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Abstract

In this paper, we present a brief review crossed modules [9], cat^1-groups [7], profinite crossed modules [6], cat^1-profinite groups [6], pullback profinite crossed modules [6] and also the pullback cat^1- profinite groups [2]. We prove that the pulback cat^1-profinite group has a left adjoint which is the induced cat^1-group.

Key Words: Crossed modules, Cat^1-groups, pullback, Profinite groups, Left, Right Adjoint, Cocomplete category.

1. Introduction

Crossed modules were introduced by J. H. C. Whitehead in [9]. Loday in [7] defined cat^1-groups and showed that the XMod category of crossed modules is equivalent to the Cat category of cat^1-groups. The manipulation of Crossed modules and cat^1-groups structures have been computer implemented using the group theory language GAP [8][1] in [3]. Enumeration of cat^1-groups of low order was also presented in [4]. Profinite crossed modules were defined Korkes and Porter in [6]. They also gave some useful completions about profinite crossed modules and cat^1-profinite groups in [6]. Pullback cat^1-profinite groups were defined in [2].

Our aim is to show that the equivalence between cat^1-groups and crossed modules due to Loday [7] takes pullback cat^1-profinite groups to the pullback profinite crossed modules.

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[1]Further information can be found at the following internet web site: www-gap.dcs.st-andrews.ac.uk/∼gap/

2. Crossed Modules and Cat^1-Groups

In this section we recall the descriptions of two equivalent categories: the category of crossed modules and their morphisms; and the category of cat^1-groups and their morphisms.

A crossed module $\mathcal{X} = (\partial : S \to R)$ consists of a group homomorphism ∂, called the *boundary* of $\mathcal{X}$, together with an action $\alpha : R \to \mathrm{Aut}(S)$ satisfying, for all $s, s' \in S$ and $r \in R$,

$$\begin{array}{lrcl} \textbf{CM1:} & \partial(s^r) & = & r^{-1}(\partial s)r \\ \textbf{CM2:} & s^{\partial s'} & = & s'^{-1}ss'. \end{array}$$

The standard examples of crossed modules are:

1. Any homomorphism $\partial : S \to R$ of abelian groups with R acting trivially on S may be regarded as a crossed module.
2. A *conjugation crossed module* is an inclusion of a normal subgroup $S \trianglelefteq R$, where R acts on S by conjugation.
3. A *central extension crossed module* has as boundary a surjection $\partial : S \to R$ with central kernel, where $r \in R$ acts on S by conjugation with $\partial^{-1}r$.
4. An *automorphism crossed module* has as range a subgroup R of the automorphism group $\mathrm{Aut}(S)$ of S which contains the inner automorphism group of S. The boundary maps $s \in S$ to the inner automorphism of S by s.
5. An *R-Module crossed module* has an R-module as source and ∂ is the zero map.
6. The direct product $\mathcal{X}_1 \times \mathcal{X}_2$ of two crossed modules has source $S_1 \times S_2$, range $R_1 \times R_2$ and boundary $\partial_1 \times \partial_2$, with $R_1,\ R_2$ acting trivially on $S_2,\ S_1$ respectively.
7. An important motivating topological example of crossed module due to Whitehead [9] is the boundary $\partial : \pi_2(X, A, x) \to \pi_1(A, x)$ from the second relative homotopy group of a based pair (X, A, x) of topological spaces, with the usual action of the fundamental group $\pi_1(A, x)$.

