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\theta-Euclidean L-fuzzy Ideals of Rings

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Abstract

The concept of fuzzy ideals is extended by introducing θ-Euclidean L-fuzzy ideals in rings. In particular, some structural theorems for a θ-Euclidean L-fuzzy ideal of R are proved.

Key Words: Fuzzy quotient ring; isomorphism of rings; θ-Euclidean L-fuzzy ideal.

1. Introduction

In this paper we define a θ-Euclidean L-fuzzy ideal on a commutative ring with identity. Then we examine θ-Euclidean L-fuzzy ideals of the ring. In particular, we give some structural theorems for a θ-Euclidean L-fuzzy ideal. We also give a theorem similar to the Factorization of Homomorphisms Theorem.

2. Preliminaries

Throughout this paper, R denotes a commutative ring with identity and ring homomorphisms preserve identities. L denotes a lattice with the least element 0 and the greatest element 1. Unless stated otherwise, L is complete and completely distributive in the sense that it satisfies the following law:

$$\bigvee\{a_i \mid i \in I\} \wedge \bigvee\{b_j \mid j \in J\} = \bigvee\{a_i \wedge b_j \mid i \in I, j \in J\}[4]$$

for all $a_i, b_j \in L$.

Definition 2.1 **[4]** An L-fuzzy ideal is a function $J\colon R \to L$ satisfying the following axioms for all $x, y \in R$,

(i) $J\ (x + y) \geq J$ (x) $\wedge\ J$ (y),

(ii) $J\ (-x) = J$ (x),

(iii) J (xy) $\geq J$ (x) $\vee\ J$ (y).

Since we are considering L-fuzzy ideals over a fixed lattice L, we shall call them fuzzy ideals only.

Definition 2.2 **[3].** Let $J\colon R \to L$ be a fuzzy ideal. The fuzzy subset $x + J\colon R \to L$ defined by $(x + J)(y) = J(y - x)$ is called a coset of the fuzzy ideal J.

The set of all cosets of a fuzzy ideal J forms a ring under the binary operations '+' and '.' defined as

$$(x + J) + (y + J) = (x + y) + J \text{ and } (x + J).(y + J) = xy + J.$$

We shall denote this ring by R/J.

Let $R_J = \{\, x \in R \mid J(x) = J(0)\,\}$. This is a $J(0)$-level cut of J and hence is an ideal of R offering us the factor ring R/R_J [2].

Theorem 2.3 **[3]** The ring R/J is isomorphic to the ring R/R_J. The isomorphic correspondence is given by $x + J \leftrightarrow x + R_J$.

Definition 2.4 **[7].** Let $f\colon R \to R'$ be a homomorphism. For the fuzzy point 0_1 of R', set Ker $f = f^{-1}(0_1)$ and call Ker f the fuzzy kernel of f.

Proposition 2.5 **[3]** If $f\colon R \to R'$ is a homomorphism and $J\colon R \to L$ and $J'\colon R' \to L$ are fuzzy ideals, then

(i) $f^{-1}(J')$ is a fuzzy ideal which is constant on Ker f,

(ii) $f^{-1}(R'_{J'}) = R_{f^{-1}(J')}$,

(iii) If f is an epimorphism, then $ff^{-1}(J') = J'$.

(iv) If J is constant on Ker f, then $f^{-1}f(J) = J$.

It may be noted that in Proposition 2.5, neither L is assumed to be complete distributive, nor $f(J)$ is claimed to be a fuzzy ideal [3].

This assumption is made in the following [3]:

Proposition 2.6 [3] If L is a complete distributive lattice and $f\colon R \to R'$ is an epimorphism, then $f(J)$ is a fuzzy ideal.

We will define now a θ-Euclidean L-fuzzy ideal on a commutative ring with identity. Strictly speaking we add an extra condition to the definition of the fuzzy ideal as follows:

Definition 2.7 Let $\theta\colon R \to L$ be a non-constant fuzzy subset of R. A function $\varphi\colon R \to L$ is called a θ-Euclidean L-fuzzy ideal if φ satisfies the following axioms.

(i) $\varphi(x+y) \geq \min\{\varphi(x), \varphi(y)\}$ for all x, y in R,

(ii) $\varphi(-x) = \varphi(x)$,

(iii) $\varphi(xy) \geq \max\{\varphi(x), \varphi(y)\}$,

(iv) For any $x, y \in R$, with $y \neq 0$, there exist elements $q, r \in R$ such that $x = yq + r$ where either $r = 0$ or else $\max\{\varphi(r), \theta(r)\} \geq \max\{\varphi(y), \theta(y)\}$.

