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MUSTAFA YILDIRIM

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Mustafa Yıldırım

Abstract

In 1975, Wenger [3] determined the fine spectra of Cesàro operator C_1 on c, the space of convergent sequences. In [6], the spectrum of the Rhaly operators on c_0 and c, under the assumption that $\lim_{n\to\infty}(n+1)a_n = L \neq 0$, has been determined. This paper presents the fine spectra of the Rhaly matrix R_a as an operator on the space c_0, with the same assumption.

Key words and phrases: Rhaly operator, spectrum and point spectrum.

1. Introduction

In this paper, $c_0,\ \ell_1,\ T^*,\ X^*,\ B(X),\ A^t,\ \pi_0(T,X)$ and $\sigma(T,X)$ respectively denote null sequences; sequences such that $\sum_k \mid x_k \mid < \infty$; the adjoint operator of T; the continuous dual of X; the linear space of all bounded linear operators, say, T on X into itself; the transposed matrix of A; the eigenvalues of T on X; and the spectrum of T on X.

In addition, we assume that given a scalar sequence of $a = (a_n)$, a Rhaly matrix $R_a = (a_{nk})$ is the lower triangular matrix where $a_{nk} = a_n,\ k \le n$ and $a_{nk} = 0$ otherwise, where

(a) $L = \lim_n (n+1)a_n$ exists, finite, and is nonzero;

(b) $a_n > 0$ for all n, and

(c) $a_i \neq a_j$ for $i \neq j$.

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(d) $a = (a_n)$ is monotone decreasing.

Let S denote the set $\{ a_n : n = 0, 1, 2, ... \}$.

In 1975, Wenger [3] determined the fine spectra of Cesàro operator C_1 on c, the space of convergent sequences. In [6], the spectrum of the Rhaly operators on c_0 and c, under the assumption that $\lim_{n\to\infty}(n+1)a_n = L \neq 0$ has been determined.

Under the above conditions, the purpose of this study is to determine the fine spectra of Rhaly operator R_a as an operator on the Banach space c_0 of convergent sequences normed by $\| x \| = \sup_{n\geq 0} | x_n |$.

If X is a Banach space, $B(X)$ denotes the collection of all bounded linear operators on X and if $T \in B(X)$, then there are three possibilities for $R(T)$, the range of T:

(I) $R(T) = X$,

(II) $\overline{R(T)} = X$, but $R(T) \neq X$,

(III) $\overline{R(T)} \neq X$

and three possibilities for T^{-1}:

(1) T^{-1} exists and is continuous,

(2) T^{-1} exists but is discontinuous,

(3) T^{-1} does not exist.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2, III_3. If an operator is in state III_2 for example, then $\overline{R(T)} \neq X$ and T^{-1} exist but is discontinuous (see [1]).

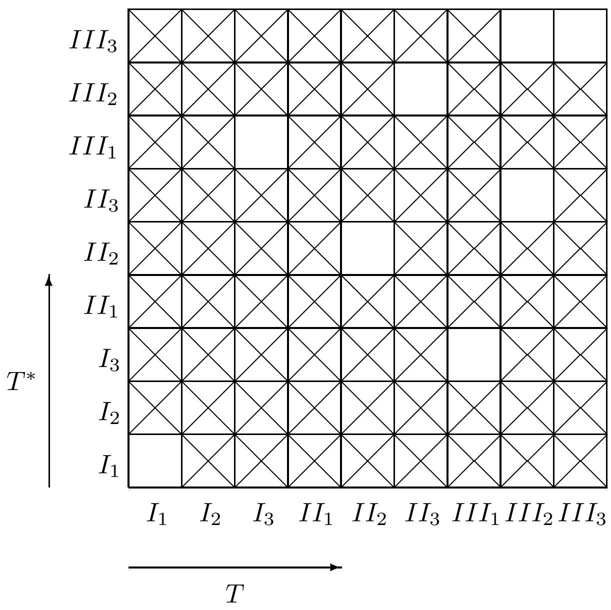

Figure1. State diagram for $B(X)$ and $B(X^*)$ for a non-reflective Banach space X

