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## New Tensor Norms and Operator Ideals Associated to Interpolation Spaces Between Sequence Spaces

*G. Castañeda, J. A. López Molina, M. J. Rivera*

### Abstract

We introduce a wide class of tensor norms  $g_{\lambda, \rho}$  which are defined with the help of interpolation spaces between perfect sequence spaces defined by a general parameter real interpolation method. We also characterize the associated  $\lambda_\rho$ -nuclear and  $\lambda_\rho$ -integral operators.

**Key Words:** Tensor norms, operator ideals,  $\lambda_\rho$ -nuclear and  $\lambda_\rho$ -integral operators.

### 1. Introduction: perfect sequence spaces and interpolation spaces

The excellent book [5] of Defant and Floret contains a complete update on the interplay between tensor norms and operator ideals in the class  $BAN$  of all Banach spaces. However after the publication of [5] some new interesting results on these topics with more or less direct relation with interpolation spaces has been appeared in the literature. These results are closely connected with the paper of Matter [14] concerning the ideal of  $(p, \sigma)$ -absolutely continuous operators and deals with the study of the typical tensor norms associated with Matter's operator ideals (see the papers of López Molina and Sánchez Pérez [13] and Arango, López Molina and Rivera [1]).

Latter papers suggest the role can be played by interpolation spaces in order to define new tensor norms and operator ideals which can be characterized by factorization properties of the involved operators. *The purpose of this paper is to develop this research program where the space  $\ell^p$  of classical tensor norms of Saphar is replaced by an interpolation space between perfect sequence Banach spaces, defined by general real interpolation*

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*methods.* This idea is not completely new. In an essential way it appears already in the Doctoral Dissertation of Harksen [6] in 1979, where it is used simply to provide *examples* of general tensor norms. However we have been able to get more deep results which extends until a characterization of the associated nuclear and integral operators. Moreover, our results point out the essential ingredients of the classical tensor norms  $g_p$  of Saphar which are hidden in his work by the very special properties of  $L^p$  spaces.

Given a tensor norm  $\alpha$  in the class  $BAN$  three main problems arise:

**Problem 1.** Characterize the operator ideal  $\mathcal{I}_\alpha$  associated with the tensor norm  $\alpha$  (the  $\alpha$ -integral operators).

**Problem 2.** Characterize the operator ideal  $\Pi_\alpha$  associated with the dual tensor norm  $\alpha'$ , i.e. compute the dual Banach space  $(E \otimes_\alpha F)'$  for  $E, F \in BAN$ .

**Problem 3.** Characterize the minimal operator ideal corresponding to  $\alpha$  (the  $\alpha$ -nuclear operators).

As we have said, we solve these problems for a wide class of tensor norms defined with help of general real interpolation spaces between perfect sequence spaces. Section 1 of the paper gives an account of definitions and results on function norms, perfect Banach sequence spaces and interpolation spaces which are needed for our purposes. Section 2 contains the construction of the tensor norm  $g_{\lambda, \rho}$  derived from interpolation spaces of the pair of perfect sequence spaces  $\lambda := (\lambda_0, \lambda_1)$  defined through the function norm  $\rho$  and solves quoted problem 2. Problem 3 is solved in section 2 and the more difficult problem 1 is solved in last section 3.

All the vector spaces we use are defined over the field  $\mathbb{R}$  of real numbers. In general our notation is standard.  $BAN$  will denote the class of all Banach spaces. If  $E \in BAN$ , we denote by  $FIN(E)$  the set of all finite dimensional subspaces  $N \subset E$ . For every  $E \in BAN$ ,  $J_E$  will be the canonical inclusion  $E \subset E''$ . In order to reduce the length of the paper we assume the reader is familiar with classical theory of tensor norms and operator ideals and its basic tools such as ultraproducts of Banach spaces as soon as Banach lattices theory. For all definitions and topics concerning these materials we refer the reader to [5], [7] and [15] respectively. However a few words on ultraproducts follows, mainly to set the notation.

Given a non void index set  $\mathfrak{D}$  and a Banach space  $E_d$  for every  $d \in \mathfrak{D}$ , we denote  $\ell^\infty((E_d)) := \{(x_d) \in \prod_{d \in \mathfrak{D}} E_d / \sup_{d \in \mathfrak{D}} \|x_d\| < \infty\}$ . Given an ultrafilter  $\mathcal{U}$  on  $\mathfrak{D}$ , we put  $Z_{\mathcal{U}} := \{(x_d) \in \ell^\infty((E_d)) / \lim_{d, \mathcal{U}} \|x_d\| = 0\}$ . Given  $(x_d) \in \ell^\infty((E_d))$ , its class in the quotient set  $\ell^\infty((E_d))/Z_{\mathcal{U}}$  is denoted by  $(x_d)_{\mathcal{U}}$ . Then the ultraproduct space  $(E_d)_{\mathcal{U}}$  is the quotient Banach space  $\ell^\infty((E_d))/Z_{\mathcal{U}}$  whose canonical quotient norm equals the norm  $\|(x_d)_{\mathcal{U}}\| := \lim_{d, \mathcal{U}} \|x_d\|$ . If  $E_d = E$  for every  $d \in \mathfrak{D}$ ,  $(E_d)_{\mathcal{U}}$  is written  $(E)_{\mathcal{U}}$  and it is called an ultrapower of  $E$ .

Given a family  $\{T_d : E_d \longrightarrow F_d \mid d \in \mathfrak{D}\}$  of continuous linear maps between Banach spaces  $E_d$  and  $F_d$  such that  $\sup_{d \in \mathfrak{D}} \|T_d\| < \infty$  we can define the canonical ultraproduct map  $(T_d)_{\mathcal{U}} : (E_d)_{\mathcal{U}} \longrightarrow (F_d)_{\mathcal{U}}$  by

$$\forall (x_d)_{\mathcal{U}} \in (E_d)_{\mathcal{U}} \quad (T_d)_{\mathcal{U}}((x_d)_{\mathcal{U}}) = (T_d(x_d))_{\mathcal{U}}. \quad (1)$$

If every  $E_d, d \in \mathfrak{D}$  is a Banach lattice,  $(E_d)_{\mathcal{U}}$  also is a Banach lattice with the canonical order given by  $(x_d)_{\mathcal{U}} \leq (y_d)_{\mathcal{U}}$  if and only if there is  $(\bar{x}_d) \in (x_d)_{\mathcal{U}}$  and  $(\bar{y}_d) \in (y_d)_{\mathcal{U}}$  such that  $\bar{x}_d \leq \bar{y}_d$  for every  $d \in \mathfrak{D}$ . Then  $(x_d)_{\mathcal{U}} \wedge (y_d)_{\mathcal{U}} = (x_d \wedge y_d)_{\mathcal{U}}$ . Moreover, it is easy to check that if  $(x_d)_{\mathcal{U}} \wedge (y_d)_{\mathcal{U}} = 0$ , we can choose representatives  $(\bar{x}_d) \in (x_d)_{\mathcal{U}}$  and  $(\bar{y}_d) \in (y_d)_{\mathcal{U}}$  such that  $\bar{x}_d \wedge \bar{y}_d = 0$  for all  $d \in \mathfrak{D}$ .

The theory of sequence spaces goes back to Köthe (see [10]). Let  $\varphi$  be the set of all sequences  $(x_i) \in \mathbb{R}^{\mathbb{N}}$  with only a finite number of non null components. A sequence space  $\lambda$  is a linear subspace of  $\mathbb{R}^{\mathbb{N}}$  containing  $\varphi$  and provided with a locally convex topology. In such a spaces the symbol  $e_i$  denotes always the sequence  $(0, 0, \dots, 0, 1, 0, 0, \dots)$  with 1 in the  $i$ -th position. Analogously, when we use Bochner spaces  $\lambda[\nu]$  of vector valued sequences where  $\nu$  is another sequence space,  $e_{nm}$  denotes the infinite matrix with 0 in all entries except 1 in the  $n$ -th file and  $m$ -th column.

If  $\lambda$  is a sequence space the  $\alpha$ -dual  $\lambda^\times$  is defined as

$$\lambda^\times := \left\{ (x_i) \in \mathbb{R}^{\mathbb{N}} \mid \sum_{i=1}^{\infty} |x_i y_i| < \infty \quad \forall (y_i) \in \lambda \right\}.$$

In such a case  $(\lambda, \lambda^\times)$  is a dual pair under the canonical bilinear form

$$\forall (x_i) \in \lambda, \forall (y_i) \in \lambda^\times \quad \langle (x_i), (y_i) \rangle = \sum_{i=1}^{\infty} x_i y_i.$$

*In this paper we shall use only sequence spaces  $\lambda$  which are Banach spaces when they are provided with the strong topology  $\beta(\lambda, \lambda^\times)$ . A sequence space is said to be perfect if  $\lambda^{\times \times} := (\lambda^\times)^\times = \lambda$ . Clearly a perfect Banach sequence space is a Banach lattice under the natural order.*

Let  $\lambda$  be a sequence space. The regular subspace  $\lambda^r$  of  $\lambda$  is defined as the closure of  $\varphi$  in  $\lambda$ . An useful fact is as follows (see lemma 3.3 in [9]):

**Lemma 1** *If a sequence space  $\lambda$  is a sublattice of  $\mathbb{R}^{\mathbb{N}}$  (with the canonical pointwise order) then*

$$\lambda^r := \left\{ (x_i) \in \lambda \ / \ (x_i) = \lim_{n \rightarrow \infty} \sum_{i=1}^n x_i e_i \right\}$$

*(the limit is taken in the topology of  $\lambda$ ).*

The next result is due to T. Komura and Y. Komura (see [9], lemma 1.3 and lemma 3.3):

**Proposition 2** *Let  $\lambda$  be a perfect space. Then  $(\lambda^r)' = (\lambda^r)^\times = \lambda^\times$ .*

Let  $\mathcal{M}$  be the set of Lebesgue measurable real functions on  $]0, \infty[$  and  $\mathcal{M}^+ \subset \mathcal{M}$  the set of  $f \in \mathcal{M}$  such that  $f(t) \geq 0$  for every  $t \in ]0, \infty[$ . Let us denote the Lebesgue measure on  $]0, \infty[$  by  $dt$ . Classical Banach spaces  $L^p(]0, \infty[, dt)$ ,  $1 \leq p \leq \infty$  will be denoted simply by  $L^p(dt)$  if there is no risk of confusion. If  $f \in \mathcal{M}^+$ ,  $L^p(f, dt)$ ,  $1 \leq p \leq \infty$  denotes the weighted space

$$L^p(f, dt) := \{g \ / \ \|f g\|_{L^p(dt)} < \infty\}.$$

The theory of general Banach function spaces and function norms has been developed by Lorentz and Luxemburg. All we need on this topic can be found, for instance, in [2]. Roughly speaking, a function norm is a map  $\rho : \mathcal{M}^+ \rightarrow [0, \infty]$  such that the set of classes of functions (modulo equality almost everywhere with respect the Lebesgue measure)

$$L^\rho := \{f : ]0, \infty[ \rightarrow \mathbb{R} \ / \ \rho(|f|) < \infty\}$$

contains all characteristic functions  $\chi_{]a,b[}$ ,  $0 < a < b < \infty$  and becomes a Banach lattice when it is endowed with the norm  $\|f\|_\rho := \rho(|f|)$ . The precise conditions to get that are not relevant for this paper. Thus we omit them. The interested reader can see all details in [2].