Example. Let Z be the ring of integers and $\varphi\colon Z \to [0,1]$ be a fuzzy subset defined by

$$\varphi(a) = \begin{cases} 1 & \text{if } a = 0, \\ 1/3 & \text{if } a \in (2) - 0, \\ 0 & \text{if } a \in Z - (2). \end{cases}$$

Let $\theta\colon Z \to [0,1]$ be a fuzzy subset defined by

$$\theta(a) = \begin{cases} 0 & \text{if } a = 0, \\ 1/3 & \text{if } a = \pm 3,\ \pm 5,\ \pm 7,\ \pm 9, ... \\ 1/\mid a \mid & \text{otherwise.} \end{cases}$$

So φ is a $[0,1]$-fuzzy ideal of Z. Also φ is a θ-Euclidean $[0,1]$-fuzzy ideal of Z.

Example. Let Z be the ring of integers and $\varphi\colon Z \to [0,1]$ be a fuzzy set defined by

$$\varphi(a) = \begin{cases} 1 & \text{if } a = 0, \\ 1/3 & \text{if } a \in (2) - 0, \\ 0 & \text{if } a \in Z - (2). \end{cases}$$

Let $\theta_1 : Z \to [0,1]$ be a fuzzy subset defined by

$$\theta_1(a) = \begin{cases} 0 & \text{if a=0,} \\ 1/\mid a \mid & \text{otherwise.} \end{cases}$$

So φ is a $[0,1]$-fuzzy ideal of Z. But φ is not a θ_1-Euclidean $[0,1]$-fuzzy ideal of Z.

Theorem 2.8 Let $f\colon R \to R'$ be an isomorphism of the rings and $\varphi'\colon R' \to L$ be a θ'-Euclidean L-fuzzy ideal of R'. Then $\varphi' \circ f\colon R \to L$ is a $\theta' \circ f$-Euclidean L-fuzzy ideal of R. Here, we mean that $(\varphi' \circ f)(x) = \varphi'(f(x))$

Proof. Let $\varphi = \varphi' \circ f$, $\theta = \theta' \circ f$ and also $a, b \in R$. Then

$$\begin{tikzcd} R \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & L \\ & R' \arrow[ur, "\varphi'"'] & \end{tikzcd}$$

Because of Proposition 2.5 [3], $\varphi\colon R \to L$ is an L-fuzzy ideal of R. So it must be shown that (iv) is satisfied. $\square$

(iv) Let $a, b \in R$. Then $f(a), f(b) \in R'$. Since φ' is a θ'-Euclidean L-fuzzy ideal of R', there exist elements $f(r), f(q) \in R'$ such that $f(a) = f(b)f(q)+f(r)$ where either $f(r) = 0$ or else $\max\{\varphi'(f(r)), \theta'(f(r))\} \geq \max\{\varphi'(f(b)), \theta'(f(b))\}$. Since f is an isomorphism, we can write

$$f(a) = f(bq) + f(r)$$

and

$$f(a) = f(bq + r),$$

thus (using one-to-oneness)

$$\Rightarrow a = bq + r.$$

First, if $f(r) = 0$, then $r = 0$, since f is one-to-one.

Otherwise if $\max\{\varphi'(f(r)), \theta'(f(r))\} \geq \max\{\varphi'(f(b)), \theta'(f(b))\}$, then $\max\{(\varphi' \circ f)(r), (\theta' \circ f)(r)\} \geq \max\{(\varphi' \circ f)(b), (\theta' \circ f)(b)\}$.

So we get

$$\max\{\varphi(r), \theta(r)\} \geq \max\{\varphi(b), \theta(b)\}.$$

Therefore $\varphi\colon R \to L$ is a θ-Euclidean L-fuzzy ideal of R.

Theorem 2.9 Let $f\colon R \to R'$ be an onto homomorphism of the rings and $\varphi\colon R \to L$ be a θ-Euclidean L-fuzzy ideal which is constant on Ker f. Also suppose that $\theta(a) = \theta(b)$ when $a - b \in \text{Ker } f$. Then $f(\varphi)\colon R' \to L$ is an $f(\theta)$-Euclidean L-fuzzy ideal of R'.

