Turkish Journal of Mathematics

Volume 27 | Number 1 Article 12

1-1-2003

Symplectic surgeries from singularities

IVAN SMITH

RICHARD THOMAS

Turk J Math
27 (2003) , 231 – 250.

Symplectic surgeries from singularities

Ivan Smith and Richard Thomas

Abstract

We describe a variety of symplectic surgeries (not a priori compatible with Kaehler structures) which are obtained by combining local Kaehler degenerations and resolutions of singularities. The effect of the surgeries is to replace configurations of Lagrangian spheres with symplectic submanifolds. We discuss several examples in detail, relating them to existence questions for symplectic manifolds with $c_1 > 0$, $c_1 = 0, c_1 < 0$ in four and six dimensions.

1. The local model

Given an isolated analytic hypersurface singularity $0 \in X_0 := \{f(z) = 0\} \subset \mathbb{C}^{n+1}$ one can form the *smoothing* $X_t = \{f(z) = t\}$ and the *resolution* $\widehat{X} \to X_0$, obtained by (repeatedly) blowing up the origin. These two associated spaces have the same link – that is, the intersection $\mathbb{S}^{2n+1} \cap f^{-1}(0)$ – and there is a smooth surgery which replaces the smoothing by the resolution, or vice-versa. Thinking of the smoothing as Kähler (and so symplectic) by restriction of the standard Kähler form on $\mathbb{C}^{n+1}$, the smoothing also looks somewhat like a resolution by symplectic parallel transport, described below. The result is that there is a canonical map or 'Lagrangian blow up' $X_0 \leftarrow X_t$ whose 'exceptional locus' (the inverse image of the singular point $0 \in X_0$) is a *Lagrangian* cycle, in fact a collection of Lagrangian spheres [21]. So the surgery replaces configurations of Lagrangian spheres by complex (symplectic) subvarieties. Symplectic parallel transport also shows that the X_ts are all isomorphic *as symplectic manifolds*, so denoting any such symplectic manifold by X, we can denote this surgery by the diagram (motivated by smoothings and resolutions in algebraic geometry)

$$\begin{array}{c} \hat{X} \\ \downarrow \\ X_0 \leftarrow X. \end{array} \tag{1}$$

More general singularities, including complete intersections and some non-isolated singularities, can be treated similarly; for simplicity we will restrict attention to isolated hypersurface singularities.

We briefly remind the reader about symplectic parallel transport [21]. The total space $\mathcal{X} = \{X_t\}_{t\in\mathbb{C}} \xrightarrow{p} \mathbb{C}$ is naturally a smooth symplectic submanifold of $\mathbb{C}^{n+2}$: $\mathcal{X} = \{(z, t) \in$

I.S. is supported by an EC Marie-Curie fellowship No. HPMF-CT–2000-01013. R.T. is supported by a Royal Society University Research Fellowship.

$^{+1} \times$ f z t $^{+2}$
X T
Tp dp
TX $X \sim X_s$ s, t /

t X X_0

f $f - t$

¨ X_0
$^{+1} \times$ ¨

ω ω

$^{+1} \times$ $\supset X$

$c_1 >$ c_1 $c_1 <$

2 g b

X $\mathcal{C}$
U $\mathcal{C}$
¨ ¨ $\mathcal{C}$

X
¨

p V p $^{+1}$

ω_X X ω $_{n+1}$
V p ω_X X X ω
E
X
σ X $- E$
E
σ
$+1$ $\times$
U E $< \varepsilon \ll$, ω $\varepsilon\sigma$
X
$X \backslash U$ ε
E $U \backslash E$ σ ω

ε E
U $\mathcal{C}$

3 r in ry ub p int

X, ω L X
L X
L_0 T L

$$\left(\left\{\sum_{i=1}^{+1} z_i^2 \right\} \backslash \quad , \frac{i}{} \sum dz_j \wedge d\overline{z}_j \right) \sim \quad T \quad \backslash \ _0, dp \wedge dq$$

z_j a_j ib_j $_j$ $a_j/|a|, -|a|b_j$ $_j$
T T T $\sum z_j^2$ t
t
t W
n b

Q $_{-1}$
$\mathcal{L}$ Q $_{-1}$
$+1$ W

$$Q \ _{-1} \quad - \ \mathcal{L} \ -^{\pi} \quad ^{+1}.$$

$\rho_\lambda \quad \pi \; \omega_{\;\; n+1} \quad \lambda^2 p \; \omega_{Q_{n-1}}$

$Q \;\; _{-1}$ [1] $Q \;\; _{-1}$
ρ_λ $H_2 \; Q \;\; _{-1}$ $\pi\lambda^2$
n $B \; \delta$ δ $^{+1}$

L 3 $b \;\; w \quad \pi^{-1} B \; \delta \; \backslash Q \;\; _{-1}, \rho_\lambda$ “ ”
$B \; \sqrt{\lambda^2 \quad \delta^2} \; \backslash B \; \lambda \quad W, \omega \;\; _{n+1} \;\; .$

P . $\mathcal{L}$ $^{+1}$ $^{+1}$
$\mathcal{L}$ $\mathcal{O} \; -$ ρ_λ $\pi \; \omega \;\; _{n+1}$
$\lambda^2 p \; \omega_{\mathbb{P}^n}$
$z \mapsto \;\; F \circ \pi \; z$ $\pi^{-1} B \; \delta \; \backslash \quad , \rho_\lambda$
$B \; \sqrt{\lambda^2 \quad \delta^2} \; \backslash B \; \lambda \; , \omega \;\; _{n+1}$ F $z \mapsto \;\; \sqrt{|z|^2 \quad \lambda^2} / |z| \; z$
F W $F \circ \pi$