We refer the reader to book [4] for the theory of interpolation spaces defined by the real method. However we present in a concise way the basic facts we shall use in this paper. A couple  $(E_0, E_1)$  of Banach spaces is said to be a compatible couple if there is a Hausdorff topological vector space  $E$  such that  $E_0 \subset E$  and  $E_1 \subset E$  with continuous inclusions. Given a compatible couple  $(E_0, E_1)$  the spaces  $E_0 \cap E_1$  and  $E_0 + E_1$  are well defined and they become Banach spaces when provided with the canonical norms

$$\forall x \in E_0 \cap E_1 \quad \|x\|_{E_0 \cap E_1} := \max\{\|x\|_{E_0}, \|x\|_{E_1}\}$$

and

$$\forall x \in E_0 + E_1 \quad \|x\|_{E_0+E_1} = \inf \{ \|x_0\|_{E_0} + \|x_1\|_{E_1} \mid x = x_0 + x_1, x_0 \in E_0, x_1 \in E_1 \}$$

respectively.

An interpolation space of the compatible couple  $(E_0, E_1)$  is a Banach space  $X$  such that  $E_0 \cap E_1 \subset X \subset E_0 + E_1$  with continuous inclusions and such that for every continuous linear map  $T : E_0 + E_1 \longrightarrow E_0 + E_1$  with the property that  $T|_{E_i} \in \mathcal{L}(E_i, E_i)$ ,  $i = 0, 1$  then  $T|_X \in \mathcal{L}(X, X)$ . In this paper we shall deal with interpolation spaces defined by the so-called  $K$ - and  $J$ -methods.

The  $J$ -functional of Peetre is defined in  $]0, \infty[ \times (E_0 \cap E_1)$  by

$$J(t; x) := \max\{\|x\|_{E_0}, t \|x\|_{E_1}\}$$

and the  $K$ -functional of Peetre is defined in  $]0, \infty[ \times (E_0 + E_1)$  as

$$K(t; x) := \inf \{ \|x_0\|_{E_0} + t \|x_1\|_{E_1} \mid x = x_0 + x_1, x_0 \in E_0, x_1 \in E_1 \}.$$

Then we define

$$(E_0, E_1)_{\rho, K} = \left\{ x \in E_0 + E_1 \mid \|x\|_{\rho, K} := \rho \left( \frac{K(t; x)}{t} \right) < \infty \right\}$$

and

$$(E_0, E_1)_{\rho, J} = \left\{ x \in E_0 + E_1 \mid \|x\|_{\rho, J} := \inf \left\{ \rho \left( \frac{J(t; u(t))}{t} \right) \mid x = \int_0^\infty \frac{u(t)}{t} dt \right\} < \infty \right\},$$

where  $u : ]0, \infty[ \longrightarrow E_0 \cap E_1$  is strongly measurable and the written Bochner integral is convergent in  $E_0 + E_1$ .

The next proposition collects important known facts about function norms and interpolation spaces that we shall need in the sequel.

**Proposition 3** (1) *Let  $\rho$  be a function norm such that the function  $h : t \in ]0, \infty[ \longrightarrow \min(1, \frac{1}{t})$  lies in  $L^\rho$ . Then*

(a)  $(E_0, E_1)_{\rho, K}$  is an interpolation space of the couple  $(E_0, E_1)$  (the interpolated space defined by the real method  $K$ ).

(b) Let  $\omega_\rho := \rho(h)$  be the norm in  $L^\rho$  of the real function  $h$ . Then the inclusion maps

$$I_{E_0 E_1}^{1\rho} : E_0 \cap E_1 \longrightarrow (E_0, E_1)_{\rho, K}, \quad I_{E_0 E_1}^{2\rho} : (E_0, E_1)_{\rho, K} \longrightarrow E_0 + E_1$$

have norms  $\|I_{E_0 E_1}^{1\rho}\| = \omega_\rho$  and  $\|I_{E_0 E_1}^{2\rho}\| = \frac{1}{\omega_\rho}$ .

(2) Let  $\rho$  be a function norm such that  $L^\rho \subset L^1(dt) + L^1\left(\frac{1}{t}, dt\right)$ . Then  $(E_0, E_1)_{\rho, J}$  is an interpolation space of the couple  $(E_0, E_1)$  (the interpolated space defined by the real method  $J$ ).

**Proof.** Consider the space  $\Phi_\rho$  of classes of real functions  $f$  defined on  $]0, \infty[$  which are measurable respect the measure  $\frac{dt}{t}$ , and such that  $\|f\|_{\Phi_\rho} := \rho\left(f\left(\frac{1}{t}\right)\right) < \infty$ .  $\Phi_\rho$  endowed with the norm  $\|\cdot\|_{\Phi_\rho}$  is a Banach space and the map  $H : L^\rho \longrightarrow \Phi_\rho$  defined by  $H(f)(t) = f\left(\frac{1}{t}\right)$  for every  $f \in L^\rho$  and  $t > 0$ , is a surjective isometry such that  $H(h)(t) = \min(1, t)$ . Then to prove (1) it is enough to apply proposition 3.3.1 in [4] and (3.3.4) and (3.3.5) of this proposition. Proposition 3.4.1 in [4] shows part (2). ■

The setting just described is too general to obtain useful duality results. Fortunately, by corollaries 3.3.6 3.4.6 in [4] we can consider a special set of function norms  $\rho$  with some nice additional properties without lost of generality respect to the class of generated interpolation spaces. Hence, *from now on we shall suppose always the function norm  $\rho$  verifies*

$$L^1(dt) \cap L^1\left(\frac{1}{t}, dt\right) \subset L^\rho \subset L^1(dt) + L^1\left(\frac{1}{t}, dt\right) \tag{2}$$

with continuous inclusion maps.

Then we define

$$L_r^\rho := \overline{L^1(dt) \cap L^1\left(\frac{1}{t}, dt\right)}^{L^\rho}$$

and we can consider the new function norm  $\rho^r$  defined by  $\rho^r(f) = \rho(f)$  if  $f \in L_r^\rho$  and  $\rho^r(f) = \infty$  if  $f \notin L_r^\rho$ .

The associated function norm to  $\rho$  is defined by

$$\rho^*(f) = \sup \left\{ \int_0^\infty f(t)g(t)dt \mid \rho(g) \leq 1, g \in L^1(dt) \cap L^1\left(\frac{1}{t}, dt\right) \right\}.$$

It is easy to check that

$$(\rho^r)^* = \rho^*. \tag{3}$$

Concerning duality we have the following proposition.

**Proposition 4** (1)

$$(L_r^\rho)' = (L^{\rho^r})' = L^{\rho^*}$$

isometrically.

(2)

$$((\lambda_0, \lambda_1)_{\rho^r, J})' = (\lambda_0', \lambda_1')_{\rho^*, K}$$

isometrically.

**Proof.** It is enough to use the space  $\Phi_\rho$  and the map  $H$  of proposition 3 and to apply (3.7.2), (3.7.3) remark 3.7.3 and theorem 3.7.6 in [4]. ■

## 2. The tensor norm $g_{\lambda, \rho}$

Let  $\lambda := (\lambda_0, \lambda_1)$  be a couple of perfect sequence spaces (which will be fixed in the sequel of the paper) such that

$$\|\mathbf{e}_i\|_{\lambda_0} = \|\mathbf{e}_i\|_{\lambda_1} = 1 = \|\mathbf{e}_i\|_{\lambda_0^\times} = \|\mathbf{e}_i\|_{\lambda_1^\times}$$

for every  $i \in \mathbb{N}$ . We shall denote by  $\lambda^\times$  the couple of its  $\alpha$ -duals:  $\lambda^\times := (\lambda_0^\times, \lambda_1^\times)$  and by  $\lambda^r$  the couple of the corresponding regular subspaces:  $\lambda^r := (\lambda_0^r, \lambda_1^r)$ . Consequently, given a function norm  $\rho$  we shall use the simplified notations

$$\lambda_{\rho, K} := (\lambda_0, \lambda_1)_{\rho, K} \quad \lambda_{\rho, J}^r := (\lambda_0^r, \lambda_1^r)_{\rho, J} \quad \lambda_{\rho^*, K}^\times := (\lambda_0^\times, \lambda_1^\times)_{\rho^*, K}.$$

Our goal in this section is to define tensor norms in the class of all Banach spaces by means of an interpolation space of the couple  $\lambda$  defined with help of  $\rho$ .

**Proposition 5** *The equalities  $\|\mathbf{e}_i\|_{\lambda_{\rho^*, K}^\times} = \rho^*(h)$  and  $\|\mathbf{e}_i\|_{\lambda_{\rho^r, J}^r} = \frac{1}{\rho^*(h)}$  hold.*

**Proof.** For every couple of Banach sequence spaces  $(\mu_0, \mu_1)$  and every function norm  $\eta$  let us denote by

$$J_{\mu_0, \mu_1}^{1\eta} : \mu_0 \cap \mu_1 \longrightarrow (\mu_0, \mu_1)_{\eta, J}, \quad J_{\mu_0, \mu_1}^{2\eta} : (\mu_0, \mu_1)_{\eta, J} \longrightarrow \mu_0 + \mu_1$$

the canonical inclusion maps.

Clearly  $h(t) := \min(1, \frac{1}{t}) \in L^\infty(dt) \cap L^\infty(t, dt)$ . By proposition 4, (1), taking topological duals in (2) we obtain  $h(t) \in L^{\rho^*}$  and by proposition 3, (1) the interpolation space



$\lambda_{\rho^*, K}^\times$  is well defined. Since  $(\lambda_0^r \cap \lambda_1^r)' = (\lambda_0^r)^\times + (\lambda_1^r)^\times = \lambda_0^\times + \lambda_1^\times$  and using the notation of proposition 3 we have

$$1 = |\langle \mathbf{e}_i, \mathbf{e}_i \rangle| \leq \|\mathbf{e}_i\|_{\lambda_0^\times + \lambda_1^\times} \|\mathbf{e}_i\|_{\lambda_0^r \cap \lambda_1^r} = \|\mathbf{e}_i\|_{\lambda_0^\times + \lambda_1^\times} \leq \|\mathbf{e}_i\|_{\lambda_0^\times} = 1$$

and hence, by proposition 3

$$\begin{aligned} 1 &= \|\mathbf{e}_i\|_{\lambda_0^\times + \lambda_1^\times} = \|I_{\lambda_0^\times, \lambda_1^\times}^{2\rho^*}(\mathbf{e}_i)\| \leq \|I_{\lambda_0^\times, \lambda_1^\times}^{2\rho^*}\| \|\mathbf{e}_i\|_{\lambda_{\rho^*, K}^\times} = \\ &= \|I_{\lambda_0^\times, \lambda_1^\times}^{2\rho^*}\| \|I_{\lambda_0^\times, \lambda_1^\times}^{1\rho^*}(\mathbf{e}_i)\| \leq \|I_{\lambda_0^\times, \lambda_1^\times}^{2\rho^*}\| \|I_{\lambda_0^\times, \lambda_1^\times}^{1\rho^*}\| \|\mathbf{e}_i\|_{\lambda_0^r \cap \lambda_1^r} = \frac{1}{\rho^*(h)} \rho^*(h) = 1 \end{aligned}$$

and we get

$$\|\mathbf{e}_i\|_{\lambda_{\rho^*, K}^\times} = \frac{1}{\|I_{\lambda_0^\times, \lambda_1^\times}^{2\rho^*}\|} = \rho^*(h).$$

