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Characterizations of Artinian and Noetherian Gamma-Rings in Terms of Fuzzy Ideals

Mehmet Ali Öztürk, Mustafa Uckun and Young Bae Jun

Abstract

Using fuzzy ideals, characterizations of Noetherian Γ-rings are given, and a condition for a Γ-ring to be Artinian is also given.

Key words and phrases: (Artinian, Noetherian) Γ-ring, fuzzy left (right) ideal, Γ-residue class ring.

1. Introduction

The notion of a fuzzy set in a set was introduced by L. A. Zadeh [6], and since then this concept has been applied to various algebraic structures. N. Nobusawa [5] introduced the notion of a Γ-ring, a concept more general than a ring. W. E. Barnes [1] weakened slightly the conditions in the definition of Γ-ring in the sense of Nobusawa. W. E. Barnes [1], S. Kyuno [3] and J. Luh [4] studied the structure of Γ-rings and obtained various generalizations analogous to corresponding parts in ring theory. Y. B. Jun and C. Y. Lee [2] applied the concept of fuzzy sets to the theory of Γ-rings. In this paper, using fuzzy ideals, we discuss characterizations of Noetherian Γ-rings, and we give a condition for a Γ-ring to be Artinian.

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2. Preliminaries

Let $M$ and $\Gamma$ be two abelian groups. If for all $x, y, z \in M$ and all $\alpha, \beta \in \Gamma$ the conditions

- $x\alpha y \in M$,
- $(x + y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha + \beta)z = x\alpha z + x\beta z$, $x\alpha(y + z) = x\alpha y + x\alpha z$,
- $(x\alpha y)\beta z = x\alpha(y\beta z)$

are satisfied, then we call $M$ a $\Gamma$-ring. By a right (resp. left) ideal of a $\Gamma$-ring $M$ we mean an additive subgroup $U$ of $M$ such that $U\Gamma \subseteq U$ (resp. $MU \subseteq U$). If $U$ is both a right and a left ideal, then we say that $U$ is an ideal of $M$. Let $U$ be an ideal of a $\Gamma$-ring $M$. If for each $a + U$, $b + U$ in the factor group $M/U$, and each $\gamma \in \Gamma$, we define $(a + U)\gamma(b + U) = a\gamma b + U$, then $M/U$ is a $\Gamma$-ring which is called the $\Gamma$-residue class ring of $M$ with respect to $U$ (see [3]). For any subsets $A$ and $B$ of a $\Gamma$-ring $M$, by $A \subseteq B$ we exclude the possibility that $A = B$. A $\Gamma$-ring $M$ is said to satisfy the left (right) ascending chain condition of left (right) ideals (or to be left (right) Noetherian) if every strictly increasing sequence $U_1 \subset U_2 \subset U_3 \subset \cdots$ of left (right) ideals of $M$ is of finite length. A $\Gamma$-ring $M$ is said to satisfy the left (right) descending chain condition of left (right) ideals (or to be left (right) Artinian) if every strictly decreasing sequence $V_1 \supset V_2 \supset V_3 \supset \cdots$ of left (right) ideals of $M$ is of finite length. A $\Gamma$-ring $M$ is said to be left (resp. right) Noetherian if $M$ satisfies the left (right) ascending chain condition on left (resp. right) ideals. $M$ is said to be Noetherian if $M$ is both left and right Noetherian. A $\Gamma$-ring $M$ is left (resp. right) Artinian if $M$ satisfies the left (right) descending chain condition on left (resp. right) ideals. $M$ is said to be Artinian if $M$ is both left and right Artinian.

We now review some fuzzy logic concepts. A fuzzy set $\mu$ in a $\Gamma$-ring $M$ is called a fuzzy left (resp. right) ideal of $M$ ([2]) if it satisfies

(FI1) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$

(FI2) $\mu(x\gamma y) \geq \mu(y)$ (resp. $\mu(x\gamma y) \geq \mu(x)$)

for all $x, y \in M$ and $\gamma \in \Gamma$. A fuzzy set $\mu$ in a $\Gamma$-ring $M$ is called a fuzzy ideal of $M$ if $\mu$ is both a fuzzy left and a fuzzy right ideal of $M$. We note from [2] that if $\mu$ is a fuzzy
left (right) ideal of a \(\Gamma\)-ring \(M\) then \(\mu(0) \geq \mu(x)\) for all \(x \in M\), and \(\mu\) is a fuzzy ideal of a \(\Gamma\)-ring \(M\) if and only if it satisfies (FI1) and

(FI3) \(\mu(x\gamma y) \geq \max\{\mu(x), \mu(y)\}\) for all \(x, y \in M\) and \(\gamma \in \Gamma\).