Proof.

$$\begin{tikzcd} R \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & L \\ & R' \arrow[ur, "f(\varphi)"'] & \end{tikzcd}$$

$$\begin{tikzcd} a \arrow[rr, mapsto] \arrow[dr] & & \varphi(a) \\ & f(a) \arrow[ur] & \end{tikzcd}$$

Let $x' \in R'$. Then there exist elements $x_0 \in R$ such that $x' = f(x_0)$. Since φ is constant on Ker f, we get $\varphi(z) = \varphi(x_0)$ for all $z \in f^{-1}(x')$. Suppose $f(z) = x'$ and

$f(x_0) = x'$ for a moment. Then $f(z - x_0) = 0$ and so we obtain $z - x_0 \in \mathrm{Ker}\, f$. That is to say,

$$\varphi(z - x_0) = \varphi(0) \Rightarrow \varphi(z) = \varphi(x_0).$$

So

$$f(\varphi)(x') = \bigvee\{\varphi(z) \mid z \in f^{-1}(x')\} = \varphi(x_0)$$

and we get $f(\varphi)(y') = \varphi(y_0)$ in a similar way. □

Is $f(\varphi)$ an $f(\theta)$-Euclidean L-fuzzy ideal of R' ?

Because of Proposition 2.6.[3], $f(\varphi)$ is an L-fuzzy ideal of R'. So it must be shown that (iv) is satisfied.

(iv) Let $x', y' \in R'$, then there exist elements $x_0, y_0 \in R$ such that $f(x_0) = x'$, $f(y_0) = y'$. Since φ is a θ-Euclidean L-fuzzy ideal of R, there exist elements q_0 , $r_0 \in R$ such that $x_0 = y_0q_0 + r_0$, where either $r_0 = 0$ or else $\max\{\varphi(r_0)\ ,\ \theta(r_0)\} \geq \max\{\varphi(y_0)\ ,\ \theta(y_0)\}$. So $f(x_0) = f(y_0q_0 + r_0)$. Therefore we get $f(x_0) = f(y_0)f(q_0) + f(r_0)$. So there exist x', y', q', $r' \in R'$, such that $f(x_0) = x'$, $f(y_0) = y'$, $f(q_0) = q'$, $f(r_0) = r'$.

Let $r_0 = 0$.Then $f(r_0) = f(0) = 0$.

Since $\theta(a) = \theta(b)$ in case $a - b \in \mathrm{Ker}\, f$, we obtain

$$f(\theta)(r') = \bigvee\{\theta(z) \mid z \in f^{-1}(r')\} = \theta(r_0)\ .$$

If $\max\{\varphi(r_0)\ ,\ \theta(r_0)\} \geq \max\{\varphi(y_0)\ ,\ \theta(y_0)\}$ then $\max\{f(\varphi)(r') = \varphi(r_0), f(\theta)(r') = \theta(r_0)\} \geq \max\{\varphi(y_0) = f(\varphi)(y'), \theta(y_0) = f(\theta)(y')\}$. That is $\max\{\ f(\varphi)(r')\ ,\ f(\theta)(r')\ \} \geq \max\{\ f(\varphi)(y')\ ,\ f(\theta)(y')\ \}$.

So $f(\varphi)$ is an $f(\theta)$-Euclidean L-fuzzy ideal of R'.

3. Fuzzy-quotient rings

Let M be a fuzzy ideal of A. For all $x \in A$ let $x + M$ be the fuzzy subset of A defined by

$$(x + M)(y) = M(y - x)$$

for all $y \in A$. The fuzzy subset $x + M$ is called a fuzzy coset of the fuzzy ideal M. The set of all such fuzzy cosets will be denoted by A/M. Two binary operations on A/M (denoted by + and .) are defined as follows: for all $x, y \in A$,

$$(x + M) + (y + M) = (x + y) + M\ ,$$

$$(x + M).(y + M) = (x.y) + M\ .$$

The above two operations are well defined and make A/M into a ring, called the fuzzy-quotient ring of A by M [6].

Theorem 3.1 [1](Factorization of Homomorphisms). Let f be a homomorphism of the ring R onto the ring R', and I be an ideal of R such that $I \subseteq \operatorname{Ker} f$. Then there exists a unique homomorphism $\bar{f}\colon R/I \to R'$ with the property that $f = \bar{f} \circ n_I$, where $n_I : R \to R/I$ is the natural homomorphism.

We can give a similar theorem to Theorem 3.1 as follows:

Theorem 3.2 Let $J\colon R \to L$ be a θ-Euclidean L-fuzzy ideal, $n\colon R \to R/R_J$ be the natural homomorphism. Also suppose that $\theta(a) = \theta(b)$ when $a - b \in \operatorname{Ker} n$. Let $\varphi\colon R/R_J \to L$ be defined as $\varphi(a + R_J) = J(a)$. Then there exists a unique $\theta^*(= n(\theta))$-Euclidean L-fuzzy ideal $\varphi\colon R/R_J \to L$ with the property that $J = \varphi \circ n$.