Now note that

$$(\lambda_0^r \cap \lambda_1^r)' = \lambda_0^\times + \lambda_1^\times \quad \text{and} \quad (\lambda_0^r + \lambda_1^r)' = \lambda_0^\times \cap \lambda_1^\times,$$

since  $\lambda_0^r \cap \lambda_1^r$  is dense in every  $\lambda_i^r, i = 0, 1$ . By proposition 2, (3) and proposition 4 we have  $(\lambda_{\rho^r, J}^r)' = \lambda_{\rho^*, K}^\times$  *isometrically*. Then we can easily check that

$$(J_{\lambda_0^r, \lambda_1^r}^{1\rho^r})' = I_{\lambda_0^\times, \lambda_1^\times}^{2\rho^*} \quad \text{and} \quad (J_{\lambda_0^r, \lambda_1^r}^{2\rho^r})' = I_{\lambda_0^\times, \lambda_1^\times}^{1\rho^*} \quad (4)$$

and hence

$$\|J_{\lambda_0^r, \lambda_1^r}^{2\rho^r}\| = \|(J_{\lambda_0^r, \lambda_1^r}^{2\rho^r})'\| = \|I_{\lambda_0^\times, \lambda_1^\times}^{1\rho^*}\| \quad (5)$$

and

$$\|J_{\lambda_0^r, \lambda_1^r}^{1\rho^r}\| = \|(J_{\lambda_0^r, \lambda_1^r}^{1\rho^r})'\| = \|I_{\lambda_0^\times, \lambda_1^\times}^{2\rho^*}\|. \quad (6)$$

Now by (5) and (6) we get

$$\begin{aligned} 1 &= \|\mathbf{e}_i\|_{\lambda_0^r + \lambda_1^r} = \|J_{\lambda_0^r, \lambda_1^r}^{2\rho^r}(\mathbf{e}_i)\|_{\lambda_0^r + \lambda_1^r} \leq \\ &\leq \|J_{\lambda_0^r, \lambda_1^r}^{2\rho^r}\| \|\mathbf{e}_i\|_{\lambda_{\rho^r, J}^r} = \|(I_{\lambda_0^\times, \lambda_1^\times}^{1\rho^*})'\| \|\mathbf{e}_i\|_{\lambda_{\rho^r, J}^r} = \|(I_{\lambda_0^\times, \lambda_1^\times}^{1\rho^*})'\| \|J_{\lambda_0^r, \lambda_1^r}^{1\rho^r}(\mathbf{e}_i)\| \leq \end{aligned}$$

$$\leq \|(I_{\lambda_0^\times, \lambda_1^\times}^{1\rho^*})'\| \|J_{\lambda_0^r, \lambda_1^r}^{1\rho^r}\| \|\mathbf{e}_i\|_{\lambda_0^r \cap \lambda_1^r} = \|I_{\lambda_0^\times, \lambda_1^\times}^{1\rho^*}\| \|I_{\lambda_0^\times, \lambda_1^\times}^{2\rho^*}\| = 1$$

and as a consequence

$$\|\mathbf{e}_i\|_{\lambda_{\rho^r, J}^r} = \frac{1}{\|(I_{\lambda_0^\times, \lambda_1^\times}^{1\rho^*})'\|} = \frac{1}{\|I_{\lambda_0^\times, \lambda_1^\times}^{1\rho^*}\|} = \frac{1}{\rho^*(h)}.$$

■

Given a sequence  $\{x_i\}_{i=1}^\infty$  in a Banach space  $E$  we define

$$\pi_{\lambda, \rho, J}((x_i)_{i=1}^\infty) := \pi_{\lambda, \rho, J}((x_i)) := \|(\|x_i\|_E)_{i=1}^\infty\|_{\lambda_{\rho^r, J}^r}$$

and

$$\varepsilon_{\lambda, \rho, K}((x_i)_{i=1}^\infty) := \varepsilon_{\lambda, \rho, K}((x_i)) := \sup_{\|x'\|_{E'} \leq 1} \|(\langle x_i, x' \rangle)_{i=1}^\infty\|_{\lambda_{\rho, K}}.$$

Remark that we use  $\rho^r$  and the couple  $(\lambda_0^r, \lambda_1^r)$  in the definition of  $\pi_{\lambda, \rho, J}((x_i)_{i=1}^\infty)$  but  $\rho$  and the couple  $(\lambda_0, \lambda_1)$  in the definition of  $\varepsilon_{\lambda, \rho, K}((x_i)_{i=1}^\infty)$ . Note also that to compute  $\varepsilon_{\lambda, \rho, K}((x_i))$  for a finite sequence  $\{x'_i\}_{i=1}^n$  of vectors in a dual space  $E'$ , by the  $\sigma(E'', E')$ -density of the closed unit ball of  $E$  in the closed unit ball of  $E''$  it is enough to take

$$\varepsilon_{\lambda, \rho, K}((x'_i)) := \sup_{\|x\|_E \leq 1} \|(\langle x'_i, x \rangle)_{i=1}^n\|_{\lambda_{\rho, K}}. \quad (7)$$

If  $\{x_i\}_{i=n}^m$  is a finite sequence in  $E$ ,  $m \geq n \geq 1$  we define

$$\pi_{\lambda, \rho, J}((x_i)_{i=n}^m) := \pi_{\lambda, \rho, J}(0, 0, \dots, 0, x_n, x_{n+1}, \dots, x_m, 0, 0, 0, \dots)$$

and analogously for  $\varepsilon_{\lambda, \rho, K}((x_i)_{i=n}^m)$ .

Now we can define the desired tensor norm. Given Banach spaces  $E$  and  $F$ , for every tensor

$$z = \sum_{i=1}^k \sum_{j=1}^{n_i} x_{ij} \otimes y_{ij} \in E \otimes F \quad (8)$$

we define

$$g_{\lambda, \rho}(z; E, F) = \inf \sum_{i=1}^n \pi_{\lambda, \rho, J}((x_{ij})_{j=1}^{n_i}) \varepsilon_{\lambda^\times, \rho^*, K}((y_{ij})_{j=1}^{n_i}), \quad (9)$$

where the inf is taken over all representations of  $z$  of type (8).

**Proposition 6**  $g_{\lambda,\rho}$  is a tensor norm in the class BAN.

**Proof.** The proof is straightforward by applying criterion 12.2 in [5]. We need only to show that  $g_{\lambda,\rho}(1 \otimes 1; \mathbb{R}, \mathbb{R}) = 1$ . By definition of  $g_{\lambda,\rho}$  and by proposition 5

$$g_{\lambda,\rho}(1 \otimes 1; \mathbb{R}, \mathbb{R}) \leq \|e_1\|_{\lambda_{\rho^r,J}^r} \|e_1\|_{\lambda_{\rho^*,K}^\times} = 1.$$

The reverse inequality is obtained by a standard method using proposition 4 (2) and noticing that for every tensor  $\sum_{i=1}^n \alpha_i \otimes \beta_i \in \mathbb{R} \otimes \mathbb{R}$ , the equality  $\varepsilon_{\lambda^\times, \rho^*, K}((\alpha_i)_{i=1}^n) = \|(\alpha_i)_{i=1}^n\|_{\lambda_{\rho^*,K}^\times}$  holds. ■

We need to have a characterization of the completion  $E \hat{\otimes}_{g_{\lambda,\rho}} F$ .

**Theorem 7** Every element  $z \in E \hat{\otimes}_{g_{\lambda,\rho}} F$  can be represented as

$$z = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij} \otimes y_{ij}, \quad x_{ij} \in E, \quad y_{ij} \in F \quad \forall i, j \in \mathbb{N} \quad (10)$$

where

$$\forall i \in \mathbb{N} \quad ((\|x_{ij}\|)_{j=1}^{\infty}) \in \lambda_{\rho^r,J}^r \quad (11)$$

and

$$\sum_{i=1}^{\infty} \pi_{\lambda,\rho,J}((x_{ij})_{j=1}^{\infty}) \varepsilon_{\lambda^\times, \rho^*, K}((y_{ij})_{j=1}^{\infty}) < \infty. \quad (12)$$

Moreover,

$$g_{\lambda,\rho}(z) := \|z\|_{E \hat{\otimes}_{g_{\lambda,\rho}} F} = \inf \sum_{i=1}^{\infty} \pi_{\lambda,\rho,J}((x_{ij})_{j=1}^{\infty}) \varepsilon_{\lambda^\times, \rho^*, K}((y_{ij})_{j=1}^{\infty})$$

where the inf is taken over all representations of  $z$  verifying (11) and (12).

**Proof.** 1) By corollary 3.6.3 and lemma 3.6.2 in [4],  $\lambda_0^r \cap \lambda_1^r$  is dense in  $\lambda_{\rho^r,J}^r$  and by lemma 1 every element of  $\lambda_{\rho^r,J}^r$  is the limit of its sections. Hence it is easy to check that every series of type (10), (11) and (12) is convergent in  $E \hat{\otimes}_{g_{\lambda,\rho}} F$  and

$$g_{\lambda,\rho} \left( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij} \otimes y_{ij} \right) \leq \sum_{i=1}^{\infty} \pi_{\lambda,\rho,J}((x_{ij})_{j=1}^{\infty}) \varepsilon_{\lambda^\times, \rho^*, K}((y_{ij})_{j=1}^{\infty}). \quad (13)$$

2) Conversely, suppose  $0 \neq z \in E \hat{\otimes}_{g_{\lambda, \rho}} F$ . Choose a Cauchy sequence  $\{z_n\}_{n=1}^{\infty} \subset E \otimes_{g_{\lambda, \rho}} F$  such that

$$\lim_{n \rightarrow \infty} g_{\lambda, \rho}(z_n - z) = 0.$$

Given  $\varepsilon > 0$  there is a strictly increasing sequence  $\{n_k\}_{k=0}^{\infty} \subset \mathbb{N}$  such that

$$g_{\lambda, \rho}(z_{n_0} - z) < \varepsilon \frac{g_{\lambda, \rho}(z)}{2} \quad \text{and} \quad \forall k \geq 1 \quad g_{\lambda, \rho}(z_{n_k} - z_{n_{k-1}}; E, F) < \varepsilon \frac{g_{\lambda, \rho}(z)}{2^{k+1}}. \quad (14)$$

Clearly

$$z = z_{n_0} + \lim_{n \rightarrow \infty} \sum_{k=1}^n (z_{n_k} - z_{n_{k-1}}) = z_{n_0} + \sum_{k=1}^{\infty} (z_{n_k} - z_{n_{k-1}}) \in E \hat{\otimes}_{g_{\lambda, \rho}} F. \quad (15)$$