3. Main results

**Theorem 3.1.** Let \(U\) be an ideal of a \(\Gamma\)-ring \(M\). If \(\mu\) is a fuzzy left (right) ideal of \(M\), then the fuzzy set \(\tilde{\mu}\) of \(M/U\) defined by

\[
\tilde{\mu}(a + U) = \sup_{x \in U} \mu(a + x)
\]

is a fuzzy left (right) ideal of the \(\Gamma\)-residue class ring \(M/U\) of \(M\) with respect to \(U\).

**Proof.** Let \(a, b \in M\) be such that \(a + U = b + U\). Then \(b = a + y\) for some \(y \in U\), and so

\[
\tilde{\mu}(b + U) = \sup_{x \in U} \mu(b + x) = \sup_{x \in U} \mu(a + y + x) = \sup_{x + y = z \in U} \mu(a + z) = \tilde{\mu}(a + U).
\]

Hence \(\tilde{\mu}\) is well-defined. For any \(x + U, y + U \in M/U\) and \(\gamma \in \Gamma\), we have

\[
\tilde{\mu}((x + U) - (y + U)) = \tilde{\mu}((x - y) + U) = \sup_{z \in U} \mu((x - y) + z)
\]

\[
= \sup_{z = u - v \in U} \mu((x - y) + (u - v))
\]

\[
= \sup_{u, v \in U} \mu((x + u) - (y + v))
\]

\[
\geq \sup_{u, v \in U} \min\{\mu(x + u), \mu(y + v)\}
\]

\[
= \min\left\{\sup_{u \in U} \mu(x + u), \sup_{v \in U} \mu(y + v)\right\}
\]

\[
= \min\left\{\tilde{\mu}(x + U), \tilde{\mu}(y + U)\right\}
\]

and

\[
\tilde{\mu}((x + U)\gamma(y + U)) = \tilde{\mu}(x\gamma y + U) = \sup_{z \in U} \mu(x\gamma y + z)
\]

\[
\geq \sup_{z \in U} \mu(x\gamma y + x\gamma z)\quad\text{because} \quad x\gamma z \in U
\]

\[
= \sup_{z \in U} \mu(x\gamma(y + z)) \geq \sup_{z \in U} \mu(y + z)
\]

\[
= \tilde{\mu}(y + U).
\]
Similarly, \( \bar{\mu}(x + U)\gamma(y + U) \geq \bar{\mu}(x + U) \). Hence \( \bar{\mu} \) is a fuzzy left (right) ideal of \( M/U \).

\[ \square \]

**Theorem 3.2.** Let \( U \) be an ideal of a \( \Gamma \)-ring \( M \). Then there is a one-to-one correspondence between the set of fuzzy left ideals \( \mu \) of \( M \) such that \( \mu(0) = \mu(u) \) for all \( u \in U \) and the set of all fuzzy left ideals \( \bar{\mu} \) of \( M/U \).

**Proof.** Let \( \mu \) be a fuzzy left ideal of \( M \). Using Theorem 3.1, we find that \( \bar{\mu} \) defined by \( \bar{\mu}(a + U) = \sup_{x \in U} \mu(a + x) \) is a fuzzy left ideal of \( M/U \). Since \( \mu(0) = \mu(u) \) for all \( u \in U \), we get

\[ \mu(a + u) \geq \min\{\mu(a), \mu(u)\} = \mu(a). \]

