho) \setminus R(\rho)))(1 + O(\rho^{(d+1)\alpha-1}))$  from which we get (7).*

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