We select representations

$$z_{n_0} = \sum_{i=1}^{v_0} \sum_{j=1}^{u_{0i}} x_{ij}^0 \otimes y_{ij}^0 \in E \otimes_{g_{\lambda, \rho}} F \quad (16)$$

and

$$\forall k \geq 1 \quad z_{n_k} - z_{n_{k-1}} = \sum_{i=1}^{v_k} \sum_{j=1}^{u_{ki}} x_{ij}^k \otimes y_{ij}^k \in E \otimes_{g_{\lambda, \rho}} F \quad (17)$$

such that

$$\sum_{i=1}^{v_0} \pi_{\lambda, \rho, J} \left( (x_{ij}^0)_{j=1}^{u_{0i}} \right) \varepsilon_{\lambda \times, \rho^*, K} \left( (y_{ij}^0)_{j=1}^{u_{0i}} \right) \leq g_{\lambda, \rho}(z_{n_0}; E, F) + \frac{\varepsilon}{2} \quad (18)$$

and

$$\forall k \geq 1 \quad \sum_{i=1}^{v_k} \pi_{\lambda, \rho, J} \left( (x_{ij}^k)_{j=1}^{u_{ki}} \right) \varepsilon_{\lambda \times, \rho^*, K} \left( (y_{ij}^k)_{j=1}^{u_{ki}} \right) \leq g_{\lambda, \rho}(z_{n_k} - z_{n_{k-1}}; E, F) + \frac{\varepsilon}{2^{k+1}}. \quad (19)$$

By (14), (15), (18) and (19) it is clear now that, with a suitable definition of vectors and indices we can write

$$z = \sum_{i=1}^{\infty} \sum_{j=1}^{d_i} \bar{x}_{ij} \otimes \bar{y}_{ij}$$

in such a way that

$$\begin{aligned}
 & \sum_{i=1}^{\infty} \pi_{\lambda, \rho, J} \left( (\bar{x}_{ij})_{j=1}^{d_i} \right) \varepsilon_{\lambda \times, \rho^*, K} \left( (\bar{y}_{ij})_{j=1}^{d_i} \right) \leq \\
 & \leq g_{\lambda, \rho}(z_{n_0}; E, F) + \frac{\varepsilon}{2} + \sum_{k=1}^{\infty} \left( g_{\lambda, \rho}(z_{n_k} - z_{n_{k-1}}; E, F) + \frac{\varepsilon}{2^{k+1}} \right) \leq \\
 & \leq g_{\lambda, \rho}(z) + g_{\lambda, \rho}(z_{n_0} - z) + \frac{\varepsilon}{2} + (1 + g_{\lambda, \rho}(z)) \sum_{k=0}^{\infty} \frac{\varepsilon}{2^{k+1}} \leq \\
 & \leq \left( 1 + \frac{\varepsilon}{2} \right) g_{\lambda, \rho}(z) + \frac{\varepsilon}{2} + \varepsilon (1 + g_{\lambda, \rho}(z))
 \end{aligned}$$

and hence,  $\varepsilon$  being arbitrary,

$$\sum_{i=1}^{\infty} \pi_{\lambda, \rho, J} \left( (\bar{x}_{ij})_{j=1}^{d_i} \right) \varepsilon_{\lambda \times, \rho^*, K} \left( (\bar{y}_{ij})_{j=1}^{d_i} \right) \leq g_{\lambda, \rho}(z). \quad \blacksquare$$

Now we characterize the topological dual of  $E \otimes_{g_{\lambda, \rho}} F$ . We need the following definition.

**Definition 8** A sequence  $\{x_n\}_{n=1}^{\infty}$  in the Banach space  $E$  is said to be weakly to be  $\lambda_{\rho, K}$ -sumable (respectively, absolutely  $\lambda_{\rho, K}$ -sumable) if  $\varepsilon_{\lambda, \rho, K}((x_n)) < \infty$  (resp., if

$$\bar{\pi}_{\lambda, \rho, K}((x_n)) := \|(\|x_n\|)_{n=1}^{\infty}\|_{\lambda_{\rho, K}} < \infty).$$

It is straightforward to see that the set  $\lambda_{\rho, K}(E)$  of all weakly  $\lambda_{\rho, K}$ -sumable sequences in  $E$  (resp., the set  $\lambda_{\rho, K}[E]$  of all absolutely  $\lambda_{\rho, K}$ -sumable sequences in  $E$ ) is a Banach space when it is provided with the norm  $\varepsilon_{\lambda, \rho, K}$  (resp., with the norm  $\pi_{\lambda, \rho, K}$ ). Let us denote  $\lambda_{\rho, K}(E)^r$  the set of elements  $(x_i) \in \lambda_{\rho, K}(E)$  which are the limit of its sections  $(x_1, x_2, \dots, x_n, 0, 0, \dots)$  in the topology of  $\lambda_{\rho, K}(E)$  if  $n$  goes to  $\infty$ .

**Definition 9** Let  $E$  and  $F$  be Banach spaces. A linear map  $T \in \mathcal{L}(E, F)$  is said to be regularly  $\lambda_{\rho, K}$ -absolutely summing if  $(T(x_n))_{n=1}^{\infty} \in \lambda_{\rho, K}[F]$  for every  $(x_n)_{n=1}^{\infty} \in \lambda_{\rho, K}(E)^r$ .

It can be proved that the set  $\mathfrak{P}_{\lambda,\rho,K}(E, F)$  of all regularly  $\lambda_{\rho,K}$ -absolutely summing maps from  $E$  into  $F$  endowed with the norm

$$\Pi_{\lambda,\rho,K}(T) = \sup \{ \bar{\pi}_{\lambda,\rho,K}((T(x_i))_{i=1}^{\infty}) / \varepsilon_{\lambda,\rho,K}((x_i)_{i=1}^{\infty}) \leq 1, (x_i) \in \lambda_{\rho,K}(E)^r \}$$

for every  $T \in \mathfrak{P}_{\lambda,\rho,K}(E, F)$ , is a Banach space. Now we have this theorem:

**Theorem 10** *If  $E$  and  $F$  are Banach spaces,  $(E \otimes_{g_{\lambda,\rho}} F)' = \mathfrak{P}_{\lambda^\times, \rho^*, K}(F, E')$ .*

**Proof.** Let  $T \in \mathfrak{P}_{\lambda^\times, \rho^*, K}(F, E')$ . We define  $\varphi_T \in (E \otimes_{g_{\lambda,\rho}} F)'$  as

$$\forall z = \sum_{i=1}^n \sum_{j=1}^{n_i} x_{ij} \otimes y_{ij} \in E \otimes_{g_{\lambda,\rho}} F \quad \langle T, z \rangle := \varphi_T(z) := \sum_{i=1}^n \sum_{j=1}^{n_i} \langle x_{ij}, T(y_{ij}) \rangle,$$

which is well defined and continuous since, by proposition 4, (2),

$$\begin{aligned} |\langle \varphi_T, z \rangle| &\leq \sum_{i=1}^n \sum_{j=1}^{n_i} \|x_{ij}\| \|T(y_{ij})\| \leq \\ &\leq \sum_{i=1}^n \left\| (\|x_{ij}\|)_{j=1}^{n_i} \right\|_{\lambda_{\rho^*, J}^r} \left\| (\|T(y_{ij})\|)_{j=1}^{n_i} \right\|_{\lambda_{\rho^*, K}^\times} \leq \\ &\leq \Pi_{\lambda^\times, \rho^*, K}(T) \left( \sum_{i=1}^n \pi_{\lambda_{\rho^*, J}}(x_{ij})_{j=1}^{n_i} \right) \varepsilon_{\lambda^\times, \rho^*, K} \left( (y_{ij})_{j=1}^{n_i} \right) \end{aligned}$$

and taking the infimum over all representations of  $z$

$$|\langle \varphi_T, z \rangle| \leq \Pi_{\lambda^\times, \rho^*, K}(T) g_{\lambda,\rho}(z).$$

Conversely, given  $\varphi \in (E \otimes_{g_{\lambda,\rho}} F)'$ , we define  $T : F \longrightarrow E'$  by

$$\forall y \in F, \quad \forall x \in E \quad \langle T(y), x \rangle = \langle \varphi, x \otimes y \rangle.$$

Let  $(y_i) \in \lambda_{\rho^*, K}^\times(F)^r$ . Fix an element  $(\eta_i) \in (\lambda_{\rho^*, K}^\times)^r$  such that  $\|(\eta_i)\|_{\lambda_{\rho^*, K}^\times} \leq 1$  and  $\eta_i > 0$  for every  $i \in \mathbb{N}$ . Given  $\varepsilon > 0$ , for every  $i \in \mathbb{N}$  we can choose  $a_i \in E$ ,  $\|a_i\| \leq 1$  such that  $\langle \varphi, a_i \otimes y_i \rangle \geq 0$  and

$$\|T(y_i)\| \leq \langle \varphi, a_i \otimes y_i \rangle + \varepsilon \eta_i.$$

Moreover, by Lemma 1 there is  $n_0 \in \mathbb{N}$  such that

$$\forall n \geq n_0, \forall h \in \mathbb{N} \cup \{0\} \quad \|(\eta_i)_{i=n}^{n+h}\|_{\lambda_{\rho^*, K}^\times} \leq \varepsilon \quad \text{and} \quad \varepsilon_{\lambda^\times, \rho^*, K}((y_i)_{i=n}^{n+h}) \leq \varepsilon. \quad (20)$$

As  $(\lambda_{\rho^r, J}^r)' = \lambda_{\rho^*, K}^\times$ , there is  $(\delta_i) \in \lambda_{\rho^r, J}^r$  such that  $\|(\delta_i)\|_{\lambda_{\rho^r, J}^r} \leq 1$  and

$$\begin{aligned} \left\| (\|T(y_i)\|)_{i=n}^{n+h} \right\|_{\lambda_{\rho^*, K}^\times} &\leq \sum_{i=n}^{n+h} \delta_i \|T(y_i)\| + \varepsilon \leq \left\langle \varphi, \sum_{i=n}^{n+h} \delta_i a_i \otimes y_i \right\rangle + \varepsilon \sum_{i=n}^{n+h} \delta_i \eta_i + \varepsilon \leq \\ &\leq \|\varphi\| g_{\lambda, \rho} \left( \sum_{i=n}^{n+h} \delta_i a_i \otimes y_i; E, F \right) + \varepsilon \|(\delta_i)_{i=n}^{n+h}\|_{\lambda_{\rho^r, J}^r} \|(\eta_i)_{i=n}^{n+h}\|_{\lambda_{\rho^*, K}^\times} + \varepsilon \leq \\ &\leq \|\varphi\| \left\| (\|\delta_i a_i\|)_{i=n}^{n+h} \right\|_{\lambda_{\rho^r, J}^r} \varepsilon_{\lambda^\times, \rho^*, K}((y_i)_{i=n}^{n+h}) + 2\varepsilon \leq \varepsilon (\|\varphi\| + 2) \end{aligned}$$

and hence  $\sum_{i=1}^\infty \|T(y_i)\| \mathbf{e}_i$  is convergent in  $\lambda_{\rho^*, K}^\times$ . As the convergence in  $\lambda_{\rho^*, K}^\times$  implies coordinatewise convergence we have necessarily

$$\left( \|T(y_i)\| \right)_{i=1}^\infty = \sum_{i=1}^\infty \|T(y_i)\| \mathbf{e}_i \in (\lambda_{\rho^*, K}^\times)^r \subset \lambda_{\rho^*, K}^\times.$$