If λ is a complex number such that $A = \lambda I - T \in I_1$ or $A = \lambda I - T \in II_1$, then $\lambda \in \rho(T, X)$. All scalar values of λ not in $\rho(T, X)$ comprise the spectrum of T. The further classification of $\sigma(T, X)$ gives rise to the fine spectrum of T. That is, $\sigma(T, X)$ can be divided into the subsets $I_2\sigma(T, X)$, $I_3\sigma(T, X)$, $II_2\sigma(T, X)$, $II_3\sigma(T, X)$, $III_1\sigma(T, X)$, $III_2\sigma(T, X)$, $III_3\sigma(T, X)$. For example, if $A = \lambda I - T$ is in a given state, III_2 (say), then we write $\lambda \in III_2\sigma(T, X)$.

Lemma : *If* $Re\frac{1}{\lambda} = \alpha$, *then*

$$\prod_{k=0}^{N-1} \mid 1 - \frac{a_k}{\lambda} \mid \ \simeq \ \frac{1}{N^{\alpha L}} \tag{1}$$

as $N \longrightarrow \infty$. *We use the notation* $a_n \simeq b_n$ *in the sense that* $\left(\frac{a_n}{b_n}\right)$, $\left(\frac{b_n}{a_n}\right)$ *are both bounded.* [6]

Theorem 1: *If* $0 < L < \infty$ *then* $S \cap (2L, \infty) \ \subseteq \ \pi_0(R_a, c_0) \ \subseteq \ S \cap [2L, \infty)$ *[6].*

Theorem 2: *If* $0 < L < \infty$ *then*

$$\left\{ \lambda \ : \ \mid \lambda - \tfrac{L}{2} \mid \ < \ \tfrac{L}{2} \right\} \ \cup \ S \ \cup \ \{L\} \ \subseteq \pi_0(R_a^*, c_0^* \cong \ell_1)$$

$$\subseteq \left(\left\{ \lambda \ : \ \mid \lambda - \tfrac{L}{2} \mid \ \leq \ \tfrac{L}{2} \right\} - \{0\} \right) \ \cup \ S$$

[6].

Theorem 3: *If* $0 < L < \infty$ *then* $\sigma(R_a, c_0) = \left\{ \lambda \ : \ \mid \lambda - \frac{L}{2} \mid \ \leq \ \frac{L}{2} \right\} \ \cup \ S$*[6].*

Theorem 4: T *has a dense range if and only if* T^* *is one-to-one.[1,II.3.7 Theorem]*

Theorem 5: $R(T^*) = X^*$ *if and only if* T *has a bounded inverse.[1,II.3.11 Theorem]*

Theorem 6: R_a *is a bounded operator on* c_0 *if and only if* $R_a^* = R_a^t$*[4].*

Main Results

Theorem A: *Let* $0 < L < \infty$*. If* $\lambda \notin S$ *and* $\alpha L > 1$ *then* $\lambda \in III_1\sigma(R_a, c_0)$ *where* $\alpha = Re\frac{1}{\lambda}$.

Proof. Since $\lambda \notin S$, $T_\lambda = \lambda I - R_a$ is a lower triangular matrix. The matrix T_λ^{-1} exists.

From Theorem 6, $R_a^* = R_a^t$ on c_0. Then $T_\lambda^* x = \theta$ implies the following:

$$x_n = (1 - \frac{a_{n-1}}{\lambda})x_{n-1} \tag{2}$$

and

$$x_n = \prod_{j=0}^{n-1}(1 - \frac{a_j}{\lambda})x_0, \ for \ n \geq 1. \tag{3}$$

Since $\alpha L > 1$, $x = (x_n)_0^\infty \in \ell_1$. Therefore T_λ^* is not one-to-one. From Theorem 4, $\overline{R(T_\lambda)} \neq c_0$. So that $T_\lambda \in III$.