A morphism between two crossed modules $\mathcal{X} = (\partial : S \to R)$ and $\mathcal{X}' = (\partial' : S' \to R')$ is a pair (σ, ρ), where $\sigma : S \to S'$ and $\rho : R \to R'$ are homomorphisms satisfying

$$\partial' \sigma = \rho\partial, \ \ \sigma(s^r) = (\sigma s)^{\rho r}.$$

In [7] Loday reformulated the notion of a crossed module as a cat^1-group, namely a group G with a pair of homomorphisms $t, h : G \to G$ having a common image R and satisfying certain axioms. We find it convenient to define a cat^1-group $\mathcal{C} = (e; t, h : G \to R)$ as a group G, two surjections $t, h : G \to R, t(r, s) = r, h(r, s) = r(\partial s)$ and an embedding $e : R \to G, er = (r, 1)$ satisfying:

$$\textbf{CAT1:} \quad te = he = \mathrm{id}_R,$$
$$\textbf{CAT2:} \quad [\ker t, \ker h] = \{1_G\}.$$

The maps t, h are often called the *source* and *target*, but we choose to call them *tail* and *head* of $\mathcal{C}$, because *source* is the GAP term for the domain of a function. A morphism $\mathcal{C} \to \mathcal{C}'$ of cat^1-groups is a pair (γ, ρ) where $\gamma : G \to G'$ and $\rho : R \to R'$ are homomorphisms satisfying

$$h'\gamma = \rho h, \quad t'\gamma = \rho t, \quad e'\rho = \gamma e.$$

3. Profinite crossed modules and cat^1-profinite groups

A profinite crossed module [6] $\mathcal{PX} = (\partial : S \to R)$ is a crossed module in which S and R profinite groups, S acts continuously on R and ∂ is a continuous group homomorphism.

The following are examples of profinite crossed modules [6]:

1. Let H be a closed normal subgroup of profinite group R with $i : H \to R$ the inclusion. Then we will say $(i : H \to R)$ is a closed normal subgroup pair. In this case, of course, R acts continuously on the left of H by conjugation and the inclusion homomorphism i makes $(i : H \to R)$ into a profinite crossed module.

2. Let R be a finitely generated profinite group, then $\mathrm{Aut}(R)$, the group of continuous automorphisms of R, is also profinite in the topology of uniform convergence. Conjugation gives a continuous homomorphism $\partial : R \to \mathrm{Aut(R)}$. Absolutely, $\mathrm{Aut(R)}$ acts continuously on R and ∂ is a profinite crossed module.

3. Suppose given a continuous morphism $\theta : M \to N$ of pseudocompact left R-modules and form the semidirect product $R \ltimes R$. This is a profinite group which we make act continuously on M via the projection from $R \ltimes N$ to R. We define a continuous morphism $\partial : M \to R \ltimes N$ by $\partial(m) = (1, \theta(m))$ where 1 denotes identity in R then $(\partial : M \to R \ltimes N)$ is a profinite crossed module.

If $\mathcal{PX} = (\partial : S \to R)$ and $\mathcal{PX}' = (\partial' : S' \to R')$ are profinite crossed modules and $(\mu, \eta) : (\partial : S \to R) \to (\partial' : S' \to R')$ is a morphism between them in which the pair (μ, η) are both continuous, then the pair (μ, η) is called a morphism of profinite crossed modules.

A cat^1-profinite group is a cat^1-group $\mathcal{C} = (e; t, h : G \to R)$ in which G is a profinite group and t and h are continuous endomorphisms of G.

A morphism of cat^1-profinite groups is a morphism $\phi : \mathcal{C} = (e; t, h : G \to R) \to \mathcal{C}' = (e'; t', h' : G' \to R')$ of the underlying cat^1-profinite groups such that ϕ is a continuous morphism of profinite groups.

To any pre cat^1-profinite group there is a canonically associated a cat^1-profinite group $\mathcal{C}$, obtained by quotienting the source group by the Peiffer subgroup $[\ker t, \ker h]$. Then the corresponding functor is denoted

$$\text{ass: (pre cat}^1\text{-profinite groups)} \to (\text{cat}^1\text{-profinite groups}),$$

and is clearly the identity when restricted to cat^1-profinite groups [5].