Again, \( \mu(a) = \mu(a + u - u) \geq \min\{\mu(a + u), \mu(u)\} = \mu(a + u) \). Hence \( \mu(a + u) = \mu(a) \) for all \( u \in U \), that is, \( \bar{\mu}(a + U) = \mu(a) \). Therefore the correspondence \( \mu \mapsto \bar{\mu} \) is injective. Now let \( \bar{\mu} \) be any fuzzy left ideal of \( M/U \) and define a fuzzy set \( \mu \) in \( M \) by \( \mu(a) = \bar{\mu}(a + U) \) for all \( a \in M \). For every \( x, y \in M \) and \( \gamma \in \Gamma \), we have

\[ \mu(x - y) = \bar{\mu}((x - y) + U) = \bar{\mu}((x + U) - (y + U)) \geq \min\{\bar{\mu}(x + U), \bar{\mu}(y + U)\} = \min\{\mu(x), \mu(y)\}, \]

and \( \mu(x\gamma) = \bar{\mu}(x\gamma + U) = \bar{\mu}(x + U)\gamma(y + U) \geq \bar{\mu}(y + U) = \mu(y) \). Thus \( \mu \) is a fuzzy left ideal of \( M \). Note that \( \mu(z) = \bar{\mu}(z + U) = \bar{\mu}(U) \) for all \( z \in U \), which shows that \( \mu(z) = \mu(0) \) for all \( z \in U \). This completes the proof. \( \square \)

**Theorem 3.3.** If every fuzzy left ideal of a \( \Gamma \)-ring \( M \) has finite number of values, then \( M \) is left Artinian.

**Proof.** Suppose that every fuzzy left ideal of a \( \Gamma \)-ring \( M \) has finite number of values and \( M \) is not left Artinian. Then there exists strictly descending chain \( U_0 \supset U_1 \supset U_2 \supset \cdots \) of left ideals of \( M \). Define a fuzzy set \( \mu \) in \( M \) by

\[ \mu(x) = \begin{cases} \frac{n}{n+1} & \text{if } x \in U_n \setminus U_{n+1}, \ n = 0, 1, 2, \cdots, \\ 1 & \text{if } x \in \bigcap_{n=0}^{\infty} U_n, \end{cases} \]

where \( U_0 \) stands for \( M \). Let us prove that \( \mu(x - y) \geq \min\{\mu(x), \mu(y)\} \) for all \( x, y \in M \). Let \( x, y \in M \). Then \( x - y \in U_n \setminus U_{n+1} \) for some \( n \) (\( n = 0, 1, 2, \cdots \)), and so either \( x \notin U_{n+1} \)

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or \( y \notin U_{n+1} \). So for definiteness, let \( y \in U_k \setminus U_{k+1} \) for \( k \leq n \). It follows that

\[
\mu(x - y) = \frac{n}{n+1} \geq \frac{k}{k+1} \geq \min\{\mu(x), \mu(y)\}.
\]

Next, let us show that \( \mu(x \gamma y) \geq \mu(y) \) for all \( x, y \in M \) and \( \gamma \in \Gamma \). There exists a non-negative integer \( n \) such that \( x \gamma y \in U_n \setminus U_{n+1} \). Then \( y \notin U_{n+1} \), and hence \( y \in U_k \setminus U_{k+1} \) for \( k \leq n \). Hence

\[
\mu(x \gamma y) = \frac{n}{n+1} \geq \frac{k}{k+1} = \mu(y).
\]

Therefore \( \mu \) is a fuzzy left ideal of \( M \) and \( \mu \) has infinite number of different values. This contradiction proves that \( M \) is a left Artinian \( \Gamma \)-ring. \( \square \)

**Theorem 3.4.** A \( \Gamma \)-ring \( M \) is left Noetherian if and only if the set of values of any fuzzy left ideal of \( M \) is a well ordered subset of \([0, 1]\).

**Proof.** Suppose that \( \mu \) is a fuzzy left ideal of \( M \) whose set of values is not a well ordered subset of \([0, 1]\). Then there exists a strictly decreasing sequence \( \{\lambda_n\} \) such that \( \mu(x_n) = \lambda_n \). Denote by \( U_n \) the set \( \{x \in M \mid \mu(x) \geq \lambda_n\} \). Then \( U_1 \subset U_2 \subset U_3 \subset \cdots \) is a strictly ascending chain of left ideals of \( M \), which contradicts that \( M \) is left Noetherian.