Let $y = (y_n) \in \ell_1$. We want to find $x = (x_n) \in \ell_1$ such that $T_\lambda^* x = y$. Let $x_0 = 0$, then we have

$$\begin{aligned} x_1 \ &= \ \frac{1}{\lambda}(y_1 - y_0) + \frac{1}{\lambda}(\lambda - a_0)x_0 \\ &= \frac{1}{\lambda}(y_1 - y_0) \end{aligned}$$

and for $n > 1$,

$$x_n = \frac{1}{\lambda}\Big\{y_n - \frac{a_{n-1}}{\lambda}y_{n-1} - \frac{a_{n-2}}{\lambda}(1-\frac{a_{n-1}}{\lambda})y_{n-2}$$

$$-\frac{a_{n-3}}{\lambda}(1-\frac{a_{n-2}}{\lambda})(1-\frac{a_{n-1}}{\lambda})y_{n-3} - \ldots - \frac{a_1}{\lambda}(1-\frac{a_{n-1}}{\lambda})(1-\frac{a_{n-2}}{\lambda})\ldots(1-\frac{a_2}{\lambda})y_1$$

$$- \prod_{j=1}^{n-1}(1-\frac{a_j}{\lambda})y_0\Big\}.$$

This defines the matrix $B = (b_{nk})$ with $n \geq 1,\ k \geq 0$, where $x = By$ as the following:

$$b_{nn} = \frac{1}{\lambda} \tag{4}$$

$$b_{n,n-1} = -\frac{a_{n-1}}{\lambda^2} \tag{5}$$

$$b_{10} = -\frac{1}{\lambda} \quad \textit{and} \quad b_{n0} = -\frac{1}{\lambda}\prod_{j=1}^{n-1}(1-\frac{a_j}{\lambda}), \quad n > 1 \tag{6}$$

$$b_{nk} = -\frac{a_k}{\lambda^2}\prod_{j=k+1}^{n-1}(1-\frac{a_j}{\lambda}), \tag{7}$$

$$b_{nk} = 0, \quad k > 1 \geq n. \tag{8}$$

By the Lemma there are possitive constants A and B such that

$$\frac{A}{n^{\alpha L}} \leq \prod_{j=1}^{n} \mid 1 - \frac{a_j}{\lambda} \mid \leq \frac{B}{n^{\alpha L}}.$$

So

$$\begin{aligned}\sum_{n=1}^{\infty} | b_{n0} | &= | b_{10} | + \sum_{n=2}^{\infty} | b_{n0} | \\ &= \frac{1}{| \lambda |} + \frac{1}{| \lambda |} \sum_{n=2}^{\infty} \prod_{j=1}^{n-1} | 1 - \frac{a_j}{\lambda} | \qquad (9) \\ &= \frac{1}{| \lambda |} + \frac{B}{| \lambda |} \sum_{n=2}^{\infty} \frac{1}{(n-1)^{\alpha L}},\end{aligned}$$

and for $k \geq 1$

$$\begin{aligned}\sum_{n=1}^{\infty} | b_{nk} | &= | b_{kk} | + | b_{k+1,k} | + \sum_{n=k+2}^{\infty} | b_{nk} | \\ &= \frac{1}{| \lambda |} + \frac{a_k}{| \lambda |^2} + \frac{a_k}{| \lambda |^2} \sum_{n=k+2}^{\infty} \prod_{j=k+1}^{n-1} | 1 - \frac{a_j}{\lambda} | \\ &\leq \frac{1}{| \lambda |} + \frac{C}{| \lambda |^2} + \frac{a_k}{| \lambda |^2} \sum_{n=k+2}^{\infty} \frac{\prod_{j=1}^{n-1} | 1 - \frac{a_j}{\lambda} |}{\prod_{j=1}^{k} | 1 - \frac{a_j}{\lambda} |} \\ &\leq \frac{1}{| \lambda |} + \frac{C}{| \lambda |^2} + \frac{a_k}{| \lambda |^2} \sum_{n=k+2}^{\infty} \frac{\frac{B}{(n-1)^{\alpha L}}}{\frac{A}{k^{\alpha L}}} \\ &\leq \frac{1}{| \lambda |} + \frac{C}{| \lambda |^2} + \frac{a_k}{| \lambda |^2} k^{\alpha L} \int_k^{\infty} \frac{1}{x^{\alpha L}} dx \\ &\leq \frac{1}{| \lambda |} + \frac{C}{| \lambda |^2} + \frac{C}{| \lambda |^2 (\alpha L - 1)}, \qquad (10)\end{aligned}$$

where $C = sup\ ka_k$.