4. Pullbacks of Profinite Crossed modules and cat^1-profinite groups

Let $\mathcal{PX} = (\partial : S \to R)$ be a profinite crossed module and $\iota : Q \to R$ be a continuous homomorphism of profinite groups. Then $\iota^*\mathcal{X} = (\partial^* : \iota^* S \to Q)$ is the pullback of $\mathcal{PX}$ by ι. So that $\iota^* S \subset Q \times S$ is the closed subgroup given by

$$\iota^* S = \{(q, s) \in Q \times S \mid \iota q = \partial s\}$$

and Q acts continuously on the right of $\iota^* S$ by

$$(q_1, s)^q = (q^{-1} q_1 q, s^{\iota q})$$

and since $\partial^*(q_1, s) = q_1$.

Let $\mathcal{PC} = (e; t, h : G \to R)$ be a cat^1-profinite group and $\iota : Q \to R$ be a continuous homomorphism. Then $e^{**}; t^{**}, h^{**} : \iota^{**} G \to Q$ is a pullback of G where $\iota^{**} G \subset Q \times G \times Q$ and

$$\iota^{**} G = \{(q_1, g, q_2) \in Q \times G \times Q \mid \iota q_1 = tg, \ \iota q_2 = hg\}$$

Now we can define tail, head and emmbedding as follows:

$$\begin{aligned} t^{**}(q_1, g, q_2) &= q_1 \\ h^{**}(q_1, g, q_2) &= q_2 \\ e^{**}(q) &= (q, e\iota q, q). \end{aligned}$$

The equivalence between the category of ProfXMod and ProfCat1 was shown in [2].

Proposition 4.1 If $\iota^*\mathcal{P}\mathcal{X}$ is the pullback of the profinite crossed module $\mathcal{X}$ over $\iota : Q \to R$ and if $\mathcal{C}, \mathcal{D}$ are the cat^1-profinite groups obtained from $\mathcal{X}, \iota^*\mathcal{X}$ respectively, then $\mathcal{D} \cong \iota^{**}\mathcal{C}$.

Proof:

$$\begin{array}{ccc} \iota^*S & \longrightarrow & S \\ {\scriptstyle \partial^\bullet}\big\downarrow & & \big\downarrow{\scriptstyle \partial} \\ Q & \xrightarrow[\iota]{} & R. \end{array}$$

Starting with the pullback profinite crossed module $\iota^*\mathcal{X} = (\partial^\bullet : \iota^*S \to Q)$, the source group of $\mathcal{D}$ is defined as the semi-direct product $Q \ltimes \iota^*S$.

$$\begin{array}{ccc} Q \ltimes \iota^*S & \longrightarrow & R \ltimes S \\ {\scriptstyle t^\bullet}\big\downarrow\big\downarrow{\scriptstyle h^\bullet} & & {\scriptstyle t}\big\downarrow\big\downarrow{\scriptstyle h} \\ Q & \xrightarrow[\iota]{} & R. \end{array}$$

The tail, head and embedding of $\mathcal{D}$ are respectively given by

$$\begin{array}{rcl} t^\bullet(q',(q,s)) & = & q' \\ h^\bullet(q',(q,s)) & = & q'\partial^\bullet(q,s) \\ & = & q'q \\ e^\bullet(q) & = & (q,(1_Q,1_S)) \end{array}$$

We then define an isomorphism of cat^1-profinite groups $(\psi, \mathrm{id}_Q) : \mathcal{D} \to \iota^{**}\mathcal{C}$,

$$\begin{array}{ccccc} & Q \ltimes \iota^*S & \xrightarrow{\psi} & \iota^{**}(R \ltimes S) & \\ {\scriptstyle e^\bullet}\big\uparrow & {\scriptstyle t^\bullet}\big\downarrow\big\downarrow{\scriptstyle h^\bullet} & & {\scriptstyle t^{**}}\big\downarrow\big\downarrow{\scriptstyle h^{**}} & \big\uparrow{\scriptstyle e^{**}} \\ & Q & \xrightarrow[\mathrm{id}]{} & Q & \end{array}$$