Proof. First we will show that this function is well-defined. Let $a + R_J = b + R_J$. So there exists $x \in R_J$ such that $a - b = x$.Using the definition of R_J , we obtain $J(x) = J(0)$. □

$$\begin{aligned} J(0) &= J(x) = J(a-b) \\ &\Rightarrow J(0) = J(a-b) \\ &\Rightarrow J(a) = J(b). \end{aligned}$$

Therefore we get $J(a) = J(b)$. This means that

$$\varphi(a + R_J) = \varphi(b + R_J).$$

So φ is well-defined.

Let $a+R_J$, $b+R_J$ be in R/R_J .

(i)

$$\begin{aligned}\varphi[(a+R_J)+(b+R_J)] &= \varphi[(a+b)+R_J]\\ &= J(a+b)\\ &\geq \min\{J(a),J(b)\}\\ &= \min\{\varphi(a+R_J)\ ,\ \varphi(b+R_J)\}.\end{aligned}$$

(ii)

$$\begin{aligned}\varphi[-(a+R_J)] &= \varphi[(-a+R_J)]\\ &= J(-a)\\ &= J(a)\\ &= \varphi[(a+R_J)].\end{aligned}$$

(iii)

$$\begin{aligned}\varphi[(a+R_J).(b+R_J)] &= \varphi[(ab)+R_J]\\ &= J(ab)\\ &\geq \max\{J(a),J(b)\}\\ &= \max\{\varphi(a+R_J)\ ,\ \varphi(b+R_J)\}.\end{aligned}$$

(iv) Let $a+R_J$, $R_J \neq b+R_J \in R/R_J$.

$$\Rightarrow b \notin R_J \Rightarrow J(b) \neq J(0)$$

$$\Rightarrow b \neq 0.$$

So a , $0 \neq b \in R$. Since J is a θ-Euclidean L-fuzzy ideal of R, there exist elements $q,r \in R$ such that $a=bq+r$, where either $r=0$ or else $\max\{J(r),\theta(r)\} \geq \max\{J(b),\theta(b)\}$.

$$\begin{aligned}a=bq+r &\Rightarrow a+R_J = bq+r+R_J\\ &\Rightarrow a+R_J = (bq+R_J)+(r+R_J).\end{aligned}$$

Therefore we obtain $a + R_J = (b + R_J).(q + R_J) + (r + R_J)$. If $r = 0$, then $r + R_J = 0 + R_J$. So $r + R_J = R_J$. Since $r, q \in R$, we get $r + R_J$, $q + R_J \in R/R_J$.

Let $r + R_J = r'$.

If $n(z) = r'$ and $n(r) = r'$, then $n(z - r) = 0'$.This means that $z - r \in \operatorname{Ker} n$. Hence we get $\theta(z) = \theta(r)$.So

$$n(\theta)(r') = \bigvee\{\theta(z) \mid z \in n^{-1}(r')\} = \theta(r).$$

If $\max\{J(r), \theta(r)\} \geq \max\{J(b), \theta(b)\}$, then $\max\{\varphi(r+R_J) = J(r), \theta(r) = n(\theta)(r')\} \geq \max\{\varphi(b + R_J) = J(b), \theta(b) = n(\theta)(b')\}$. So $\max\{\varphi(r + R_J), \theta^*(r + R_J)\} \geq \max\{\varphi(b + R_J), \theta^*(b + R_J)\}$. Finally, if J is a θ-Euclidean L-fuzzy ideal of R, then there exists a θ^*-Euclidean L-fuzzy ideal from R/R_J to L. Also for each $a \in R$, $J(a) = \varphi(a + R_J) = \varphi(n(a)) = (\varphi \circ n)(a)$. It means that $J = \varphi \circ n$.

Now let us show that this factorization is unique. Suppose that $\varphi' \circ n = J$ for some other $\theta^*(= n(\theta))$-Euclidean L-fuzzy ideal $\varphi'\colon R/R_J \to L$. But then

$$\varphi(a + R_J) = J(a) = (\varphi' \circ n)(a) = \varphi'(a + R_j)$$

for all $a \in R$. Hence we obtain $\varphi = \varphi'$. So φ is a unique $\theta^*(= n(\theta))$-Euclidean L-fuzzy ideal from R/R_J into L with the property that $J = \varphi \circ n$.

Corollary 3.3 Let $J\colon R \to L$ be a θ-Euclidean L-fuzzy ideal. Suppose that $\theta(a) = \theta(b)$ when $a - b \in \operatorname{Ker} n$.Then there exists a θ^*-Euclidean L-fuzzy ideal from R/J to L.

Proof. Since $J\colon R \to L$ is a θ-Euclidean L-fuzzy ideal and from Theorem 3.2. , $\varphi\colon R/R_J \to L$ is a θ^*-Euclidean L-fuzzy ideal. Also the rings R/J and R/R_J are isomorphic. So there exists a θ^*-Euclidean L-fuzzy ideal from R/J to L. □

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