Since $\alpha L > 1$ we have $\sup_k \sum_n \mid b_{nk} \mid < \infty$, hence we have $B \in B(\ell_1)$, so that T_λ^* is shown to be onto. From Theorem 5, $T_\lambda \in (1)$. As a result, $T_\lambda \in III_1$ and $\lambda \in III_1\sigma(R_a, c_0)$. □

Theorem B: *Let $0 < L < \infty$. If $\lambda \notin S$ and $\alpha L = 1$ then $\lambda \in II_2\sigma(R_a, c_0)$.*

Proof. Since $\lambda \notin S$, T_λ is a lower triangular matrix. So T_λ is one-to-one; i.e.$T_\lambda \in (1) \cup (2)$.

Consider the adjoint operator T_λ^*. Then if $T_\lambda^* x = \theta$, then

$$x_n = \prod_{j=0}^{n-1}(1 - \frac{a_j}{\lambda})x_0 \quad for\ n \geq 1.$$

Since $\alpha L = 1$, we have

$$x = (x_0, x_1, \dots) \in \ell_1 \iff x_0 = 0 \iff x = \theta.$$

Hence T_λ^* is one-to-one; i.e.$T_\lambda^* \in (1) \cup (2)$. Now if we look at Fig 1, then we obtain $T_\lambda \in I_1 \cup II_2$. From Theorem 3, since $\lambda \in \sigma(R_a, c_0)$, we get $T_\lambda \notin I_1$, so $T_\lambda \in II_2$. Hence we obtain $\lambda \in II_2\sigma(R_a, c_0)$. □

Theorem C: *Let $0 < L < \infty$. If $\lambda = a_m$ for at least one m ($m = 0, 1, \dots$), then $\lambda = a_m \in III_3\sigma(R_a, c_0)$.*

Proof. Consider the system

$$(\lambda I - R_a^*)x = 0.$$

Suppose that $\lambda = a_0$. Then we have

$$(\lambda I - R_a^*)_0 x = 0,$$

which yields

$$(a_0 - a_0)x_0 - \sum_{k=1}^{\infty} a_k x_k = 0,$$

or

$$\sum_{n=1}^{\infty} a_n x_n = 0.$$

This in turn implies that

$$a_1 x_1 = -\sum_{k=2}^{\infty} a_k x_k.$$

$(\lambda I - R_a^*)_1 x = 0$ yields

$$0 = (a_0 - a_1)x_1 - \sum_{k=2}^{\infty} a_k x_k = a_0 x_1 - a_1 x_1 + a_1 x_1 = a_0 x_1,$$

which implies that $x_1 = 0$. By induction one can show that $x_n = 0$ for all $n > 0$.

If $\lambda = a_m, m > 0$, then

$$(\lambda I - R_a^*)_m x = 0,$$

which becomes

$$a_m x_m - \sum_{k=m+1}^{\infty} a_k x_k = 0,$$

which implies that

$$\sum_{k=m+1}^{\infty} a_k x_k = 0,$$

or that

$$a_{m+1} x_{m+1} = -\sum_{k=m+2}^{\infty} a_k x_k.$$

$$(\lambda I - R_a^*)_{m+1} x = 0$$

becomes

$$a_m x_{m+1} - \sum_{k=m+1}^{\infty} a_k x_k = 0,$$

or

$$\begin{aligned} 0 &= a_m x_{m+1} - a_{m+1}x_{m+1} - \textstyle\sum_{k=m+1}^{\infty} a_k x_k \\ &= a_m x_{m+1} - a_{m+1}x_{m+1} + a_{m+1}x_{m+1} = a_m x_{m+1}, \end{aligned}$$

which implies that $x_{m+1} = 0$. Again by induction it can be shown that $x_n = 0$ for each $n > m$.

Therefore in each case x is a finite sequence and $x \in \ell_1$. Hence $T_{a_n}^*$ is not 1-1, and thus T_{a_n} does not have dense range. Therefore $T_{a_n} \in III$.

Since $\lambda = a_n$ for each n, $T_{a_n}^{-1}$ does not exist. □

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Mustafa YILDIRIM
Department of Mathematics
Cumhuriyet University
58140 Sivas-TURKEY
e-mail: yildirim@cumhuriyet.edu.tr

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