Then  $T \in \mathfrak{P}_{\lambda^\times, \rho^*, K}(F, E')$ . A similar computation using theorem 7 shows  $\|(\|T(y_i)\|)_{i=1}^\infty\|_{\lambda_{\rho^*, K}^\times} \leq \|\varphi\| \varepsilon_{\lambda^\times, \rho^*, K}((y_i)_{i=1}^\infty)$  and hence  $\Pi_{\lambda^\times, \rho^*, K}(T) \leq \|\varphi\|$ . ■

### 3. $\lambda_\rho$ -nuclear operators

Let  $E, F \in BAN$ . An operator  $T \in \mathcal{L}(E, F)$  is said to be  $\lambda_\rho$ -nuclear if there is  $z = \sum_{i=1}^\infty \sum_{j=1}^\infty x'_{ij} \otimes y_{ij} \in E' \hat{\otimes}_{g_{\lambda, \rho}} F$  verifying (12) such that

$$\forall x \in E \quad T(x) = \sum_{i=1}^\infty \sum_{j=1}^\infty \langle x'_{ij}, x \rangle y_{ij}. \quad (21)$$

$\mathcal{N}_{\lambda, \rho}(E, F)$  will denote the set of all  $\lambda_\rho$ -nuclear operators from  $E$  into  $F$ . It is easy to see that, for every  $T \in \mathcal{N}_{\lambda, \rho}(E, F)$

$$\mathbf{N}_{\lambda, \rho}(T) := \inf \{ g_{\lambda, \rho}(z) \mid z \in E' \hat{\otimes}_{g_{\lambda, \rho}} F \text{ and (21) holds} \}$$

is a norm in  $\mathcal{N}_{\lambda, \rho}(E, F)$  and that  $(\mathcal{N}_{\lambda, \rho}, \mathbf{N}_{\lambda, \rho})$  is a Banach operator ideal. Our aim in this section is to get a characterization of  $\lambda_\rho$ -nuclear operators. We have

**Theorem 11** *An operator  $T \in \mathcal{L}(E, F)$  is  $\lambda_\rho$ -nuclear if and only if  $T$  has a factorization*

$$\begin{array}{ccc}
 E & \xrightarrow{T} & F \\
 A \downarrow & & \uparrow B \\
 \ell^\infty[\ell^\infty] & \xrightarrow{D} & \ell^1[(\lambda^r)_{\rho^r, J}]
 \end{array}$$

where  $D$  is a positive diagonal operator and  $A$  and  $B$  are continuous linear maps. Moreover,

$$\mathbf{N}_{\lambda_\rho}(T) := \inf \{ \|A\| \|D\| \|B\| \},$$

where the inf is taken over all possible factorizations as above.

**Proof.** a) Suppose  $T \in \mathcal{N}_{\lambda_\rho}(E, F)$ . By theorem 7 and by definition of the norm in  $\mathcal{N}_{\lambda_\rho}(E, F)$  given  $\varepsilon > 0$ , there is a representation

$$z = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x'_{ij} \otimes y_{ij} \quad (22)$$

and a sequence  $\{n_i\}_{i=1}^{\infty} \subset \mathbb{N}$  such that  $x'_{ij} = 0$  for all  $j > n_i$  and  $\varepsilon_{\lambda^{\times, \rho^*}, K}((y_{ij})_{j=1}^{\infty}) = 1$  for every  $i \in \mathbb{N}$ ,

$$\sum_{i=1}^{\infty} \pi_{\lambda, \rho^r, J}((x'_{ij})_{j=1}^{\infty}) \varepsilon_{\lambda^{\times, \rho^*}, K}((y_{ij})) = \sum_{i=1}^{\infty} \pi_{\lambda, \rho^r, J}((x'_{ij})_{j=1}^{\infty}) < \mathbf{N}_{\lambda_\rho}(T) + \varepsilon. \quad (23)$$

We define  $A : E \rightarrow \ell^\infty[\ell^\infty]$ , as

$$\forall x \in E \quad A(x) = \left( \left( \frac{\langle x'_{ij}, x \rangle}{\|x'_{ij}\|} \right)_{j=1}^{\infty} \right)_{i=1}^{\infty}. \quad (24)$$

Clearly

$$\|A\| = \sup_{\|x\| \leq 1} \sup_{i \in \mathbb{N}} \sup_{j \in \mathbb{N}} \frac{|\langle x'_{ij}, x \rangle|}{\|x'_{ij}\|} \leq 1. \quad (25)$$



The linear map  $D : \ell^\infty[\ell^\infty] \rightarrow \ell^1[\lambda_{\rho^r, J}^r]$  defined as

$$\forall (\eta_{ij}) \in \ell^\infty[\ell^\infty] \quad D((\eta_{ij})) = \left( (\eta_{ij} \|x'_{ij}\|)_{j=1}^\infty \right)_{i=1}^\infty$$

verifies

$$\begin{aligned} \|D((\eta_{ij}))\| &= \sum_{i=1}^\infty \|(\|\eta_{ij} x_{ij}\|)_{j=1}^\infty\|_{\lambda_{\rho^r, J}^r} \leq \\ &\leq \|((\eta_{ij}))\|_{\ell^\infty[\ell^\infty]} \sum_{i=1}^\infty \pi_{\lambda_{\rho^r, J}}((x_{ij})_{j=1}^\infty) \leq \|((\eta_{ij}))\|_{\ell^\infty[\ell^\infty]} (\mathbf{N}_\lambda(T) + \varepsilon), \end{aligned}$$

and by the properties of the chosen representation of  $z$ , we see that  $D((\eta_{ij}))$  actually lies in  $\ell^1[\lambda_{\rho^r, J}^r]$ . Then  $D$  is well defined and  $\|D\| \leq (\mathbf{N}_\lambda(T) + \varepsilon)$ .

Finally, we define  $B : \ell^1[\lambda_{\rho^r, J}^r] \rightarrow F$  as

$$\forall (\beta_{ij}) \in \ell^1[\lambda_{\rho^r, J}^r] \quad B((\beta_{ij})) = \sum_{i=1}^\infty \sum_{j=1}^\infty \beta_{ij} y_{ij}.$$

We demonstrate that this definition is meaningful. By the property of sectional convergence in  $\lambda_{\rho^r, J}^r$  (see the beginning of the proof the theorem 7) every series  $\sum_{j=1}^\infty \beta_{ij} y_{ij}$ ,  $i \in \mathbb{N}$  is convergent in  $F$ . Then given  $\delta > 0$ , there is  $n$  in  $\mathbb{N}$  such that for every  $h \in \mathbb{N}$

$$\begin{aligned} \left\| \sum_{i=n+1}^{n+h} \sum_{j=1}^\infty \beta_{ij} y_{ij} \right\| &= \sup_{\|y'\|_{F'} \leq 1} \left| \left\langle \sum_{i=n+1}^{n+h} \sum_{j=1}^\infty \beta_{ij} y_{ij}, y' \right\rangle \right| \leq \\ &\leq \sup_{\|y'\|_{F'} \leq 1} \sum_{i=n+1}^{n+h} \left\| \sum_{j=1}^\infty \beta_{ij} \mathbf{e}_j \right\|_{\lambda_{\rho^r, J}^r} \left\| \sum_{j=1}^\infty \langle y_{ij}, y' \rangle \mathbf{e}_j \right\|_{\lambda_{\rho^*, K}^\times} \leq \\ &\leq \varepsilon_{\lambda^\times, \rho^*, K} ((y_{ij})_{j=1}^\infty) \sum_{i=n+1}^{n+h} \left\| \sum_{j=1}^\infty \beta_{ij} \mathbf{e}_j \right\|_{\lambda_{\rho^r, J}^r} \leq \delta \end{aligned}$$

since  $((\beta_{ij})) \in \ell^1[\lambda_{\rho^r, J}^r]$ . Hence we obtain easily  $\|D\| \leq 1$ . Clearly  $T = BDA$  and  $\|B\| \|D\| \|A\| \leq \mathbf{N}_{\lambda, \rho}(T) + \varepsilon$ .

b) Conversely, let  $T = BDA$  be a factorization as the given one in the diagram. For every  $i, j \in \mathbb{N}$ ,  $\mathbf{e}_{ij} \in (\ell^\infty[\ell^\infty])'$ . Define  $u'_{ij} \in E' := A'(\mathbf{e}_{ij}) \in E'$  for each  $i, j \in \mathbb{N}$ . Then

$$\forall x \in E \quad A(x) = \left( (\langle A(x), \mathbf{e}_{ij} \rangle)_{j=1}^\infty \right) = \left( (\langle x, A'(\mathbf{e}_{ij}) \rangle)_{j=1}^\infty \right) = \left( (\langle x, u'_{ij} \rangle)_{j=1}^\infty \right)$$

and  $A(x) = ((\langle u'_{ij}, x \rangle))$ .

On the other hand, let  $((b_{ij}))$  such that

$$\forall ((b_{ij})) \in \ell^\infty[\ell^\infty] \quad D((b_{ij})) = (b_{ij}\beta_{ij}) \in \ell^1[\lambda_{\rho^r, J}^r].$$

As the double sequence  $\mathbf{u} := ((u_{ij}))$  with  $u_{ij} = 1$  for every  $i, j \in \mathbb{N}$  belongs to  $\ell^\infty[\ell^\infty]$ , for every  $i \in \mathbb{N}$  we have  $(b_{ij})_{j=1}^\infty \in \lambda_{\rho^r, J}^r$  and

$$\|D\| = \sum_{i=1}^\infty \|(b_{ij})_{j=1}^\infty\|_{\lambda_{\rho^r, J}^r} < \infty. \quad (26)$$

Finally, put  $x'_{ij} := b_{ij}u'_{ij} \in E'$  and  $y_{ij} := B(\mathbf{e}_{ij})$  for every  $i, j \in \mathbb{N}$ . By proposition 4, (2)  $B' : F' \longrightarrow \ell^\infty[\lambda_{\rho^*, K}^\times]$ . Hence by (26),

$$\begin{aligned} & \sum_{i=1}^\infty \pi_{\lambda, \rho^r, J} \left( (x'_{ij})_{j=1}^\infty \right) \varepsilon_{\lambda^\times, \rho^*, K} \left( (y_{ij})_{j=1}^\infty \right) = \\ &= \sum_{i=1}^\infty \left\| \left( \|b_{ij}u'_{ij}\|_{j=1}^\infty \right)_{j=1}^\infty \right\|_{\lambda_{\rho^r, J}^r} \sup_{\|y'\| \leq 1} \left\| \left( \langle B(\mathbf{e}_{ij}), y' \rangle \right)_{j=1}^\infty \right\|_{\lambda_{\rho^*, K}^\times} \leq \\ &\leq \|A\| \sum_{i=1}^\infty \left\| \left( \|b_{ij}\|_{j=1}^\infty \right)_{j=1}^\infty \right\|_{\lambda_{\rho^r, J}^r} \sup_{\|y'\| \leq 1} \left\| \left( \langle \mathbf{e}_{ij}, B'(y') \rangle \right)_{j=1}^\infty \right\|_{\lambda_{\rho^*, K}^\times} \leq \\ &\leq \|A\| \sum_{i=1}^\infty \left\| \left( \|b_{ij}\|_{j=1}^\infty \right)_{j=1}^\infty \right\|_{\lambda_{\rho^r, J}^r} \sup_{\|y'\| \leq 1} \|B'\| \|y'\| \leq \|A\| \|B\| \|D\| \end{aligned} \quad (27)$$

and we obtain that  $\sum_{i=1}^\infty \sum_{j=1}^\infty x'_{ij} \otimes y_{ij} \in E \hat{\otimes}_{g_{\lambda, \rho}} F$ . On the other hand, by continuity of the involved maps, we easily obtain

$$\forall x \in E \quad T(x) = BDA(x) = \left\langle \sum_{i=1}^\infty \sum_{j=1}^\infty x'_{ij} \otimes y_{ij}, x \right\rangle$$

and hence  $T \in \mathcal{N}_{\lambda, \rho}(E, F)$ . The formula for  $\mathbf{N}_{\lambda, \rho}(T)$  follows from part a) and (29). ■

#### 4. $\lambda_\rho$ -integral operators

According to the general theory of tensor norms and operator ideals (see [5]) we have the following definition.