Conversely, assume that the set of values of any fuzzy left ideal of \( M \) is a well ordered subset of \([0, 1]\) and \( M \) is not a left Noetherian \( \Gamma \)-ring. Then there exists a strictly ascending chain

\[
U_1 \subset U_2 \subset U_3 \subset \cdots
\]

(3.1)

of left ideals of \( M \). Note that \( U := \bigcup_{i \in \mathbb{N}} U_i \) is a left ideal of \( M \), where \( \mathbb{N} \) is the set of all natural numbers. Define a fuzzy set \( \mu \) in \( M \) by

\[
\mu(x) = \begin{cases} 
0 & \text{if } x \notin U_i, \\
\frac{1}{k} & \text{where } k = \min\{i \in \mathbb{N} \mid x \in U_i\}.
\end{cases}
\]

It can be easily seen that \( \mu \) is a fuzzy left ideal of \( M \). Since the chain (3.1) is not terminating, \( \mu \) has a strictly descending sequence of values, contradicting that the value set of any fuzzy left ideal is well ordered. Consequently, \( M \) is left Noetherian. \( \square \)
Lemma 3.5. ([2, Theorem 3]) A fuzzy set $\mu$ in a $\Gamma$-ring $M$ is a fuzzy left (right) ideal of $M$ if and only if for every $\lambda \in [0, 1]$, the set $U(\mu; \lambda) := \{ x \in M \mid \mu(x) \geq \lambda \}$ is a left (right) ideal of $M$ when it is nonempty.

Lemma 3.6. Let $S = \{ \lambda_n \in (0, 1) \mid n \in \mathbb{N} \} \cup \{ 0 \}$, where $\lambda_i > \lambda_j$ whenever $i < j$. Let $\{ U_n \mid n \in \mathbb{N} \}$ be a family of left ideals of a $\Gamma$-ring $M$ such that $U_1 \subset U_2 \subset U_3 \subset \cdots$. Then a fuzzy set $\mu$ in $M$ defined by

$$
\mu(x) = \begin{cases}
\lambda_1 & \text{if } x \in U_1, \\
\lambda_n & \text{if } x \in U_n \setminus U_{n-1}, \ n = 2, 3, \cdots, \\
0 & \text{if } x \in M \setminus \bigcup_{n=1}^{\infty} U_n,
\end{cases}
$$

is a fuzzy left ideal of $M$.

Proof. Using Lemma 3.5, the proof is straightforward.

Theorem 3.7. Let $S = \{ \lambda_1, \lambda_2, \cdots, \lambda_n, \cdots \} \cup \{ 0 \}$ where $\{ \lambda_n \}$ is a fixed sequence, strictly decreasing to 0 and $0 < \lambda_n < 1$. Then a $\Gamma$-ring $M$ is left Noetherian if and only if for each fuzzy left ideal $\mu$ of $M$, $\text{Im}(\mu) \subset S$ implies that there exists $n_0 \in \mathbb{N}$ such that $\text{Im}(\mu) \subset \{ \lambda_1, \lambda_2, \cdots, \lambda_{n_0} \} \cup \{ 0 \}$.

Proof. If $M$ is left Noetherian, then $\text{Im}(\mu)$ is a well ordered subset of $[0, 1]$ by Theorem 3.4 and so the condition is necessary by noticing that a set is well ordered if and only if it does not contain any infinite descending sequence. Conversely, if possible let $M$ be not left Noetherian. Then there exists a strictly ascending chain of left ideals of $M$ $U_1 \subset U_2 \subset U_3 \subset \cdots$. Define a fuzzy set $\mu$ in $M$ by

$$
\mu(x) = \begin{cases}
\lambda_1 & \text{if } x \in U_1, \\
\lambda_n & \text{if } x \in U_n \setminus U_{n-1}, \ n = 2, 3, \cdots, \\
0 & \text{if } x \in M \setminus \bigcup_{n=1}^{\infty} U_n,
\end{cases}
$$

Then, by Lemma 3.6, $\mu$ is a fuzzy left ideal of $M$. This contradicts our assumption. Hence $M$ is left Noetherian.

References


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