where

$$\psi(q',(q,s)) = (q',(\iota q',s),q'q).$$

First note that $\psi(q',(q,s)) \in \iota^{**}(R \ltimes S)$ because

$$t(\iota q',s) = \iota q'$$

and

$$h(\iota q',s) = (\iota q')(\partial s) = (\iota q')(\iota q) = \iota(q'q).$$

We verify that ψ is a homomorphism as follows:

$$\begin{array}{rcl} \psi((q_1',(q_1,s_1))(q_2',(q_2,s_2)) & = & \psi(q_1'q_2',(q_1^{q_2'}q_2, s_1^{\iota q_2'}s_2)) \\ & = & (q_1'q_2',(\iota(q_1'q_2'), s_1^{\iota q_2'}s_2), q_1'q_1q_2'q_2) \\ \psi(q_1',(q_1,s_1))\psi(q_2',(q_2,s_2)) & = & (q_1',(\iota q_1',s_1),q_1'q_1)(q_2',(\iota q_2',s_2),q_2'q_2) \\ & = & (q_1'q_2',(\iota q_1',s_1)(\iota q_2',s_2),q_1'q_1q_2'q_2) \\ & = & (q_1'q_2',((\iota q_1')(\iota q_2'), s_1^{\iota q_2'}s_2), q_1'q_1q_2'q_2). \end{array}$$

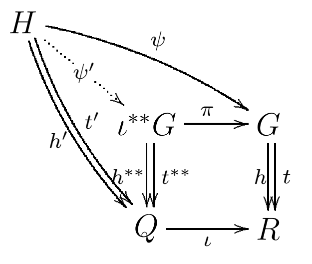
The inverse of ψ is given by $\psi^{-1}(q_1,(r,s),q_2) = (q_1,(q_1^{-1}q_2,s))$.

Then

$$\begin{array}{rcl} t^{**}\psi(q',(q,s)) & = & t^{**}(q',(\iota q',s),q'q) \\ & = & q' \\ & = & t^{\bullet}(q',(q,s)), \\ h^{**}\psi(q',(q,s)) & = & h^{**}(q',(\iota q',s),q'q) \\ & = & q'q \\ & = & h^{\bullet}(q',(q,s)), \\ \psi e^{\bullet}(q) & = & \psi(q,(1_Q,1_S)) \\ & = & (q,(\iota q,1_s),q) \\ & = & e^{**}(q), \end{array}$$

so the diagram commutes and the proof is complete. □

The universal property of induced cat^1-profinite group is the following. Let $\mathcal{C} = (e;t,h : G \to R)$ be a cat^1-group and let $\iota^{**}\mathcal{C} = (e^{**};t^{**},h^{**} : \iota^{**}G \to Q)$ be induced by the homomorphism $\iota : Q \to R$, is given by the diagram



where the pair (π, ι) is a morphism of cat^1-group such that for any cat^1-group $\mathcal{H} = (e'; t', h' : H \to Q)$ and any morphism of cat^1-group $(\psi, \iota) : \mathcal{C} \to \mathcal{H}$ there is a unique morphism $((\psi', 1) : \iota^{**}\mathcal{C} \to \mathcal{H}))$ of cat^1-profinite groups such that $\pi\psi' = \psi$.

5. Construction of the left adjoint

The construction of left adjoint of pullback cat^1-groups were given in [1].

Proposition 5.1 The category of cat^1-profinite groups is cocomplete.