**Definition 12** *Given Banach spaces  $E$  and  $F$  an operator  $T \in \mathcal{L}(E, F)$  is said to be a  $\lambda_\rho$ -integral operator if it belongs to the maximal normed operator ideal  $(\mathcal{I}_{\lambda, \rho}, \mathbf{I}_{\lambda, \rho})$  associated with the tensor norm  $g_{\lambda, \rho}$ .*

A first result in order to get a complete characterization of  $\lambda_\rho$ -integral operators is provided by this theorem:

**Theorem 13** *Let  $H$  be a closed sublattice of a space  $L^\infty(\Omega, \mu)$ . Let  $T : H \longrightarrow \ell^1[\lambda_{\rho^r, J}^r]$  be a positive operator. Then  $T$  is  $\lambda_\rho$ -integral.*

**Proof.** Let  $I_1 : \ell^1[\lambda_{\rho^r, J}^r] \longrightarrow (\ell^1[\lambda_{\rho^r, J}^r])''$  be the canonical inclusion map. By the representation theorem of maximal operator ideals (theorem 17.5 in [5]) we only need to show that  $I_1 T$  is  $\lambda_\rho$ -integral from  $H$  into  $(\ell^1[\lambda_{\rho^r, J}^r])''$ . By proposition 4, (2)  $(\lambda_{\rho^r, J}^r)' = \lambda_{\rho^*, K}^\times$ . Then,  $(\lambda_{\rho^*, K}^\times)^{\times \times} \subset (\lambda_{\rho^r, J}^r)^\times \subset (\lambda_{\rho^r, J}^r)' = \lambda_{\rho^*, K}^\times$ . Hence  $\lambda_{\rho^*, K}^\times$  is perfect and by proposition 2, (2) we have  $(\ell^1[\lambda_{\rho^r, J}^r])' = \ell^\infty[\lambda_{\rho^*, K}^\times]$  and

$$(c_0[(\lambda_{\rho^*, K}^\times)^r])' = \ell^1[(\lambda_{\rho^*, K}^\times)^\times] \subset \ell^1[(\lambda_{\rho^r, J}^r)'] \subset (\ell^1[\lambda_{\rho^r, J}^r])''.$$

Let  $I_2 : \ell^1[(\lambda_{\rho^*, K}^\times)^\times] \longrightarrow (\ell^1[\lambda_{\rho^r, J}^r])''$  and  $I_3 : \ell^1[\lambda_{\rho^r, J}^r] \longrightarrow \ell^1[(\lambda_{\rho^*, K}^\times)^\times]$  be the inclusion maps. Clearly  $I_1 T = I_2 I_3 T$  and hence it is enough to prove that  $I_3 T \in \mathcal{I}_{\lambda, \rho}(H, \ell^1[(\lambda_{\rho^*, K}^\times)^\times])$ . Once again by the representation theorem of maximal operator ideals we only need to show that  $I_3 T \in (H \otimes_{g'_{\lambda, \rho}} c_0[(\lambda_{\rho^*, K}^\times)^r])'$ .

Every  $\mathbf{x} \in c_0[(\lambda_{\rho^*, K}^\times)^r]$  is a scalar double sequence  $(x_{ij})_{i, j=1}^\infty$  such that

$$\lim_{i \rightarrow \infty} \|(x_{ij})_{j=1}^\infty\|_{(\lambda_{\rho^*, K}^\times)^r} = 0$$

and

$$\|\mathbf{x}\| = \sup_{i \in \mathbb{N}} \|(x_{ij})_{j=1}^\infty\|_{(\lambda_{\rho^*, K}^\times)^r}.$$

By lemma 1 the linear span  $\mathcal{T}$  of the set  $\{\mathbf{e}_{ij}, i \in \mathbb{N}, j \in \mathbb{N}\}$  is dense in  $c_0[(\lambda_{\rho^*, K}^\times)^r]$ . Hence by density lemma (see [5]), our theorem will be proved if we show that  $I_3 T \in (H \otimes_{g'_{\lambda, \rho}} \mathcal{T})'$ .

Let

$$z = \sum_{i=1}^n f_i \otimes \mathbf{x}_i \in H \otimes T.$$

Given  $\varepsilon > 0$  there are finite dimensional subspaces  $M \subset H, N \subset T$  such that  $z \in M \otimes N$  and

$$g'_{\lambda, \rho}(z; M, N) \leq g'_{\lambda, \rho}(z; H, T) + \varepsilon.$$

Let  $\{\bar{f}_j\}_{j=1}^k$  and  $\{\mathbf{y}^j := (\eta_{nm}^j)\}_{j=1}^h$  be bases of  $M$  and  $N$ , respectively. Let  $s \in \mathbb{N}$  be such that  $\eta_{nm}^j = 0$  for every  $j = 1, 2, \dots, h$  and every  $n > s$  and  $m > s$ . Let us  $Q_{vt} : \ell^1[\lambda_{\rho^r, J}^r] \rightarrow \mathbb{R}$  (resp.  $P_{vt} : \ell^1[(\lambda_{\rho^*, K}^{\times})^{\times}] \rightarrow \mathbb{R}$ ) denote the canonical projection onto the  $(v, t)$ -axis of  $\ell^1[\lambda_{\rho^r, J}^r]$  (resp.  $\ell^1[(\lambda_{\rho^*, K}^{\times})^{\times}]$ ),  $v, t \in \mathbb{N}$ . Then, if

$$f = \sum_{i=1}^k \alpha_i \bar{f}_i \in M \quad \text{y} \quad \mathbf{x} = \sum_{j=1}^h \beta_j \mathbf{y}^j \in N$$

we have

$$\begin{aligned} \langle f \otimes \mathbf{x}, I_3 T \rangle &= \sum_{i=1}^k \sum_{j=1}^h \alpha_i \beta_j \langle \bar{f}_i \otimes \mathbf{y}^j, I_3 T \rangle = \\ &= \sum_{i=1}^k \sum_{j=1}^h \alpha_i \beta_j \langle I_3 T(\bar{f}_i), \mathbf{y}^j \rangle = \sum_{i=1}^k \sum_{j=1}^h \alpha_i \beta_j \sum_{n=1}^s \sum_{m=1}^s \eta_{nm}^j \langle I_3 T(\bar{f}_i), \mathbf{e}_{nm} \rangle = \\ &= \sum_{i=1}^k \sum_{j=1}^h \alpha_i \beta_j \sum_{n=1}^s \sum_{m=1}^s \eta_{nm}^j P_{nm} I_3 T(\bar{f}_i) = \\ &= \left\langle \sum_{n=1}^s \sum_{m=1}^s P_{nm} I_3 T \otimes \mathbf{e}_{nm}, \left( \sum_{i=1}^k \alpha_i \bar{f}_i \right) \otimes \sum_{j=1}^h \beta_j \mathbf{y}^j \right\rangle = \\ &= \left\langle \sum_{n=1}^s \sum_{m=1}^s P_{nm} I_3 T \otimes \mathbf{e}_{nm}, f \otimes \mathbf{x} \right\rangle = \langle U, f \otimes \mathbf{x} \rangle \end{aligned}$$

where  $U$  is the tensor

$$U := \sum_{n=1}^s \sum_{m=1}^s P_{nm} I_3 T \otimes \mathbf{e}_{nm} \in H' \otimes \ell^1[\lambda_{\rho^r, J}^r].$$

The last result is extended by linearity and we get

$$\forall z \in M \otimes N \quad \langle z, I_3 T \rangle = \langle U, z \rangle.$$

On the other hand, given  $\delta > 0$ , for every  $n, m = 1, 2, \dots, s$  there is  $g_{nm} \in H \subset L^\infty(\Omega, \mu)$  such that  $\|g_{nm}\| \leq 1$  y  $\|P_{nm} I_3 T\|_{H'} \leq |P_{nm} I_3 T(g_{nm})| + \delta$ . Putting  $g := \max\{|g_{nm}| / n, m = 1, 2, \dots, r\}$ , as  $H$  is a sublattice of  $L^\infty(\Omega, \mu)$  and  $I_3 T$  is a positive operator, we have  $g \in H$ ,  $\|g\|_H \leq 1$  and

$$\begin{aligned} \pi_{\rho, \lambda, J}((P_{nm} I_3 T)_{m=1}^s) &= \|(\|P_{nm} I_3 T\|_{H'})_{m=1}^s\|_{\lambda_{\rho^r, J}^r} \leq \\ &\leq \|(|P_{nm} I_3 T(g_{nm})|)_{m=1}^s\|_{\lambda_{\rho^r, J}^r} + \delta \left\| \sum_{m=1}^s \mathbf{e}_m \right\|_{\lambda_{\rho^r, J}^r} \leq \\ &\leq \|(P_{nm} I_3 T(g))_{m=1}^s\|_{\lambda_{\rho^r, J}^r} + \delta \left\| \sum_{m=1}^s \mathbf{e}_m \right\|_{\lambda_{\rho^r, J}^r} = \\ &= \left\| \sum_{m=1}^s Q_{nm} T(g) \mathbf{e}_{nm} \right\|_{\lambda_{\rho^r, J}^r} + \delta \left\| \sum_{m=1}^s \mathbf{e}_m \right\|_{\lambda_{\rho^r, J}^r}. \end{aligned} \quad (28)$$

Moreover, by proposition 4, (2) and (7), for every  $1 \leq n \leq s$

$$\begin{aligned} \varepsilon_{\rho^*, \lambda^*, K}((\mathbf{e}_{nm})_{m=1}^s) &= \\ &= \sup \left\{ \|(\langle \mathbf{e}_{nm}, (v_{mn}) \rangle)_{m=1}^s\|_{\lambda_{\rho^*, K}^*} / \|(v_{nm})\|_{\ell^\infty[\lambda_{\rho^*, K}^*]} \leq 1 \right\} = \\ &= \sup \left\{ \left\| \sum_{m=1}^s v_{nm} \mathbf{e}_{nm} \right\|_{\lambda_{\rho^*, K}^*} / \|(v_{nm})\|_{\ell^\infty[\lambda_{\rho^*, K}^*]} \leq 1 \right\} \leq 1. \end{aligned} \quad (29)$$