Proof: Let $F : \mathbf{C} \to (\mathrm{cat}^1 - \mathrm{profinitegroups})$. We wish to construct colim F. For each object c of $\mathbf{C}$, we write

$$F(c) = (e_c; t_c, h_c : F_1(c) \to F_0(c),).$$

Then we form

$$\begin{aligned} F' &= (e'; t', h' : \mathrm{colim}_c F_1(c) \to \mathrm{colim}_c F_0(c)) \\ &= (e'; t', h' : F_1' \to F_0'), \end{aligned}$$

where F_1', F_0' are the colimits in the category of profinite groups, so that $t'e' = h'e' = 1$. So F' is a pre-cat^1-profinite group. The required colimit is then the associated cat^1-profinite group **ass** F',

$$\mathbf{ass}F' = (e''; t'', h'' : F_1'/[\ker t', \ker h'] \to F_0').$$

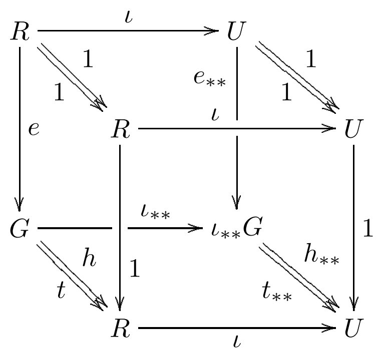
□

Proposition 5.2 The functor $\iota^{**} : Cat^1ProGrp/U \to Cat^1ProGrp/R$ has a left adjoint $\iota_{**} : Cat^1ProGrp/R \to Cat^1ProGrp/U$.

Proof:
We can give the left adjoint construction as follows. Let $\mathcal{C} = (e; t, h : G \to R)$ be a cat^1-profinite group over R and $\iota : R \to U$ is a morphism of profinite groups. Then the induced cat^1-profinite group is $\iota_{**}\mathcal{C} = (e_{**}; t_{**}, h_{**} : \iota_{**}G \to U)$ is given by the pushout

$$\begin{array}{ccc} (1; 1, 1 : R \to R) & \xrightarrow{\iota} & (1; 1, 1 : U \to U) \\ \downarrow & & \downarrow \\ (e; t, h : G \to R) & \longrightarrow & (e_{**}; t_{**}, h_{**} : \iota_{**}G \to U). \end{array}$$

We draw the above diagram as a three dimensional diagram as follows



in the category of cat^1-profinite groups. For computational purposes note that, by the previous proposition 5.1, $\iota_{**}G = (G *_R U)/[\ker t_{**}, \ker h_{**}]$, where $*_R$ denotes coproduct of profinite groups, that is, a free product with amalgamation over R. □

References

[1] Alp, M., Left adjoint of pullback cat^1-groups, *Turkish Journal of Mathematics,* Vol. 23 No. 2 243-249 (1999).

[2] Alp, M., Pullbacks of Profinite Crossed modules and cat^1-profinite groups, *Journal of Institute of Mathematics Computer Sciences,* (To appear).

[3] Alp, M., and Wensley, C. D., XMOD, Crossed modules and cat1-groups in GAP, version 1.3 *Manual for the XMOD share package,* 1-80, (1997).

[4] Alp, M., and Wensley, C. D., Enumeration of cat^1-group of low order, *International Journal of Algebra and Computation,* Vol. 10, No. 4 407- 424 (2000).

[5] Brown, R. and Loday, J. L., Homotopical excision, and Hurewicz theorems, for n-cubes of spaces, *Proc. London Math. Soc.,* (3) 54 176-192, (1987).

[6] Korkes , F. J., and Porter, T., Profinite crossed modules, *U. W. B. Pure Mathematics Preprint* No. 86.11, pp. 1-37, (1986).

[7] Loday, J. L., Spaces with finitely many non-trivial homotopy groups, *J.App.Algebra,* 24 179-202, (1982).

[8] Schönert, M. et al, GAP: Groups, Algorithms, and Programming, Lehrstuhl D für Mathematik, Rheinisch Westfölische Technische Hochschule, Aachen, Germany, third edition, 1993.

[9] Whitehead, J. H. C., On adding relations to homotopy groups, *Ann. Math.,* 47 806-810, (1946).

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