Hence, by definition of  $g_{\lambda, \rho}$  and using (29) y (28) we get

$$\begin{aligned} |\langle z, I_3 T \rangle| &= |\langle U, z \rangle| \leq g'_{\lambda, \rho}(z; M, N) g_{\lambda, \rho}(U) \leq \\ &\leq (g'_{\lambda, \rho}(z; H, T) + \varepsilon) \sum_{n=1}^s \pi_{\rho, \lambda, J}((P_{nm} I_3 T)_{m=1}^s) \varepsilon_{\rho^*, \lambda^*, K}((\mathbf{e}_{nm})_{m=1}^s) \leq \end{aligned}$$

$$\begin{aligned}
 &\leq (g'_{\lambda,\rho}(z; H, T) + \varepsilon) \left( \sum_{n=1}^s \left\| \sum_{m=1}^s Q_{nm} T(g) \mathbf{e}_{nm} \right\|_{\lambda_{\rho^r, J}^r} + \delta \sum_{n=1}^s \left\| \sum_{m=1}^s \mathbf{e}_m \right\|_{\lambda_{\rho^r, J}^r} \right) \leq \\
 &\leq (g'_\lambda(z; H, T) + \varepsilon) \left( \sup_{\|w\|_H \leq 1} \sum_{n=1}^s \left\| \sum_{m=1}^s Q_{nm} T(w) \mathbf{e}_{nm} \right\|_{\lambda_{\rho^r, J}^r} + \delta \sum_{n=1}^s \left\| \sum_{m=1}^s \mathbf{e}_m \right\|_{\lambda_{\rho^r, J}^r} \right) \leq \\
 &\leq (g'_{\lambda,\rho}(z; H, T) + \varepsilon) \left( \|T\| + \delta \sum_{n=1}^s \left\| \sum_{m=1}^s \mathbf{e}_m \right\|_{\lambda_{\rho^r, J}^r} \right).
 \end{aligned}$$

But  $\delta > 0$  being arbitrarily and independent of the subspaces  $M$  and  $N$ , we obtain

$$|\langle z, I_3 T \rangle| = |\langle U, z \rangle| \leq (g'_{\lambda,\rho}(z; H, T) + \varepsilon) \|T\|.$$

Finally,  $\varepsilon > 0$  being arbitrary, we have

$$|\langle z, I_3 T \rangle| = |\langle U, z \rangle| \leq g'_{\lambda,\rho}(z; H, T) \|T\|. \quad \blacksquare$$

**Definition 14** *A Banach lattice  $X$  is said to be lattice finitely representable in a lattice  $X$  if, for every finite dimensional sublattice  $M$  of  $X$  and for every  $\varepsilon > 0$ , there are a finite dimensional sublattice  $N$  of  $X$  and a lattice isomorphism  $J : M \rightarrow N$  such that  $\|J\| \|J^{-1}\| \leq 1 + \varepsilon$ .*

A deep result of Conroy and Moore (see [3] for example) asserts that the bidual lattice  $E''$  of every Banach lattice  $E$  is lattice finitely representable in  $E$ , a result which can be looked as the lattice version of the principle of local reflexivity. On the other hand finite representability is closely related to ultraproducts since a Banach lattice  $E$  is lattice finitely representable in a Banach lattice  $F$  if and only if  $E$  is order isometric to a sublattice of an ultrapower  $(F)_\mathcal{U}$  (see proposition 4.5 in [16] for instance).

Now we shall need two auxiliary results. First one is an extension of a theorem of Hollstein [8] which has been proved in [12] and concerns to arbitrary tensor products with  $\mathcal{L}^{\infty, \delta}$  spaces of Lindenstrauss. We recall that given  $1 \leq \delta < \infty$ , a Banach space  $E$  is said to be an  $\mathcal{L}^{\infty, \delta}$ -space if for every finite dimensional subspace  $P \subset E$  and  $\varepsilon > 0$  there is a finite dimensional subspace  $Q$ ,  $P \subset Q \subset E$  such that the Banach-Mazur distance  $d(Q, \ell_{\dim(Q)}^\infty) < \delta$ . It is known that the spaces  $L^\infty(\mu)$  are  $\mathcal{L}^{\infty, \delta}$ -spaces for every  $\delta > 1$ . Then we have the following.

**Proposition 15** *Let  $E$  be an  $\mathcal{L}^{\infty, \delta}$ -space,  $F$  a Banach space and  $H$  a closed subspace of  $F$ . For every finitely generated tensor norm  $\alpha$  we have the isomorphism*

$$(E \otimes_{\alpha} F)/(E \otimes H) \simeq E \otimes_{\alpha} F/H$$

and for every  $v \in E \otimes_{\alpha} F/H$  there is  $u \in E \otimes F$  such that  $(I_E \otimes K_H)(u) = v$  and  $\alpha(u; E, F) \leq \alpha(v; E, F/H)$  ( $I_E$  denotes the identity map on  $E$  and  $K_H : F \rightarrow F/H$  denotes the canonical quotient map).

The second result is a sort of approximation of finite dimensional vector subspaces of a Banach lattice  $G$  by finite dimensional sublattices:

**Lemma 16** (see [16], Lemma 4.4) *Let  $G$  be an order complete Banach lattice and let  $X \subset G$  be a finite dimensional Banach subspace of  $G$ . Then for every  $\varepsilon > 0$  there is a finite dimensional Banach sublattice  $Y$  of  $G$  and an operator  $A : X \rightarrow Y$  such that*

$$\forall x \in X \quad \|A(x) - x\| \leq \varepsilon \|x\|.$$

In particular,  $\|A\| \leq 1 + \varepsilon$ .

Now theorem 13 can be substantially generalized in the following way:

**Theorem 17** *Let  $H$  be a closed sublattice of a space  $L^{\infty}(\mu)$ . Let  $X$  be a Banach lattice which is lattice finitely representable in the Bochner lattice  $\ell^1[\lambda_{\rho^r, J}^r]$ . Then, every lattice homomorphism  $T : H \rightarrow X$  is  $\lambda_{\rho}$ -integral.*

**Proof.** By the representation theorem of maximal ideals (theorem 17.5 in [5]) it is enough to see that  $T \in (H \otimes_{g'_{\lambda, \rho}} X')'$ . Fix a representation

$$z = \sum_{i=1}^n h_i \otimes x'_i$$

of an element  $z \in H \otimes X'$ . Given  $\varepsilon > 0$  we choose  $M \in FIN(H)$  and  $N \in FIN(X')$  such that  $z \in M \otimes N$  and

$$g'_{\lambda, \rho}(z; M, N) \leq g'_{\lambda, \rho}(z; H, X') + \varepsilon. \tag{30}$$

We identify  $M$  with  $J_M(M) \subset J_H(H) \subset H''$ .  $H''$  is order isometric to some space  $L^{\infty}(\nu)$ ,  $H$  being an abstract  $M$ -space. Given  $\delta > 0$  since  $H''$  is order complete, by lemma

16, there is a finite dimensional sublattice  $H_1 \subset H''$  and a linear operator  $A : M \longrightarrow H_1$  such that

$$\forall h \in M \quad \|A(h) - h\| \leq \delta \|h\| \quad (31)$$

and

$$\|A\| \leq 1 + \delta. \quad (32)$$

If  $T'' : H'' \longrightarrow X''$  is the bitransposed map of  $T$  we have

$$\begin{aligned} |\langle T, z \rangle| &= \left| \sum_{i=1}^n \langle T(h_i), x'_i \rangle \right| \leq \\ &\leq \left| \sum_{i=1}^n \langle T''(h_i - A(h_i)), x'_i \rangle \right| + \left| \sum_{i=1}^n \langle T''A(h_i), x'_i \rangle \right| \leq \\ &\leq \|T''\| \sum_{i=1}^n \|h_i - A(h_i)\| \|x'_i\| + \left| \sum_{i=1}^n \langle T''A(h_i), x'_i \rangle \right|. \end{aligned} \quad (33)$$

Using Conroy and Moore's theorem,  $H''$  is lattice finitely representable in  $H$ . Then there is a sublattice  $H_2 \subset H$  and a lattice isomorphism  $B : H_1 \longrightarrow H_2$  such that  $\|B\| \|B^{-1}\| \leq 1 + \delta$ . By a result of Ando (see for instance theorem 1.4.19 in [15])  $T''$  is a lattice homomorphism also. Hence  $X_1 := T''B^{-1}(H_2) = T''(H_1)$  is a finite dimensional sublattice of  $X''$ . Once again by the theorem of Conroy and Moore, there exists a finite dimensional sublattice  $X_2 \subset X$  and a lattice isomorphism  $C : X_1 \longrightarrow X_2$  such that  $\|C\| \|C^{-1}\| \leq 1 + \delta$ . By our hypothesis, there is also a finite dimensional sublattice  $Z \subset \ell^1[\lambda_{\rho^r, J}^r]$  and a lattice isomorphism  $J : X_2 \longrightarrow Z$  such that  $\|J\| \|J^{-1}\| \leq 1 + \delta$ .

Define  $R := JCT''B^{-1} : H_1 \longrightarrow Z$ . Let  $V_Z : Z \longrightarrow \ell^1[\lambda_{\rho^r, J}^r]$  be the canonical inclusion,  $\bar{h}_i := BA(h_i)$  for every  $i = 1, 2, \dots, n$  and  $K : X''' \longrightarrow X'''/T''(H_1)^\perp = (T''(H_1))' = X'_1$  the canonical quotient map. We have

$$\begin{aligned} \left\langle \sum_{i=1}^n T''A(h_i), x'_i \right\rangle &= \sum_{i=1}^n \langle T''B^{-1}, BA(h_i) \otimes x'_i \rangle = \\ &= \sum_{i=1}^n \langle T''B^{-1}(\bar{h}_i), x'_i \rangle = \sum_{i=1}^n \langle (JC)^{-1}(JC)T''B^{-1}(\bar{h}_i), x'_i \rangle = \end{aligned}$$



$$\begin{aligned}
 &= \sum_{i=1}^n \langle (JC)^{-1}R(\bar{h}_i), K(x'_i) \rangle = \sum_{i=1}^n \langle R(\bar{h}_i), ((JC)^{-1})'K(x'_i) \rangle = \\
 &= \left\langle R, \sum_{i=1}^n \bar{h}_i \otimes ((JC)^{-1})'K(x'_i) \right\rangle. \tag{34}
 \end{aligned}$$

By theorem 13, the map  $V_Z R : H_1 \longrightarrow \ell^1[\lambda_{\rho^r, J}^r]$  is  $\lambda_\rho$ -integral and its integral norm verifies the equality settled in this theorem, all the involved operators in its definition being *positive*. Hence, if  $Q = V_Z' : (\ell^1[\lambda_{\rho^r, J}^r])' \longrightarrow Z' = (\ell^1[\lambda_{\rho^r, J}^r])'/Z^\perp$  is the canonical quotient map and  $\mathbf{z}_i \in (\ell^1[\lambda_{\rho^r, J}^r])'$  are elements such that  $Q(\mathbf{z}_i) = ((JC)^{-1})'K(x'_i)$  and the inequality of proposition 15 holds for every  $i = 1, 2, \dots, n$ , by (34) we have

$$\begin{aligned}
 &\left\langle \sum_{i=1}^n T''A(h_i), x'_i \right\rangle = \sum_{i=1}^n \langle R(\bar{h}_i), Q(\mathbf{z}_i) \rangle = \sum_{i=1}^n \langle V_Z R(\bar{h}_i), \mathbf{z}_i \rangle = \\
 &= \left\langle V_Z R, \sum_{i=1}^n \bar{h}_i \otimes \mathbf{z}_i \right\rangle \leq \|V_Z R\| g'_{\lambda, \rho} \left( \sum_{i=1}^n \bar{h}_i \otimes \mathbf{z}_i; H_2, (\ell^1[\lambda_{\rho^r, J}^r])' \right). \tag{35}
 \end{aligned}$$

As  $H_2$  is an abstract  $M$ -space, the dual  $H_2'$  is an abstract  $L$ -space and by the classical Bonhnenblust-Nakano-Kakutani's theorem and proposition II.5.3 in [11],  $H_2$  is an  $\mathcal{L}^{\infty, 1+\delta}$ -space. Using now proposition 15 and the metric property of mappings, by (35), (32) and (30) we obtain

$$\begin{aligned}
 &\left\langle \sum_{i=1}^n T''A(h_i), x'_i \right\rangle \leq (1 + \delta) \|R\| g'_{\lambda, \rho} \left( \sum_{i=1}^n \bar{h}_i \otimes Q(\mathbf{z}_i); H_2, Z' \right) \leq \\
 &\leq (1 + \delta) \|J\| \|T\| \|C\| \|B^{-1}\| g'_{\lambda, \rho} \left( \sum_{i=1}^n \bar{h}_i \otimes ((JC)^{-1})'K(x'_i); H_2, Z' \right) \leq \\
 &\leq (1 + \delta) \|J\| \|T\| \|C\| \|B^{-1}\| \|(JC)^{-1}\| \|K\| g'_{\lambda, \rho} \left( \sum_{i=1}^n \bar{h}_i \otimes x'_i; H_2, N \right) \leq \\
 &\leq (1 + \delta)^3 \|T\| \|B^{-1}\| g'_{\lambda, \rho} \left( \sum_{i=1}^n BA(h_i) \otimes x'_i; H_2, N \right) \leq
 \end{aligned}$$

$$\leq (1 + \delta)^3 \|T\| \|B^{-1}\| \|B\| \|A\| g'_{\lambda, \rho}(z; M, N) \leq (1 + \delta)^5 \|T\| (g'_{\lambda, \rho}(z; H, X') + \varepsilon). \quad (36)$$

Turning to (33), using (31) finally we get

$$|\langle T, z \rangle| \leq \delta \|T\| \sum_{i=1}^n \|h_i\| \|x'_i\| + (1 + \delta)^5 \|T\| (g'_{\lambda, \rho}(z; H, X') + \varepsilon).$$

Since  $\delta$  and  $\varepsilon$  are arbitrary we obtain

$$|\langle T, z \rangle| \leq \|T\| g'_{\lambda, \rho}(z; H, X')$$

as desired. ■

**Theorem 18** *Let  $E, F \in BAN$  and  $T \in \mathcal{L}(E, F)$ . Then  $T$  is  $\lambda_\rho$ -integral if and only if there is a measure space  $(\Omega, \mathcal{A}, \mu)$  such that  $J_F T$  can be factorized as*

$$\begin{array}{ccccc} E & \xrightarrow{T} & F & \xrightarrow{J_F} & F'' \\ \downarrow A & & & & \uparrow B \\ L^\infty(\Omega, \mu) & \xrightarrow{D} & & & X \end{array}$$

where  $X$  is a lattice finitely representable in  $\ell^1[\lambda_{\rho^r, J}]$  Banach lattice and  $D$  is a lattice homomorphism. Moreover,

$$\mathbf{I}_{\lambda, \rho}(T) = \inf\{\|B\| \|D\| \|A\|\}$$

over all factorizations of that type.

**Proof.** The sufficient condition follows from theorem 17 and elementary properties of operator ideals. Hence

$$\begin{aligned} \mathbf{I}_{g_{\lambda, \rho}}(T) &= \sup_{g'_{\lambda, \rho}(z; E, F') \leq 1} |\langle T, z \rangle| = \sup_{g'_{\lambda, \rho}(z; E, F') \leq 1} |\langle J_F T, z \rangle| = \\ &= \mathbf{I}_{g_{\lambda, \rho}}(J_F T) \leq \|B\| \|D\| \|A\|. \end{aligned}$$

To prove the necessary condition we define

$$\mathcal{D} = \{(M, N) \mid M \in FIN(E), N \in FIN(F')\}$$

and the partial order in  $\mathcal{D}$  given by  $(M_1, N_1) \leq (M_2, N_2) \iff M_1 \subset M_2$  and  $N_1 \subset N_2$ .  
The family

$$\mathcal{H} = \{ \mathcal{X}(M, N) / (M, N) \in \mathcal{D} \},$$

where

$$\forall (M_0, N_0) \in \mathcal{D} \quad \mathcal{X}(M_0, N_0) = \{ (M, N) \in \mathcal{D} \mid (M_0, N_0) \leq (M, N) \},$$

is a filter basis in  $\mathcal{D}$ . Let  $\mathcal{U}$  be an ultrafilter finer than  $\mathcal{H}$ .

Let  $T \in \mathfrak{I}_{\lambda, \rho}(E, F)$ . By the theorem of representation of maximal operator ideals,  $\phi = J_F T \in (E \otimes_{g'_{\lambda, \rho}} F)'$  and hence for every  $(M, N) \in \mathcal{D}$  and every  $z \in M \otimes N$  we have  $g'_{\lambda, \rho}(z; E, F) \leq g'_{\lambda, \rho}(z; M, N)$ . This means that the restriction  $\phi|_{M \otimes N}$  to  $M \otimes N$  of  $\phi$  belongs to  $(M \otimes_{g'_{\lambda, \rho}} N)' = M' \otimes_{g'_{\lambda, \rho}} N'$  and hence it is a  $\lambda_\rho$ -nuclear operator that furthermore,  $g'_{\lambda, \rho}$  being a finitely generated tensor norm, verifies

$$\mathbf{N}_{\lambda, \rho}(\phi|_{M \otimes N}) = \mathbf{I}_{\lambda, \rho}(\phi|_{M \otimes N}) \leq \mathbf{I}_{\lambda, \rho}(\phi).$$

By theorem 11, given  $\varepsilon > 0$  there is a factorization

$$\begin{array}{ccc} M & \xrightarrow{\phi|_{M \otimes N}} & N \\ A_{MN} \downarrow & & \uparrow B_{MN} \\ \ell^\infty[\ell^\infty] & \xrightarrow{D_{MN}} & \ell^1[\lambda_{\rho^r, J}^r] \end{array}$$

such that  $\|A_{MN}\| = 1, \|B_{MN}\| = 1$  and

$$\begin{aligned} \|B_{MN}\| \|D_{MN}\| \|A_{MN}\| &= \|D_{MN}\| \leq \mathbf{N}_{\lambda, \rho}(\phi|_{M \otimes N})(1 + \varepsilon) = \\ &= \mathbf{I}_{g_{\lambda, \rho}}(\phi|_{M \otimes N})(1 + \varepsilon) \leq \mathbf{I}_{g_{\lambda, \rho}}(\phi)(1 + \varepsilon). \end{aligned} \tag{37}$$

Put  $M_{MN} := M$  and  $N_{MN} := N$  for every  $(M, N) \in \mathcal{D}$ . Let  $G_E : E \rightarrow (M_{MN})_{\mathcal{U}}$  and  $G_{F'} : F' \rightarrow (N_{MN})_{\mathcal{U}}$  be the natural isometric embeddings into the corresponding ultraproducts, defined by  $G_E(x) = (z_{MN})_{\mathcal{U}}$  where  $z_{MN} = x$  if  $x \in M$  and  $z_{MN} = 0$  in other case and analogously for  $G_{F'}$ . We form the ultraproduct of spaces and operators

given in the previous diagram. If  $J_{\mathcal{U}} : (N'_{MN})_{\mathcal{U}} \rightarrow ((N_{MN})_{\mathcal{U}})'$  is the natural inclusion, we get the diagram

$$\begin{array}{ccccc}
 E & \xrightarrow{J_{FT}} & & & F'' \\
 \downarrow G_E & & & & \uparrow (G_{F'})' \\
 (M_{MN})_{\mathcal{U}} & \xrightarrow{(\phi|_{M \otimes N})_{\mathcal{U}}} & (N'_{MN})_{\mathcal{U}} & \xrightarrow{J_{\mathcal{U}}} & ((N_{MN})_{\mathcal{U}})' \\
 \downarrow A & & \uparrow B & & \\
 (\ell^\infty[\ell^\infty])_{\mathcal{U}} & \xrightarrow{D} & (\ell^1[\lambda_{\rho^r, J}^r])_{\mathcal{U}} & & 
 \end{array}$$

where we have defined  $A := (A_{MN})_{\mathcal{U}}$ ,  $D := (D_{MN})_{\mathcal{U}}$  and  $B := (B_{MN})_{\mathcal{U}}$ .

As  $H := (\ell^\infty[\ell^\infty])_{\mathcal{U}}$  is an abstract  $M$ -space, its bidual  $H''$  is a lattice order isometric to some space  $L^\infty(\Omega, \mu)$ . Let  $P : F'''' \rightarrow F''$  be canonical projection map. Clearly  $\|P\| \leq 1$  and  $PJ_{F''}J_{FT} = J_{FT}$ . Hence, if  $A_1 := J_H A G_E$ ,  $B_1 := P(G_{F'})''(J_{\mathcal{U}})''B''$  and  $Z := (\ell^1[\lambda_{\rho^r, J}^r])_{\mathcal{U}}$  we get the commutative diagram

$$\begin{array}{ccc}
 E & \xrightarrow{J_{FT}} & F'' \\
 \downarrow A_1 & & \uparrow B_1 \\
 L^\infty(\Omega, \mu) & \xrightarrow{D''} & Z''
 \end{array}$$

Since  $Z$  is lattice finitely representable in  $\ell^1[\lambda_{\rho^r, J}^r]$  by Conroy Moore's theorem,  $Z''$  is lattice finitely representable in  $\ell^1[\lambda_{\rho^r, J}^r]$  also. Moreover  $D$  is a positive and preserving disjointness linear map, (remember the comments given in section 1 when the order in an ultraproduct of Banach lattices was defined). Then  $D$  is a lattice homomorphism (theorem 1.3.11 in [15]). By a result of Ando (theorem 1.4.19 in [15])  $D''$  is a lattice homomorphism also and so we get the desired factorization. Furthermore

$$\|A_1\| \|D''\| \|B_1\| \leq \|G_E\| \|A\| \|D\| \|(G_{F'})'\| \|B\| =$$

$$= \lim_{(M,N), \mathcal{U}} \|B_{MN}\| \|D_{MN}\| \|A_{MN}\| \leq \mathbf{I}_{g_{\lambda\rho}}(\phi)(1 + \varepsilon)$$

and  $\varepsilon$  being arbitrary, having in mind the result of the sufficient condition in the present theorem, we get the desired formula to compute  $\mathbf{I}_{g_{\lambda\rho}}(\phi)$ . ■

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