$\times \quad ^{+1}$ ∎

$B \; \sqrt{\lambda^2 \quad \delta^2}$ $\pi^{-1} B \; \delta$

¨

x b
λ

ﬂ
T u, v $^{+1} \times \quad ^{+1} \; | \; |u|$ $, \langle u, v \rangle$

$du \wedge dv$ F
$\omega \quad c_1 \; TX, \omega \quad H^2 \; X,$
ω
c_1

---

[1] O g ( ) m m $\mathbb{P}^n$ g
q g $\mathbb{C}^{n+1}$ g m $\sum dx_j \wedge dy_j$,
g $\mathbb{P}^1 \subset \mathbb{P}^n$ π

L **3 2** X, ω b F 6- L X . v T ^{3} $|$ $|v| \le \mu$ b X $\mu > \frac{1}{2\pi}$ Y b b L - c_1 Y .

P . X L c_1 X H^2 X, L $\sim$ H^2 Y, E H^2 Y $E \sim$ $^1 \times$ ^{1} c_1 Y c_1 X $-$ E ρ_λ ω_X $-\pi\lambda^2 E$ ω_X H^2 Y π π λ $/\sqrt{\pi}$ B $\sqrt{\ /\pi \quad \delta^2}$ $, \omega$ $_3$ v T ^{3} $|$ $|v| \le$ $/$ π $\delta^2/$ $|z|^2 \le R \Rightarrow |v|$ $|\Re$ z $||\Im$ z $| \le R/$ ∎

¨

$-$ c_1

¨

$^2 \times$ $^1, \frac{1}{\pi}$ $-$ $\omega_{FS} \oplus$ ω_{FS} B $\sqrt{\ /\pi}$ ^{2} $/\pi$ ^{3} $^2 \times$ ^{1}

c_1 ω $^2 \times$ ^{1} ¨ ^{1}

Qu sti n 3 μ T ^{3} $^2 \times$ $^1, \frac{1}{\pi}$ $-$ $\omega_{FS} \oplus$ ω_{FS} $\mu >$ $/$ π

¨ $^2 \times$ ^{1} b_2 H_2 H_4

c_1^3

b_2 2

/ π

2 × 1

1 × 1

$\mathcal{O}$ − , − fl

4 S r s uti ns

Q_2 $Q_2 \sim$ 1 × 1

Q_2

fl

xy zw 4

1 x/z w/y x/w z/y

..

r 4 X b x- j

$L_1, \ldots, L$ w $\sum_X \lambda_i L_i$ L_i H_3 X, (w λ_i / . .

k .

D 1 $|\lambda_i|$

σ D

ω_X $\varepsilon\sigma$ $< \varepsilon \ll$ □

---

2 , g [4]; $b_2 = 4$,

$c_1^3 =$ 6, g $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ g (, ,)

$X \times$ [1]

$X \supset$ [3]

$\omega_X \oplus -\omega_{FS}$ L H_3

L_i

ω

$H_4 \times H_4$ A, B A B ω

$h^{2,0}, h^{1,1} -$

b_2 X

ω

$\beta\omega_{\mathbb{P}^2} \oplus \alpha\omega_{\mathbb{P}^1}$ $^2 \times$ ^{1} $\alpha, \beta >$

L

^{1} $\beta > \alpha$ $\beta^2\pi^2/$ ^{2}

$\sqrt{\beta}$ ^{2}

$\sqrt{\alpha}$ ^{1} $\pi\alpha$ L

fl D $-D$

Ω_ε π $\omega \pm \varepsilon$ D ε

$^1 \times$ ^{2}

Pr p siti n 4 2 x b w Ω_ε ε $\beta > \alpha.$

P .

fl

$\alpha\omega_{\mathbb{P}^1} \oplus -\beta\omega_{\mathbb{P}^2}$

^{4} $\mathcal{O}$ $\oplus \mathcal{O}$ ^{1}

$\mathcal{O}$ $-$ ^{2} $^2_J \times$ ^{4}

$^4, \sum dx_i \otimes dx_i$ j ^2_J

j $\times$ ^{4}

$^4, j$ ^{4}

$$^2_J \times \quad ^4 \backslash \qquad T \quad ^3 \qquad j, v \quad v/|v|, jv \;.$$

L ^2_J

U w, w $^1 \times$ ^{3} ^{3} ∂B ^{2}

B^4 v ^{2}

U $v \times$ ^{2} ^{1} $v, e^{i\theta v}$

^{1} ^{2}

$$T\ ^3|_{\mathbb{S}^1} \sim \{\ e^{i\theta v}, \lambda j e^{i\theta v}\ |\ \lambda\ ,\infty\ , j\ \ ^2_J\}\ \ T\ ^3.$$

$$^2_J \times\ \langle v, iv\rangle\ \ ^2_J \times\ ^4.$$

$|\ ^2|$ $v \times$ ^{2}

$v \times$ ^{2} $w \times$ ^{2} v / w

^2_J $|\ ^2|\ ^3$

^2_J $|\ ^2|$

$|\ ^2|$

$|\ ^2|^3$

R w, w $^1 \times$ $^2\,|\,|w| \leq$ U

$\mathcal{O}$ $-$ ^{1}

D^4 $|R|$

$|R|^2 \cdot |\ ^2|$ $-$ R fl

w w t jw t

$$R\ \ \{\ w,\ \ _{1-w}\ w\ \ ^1 \times B^4\,|\,|w| \leq\ \}.$$

fl ^{1} U ^{3}

$|R\ |$ $|R|$ ^{1} ^2_J

^{1} $|R|$ $|R\ |$ $\mathcal{O}$ $-$

$|\ ^2|\cdot\ ^2_J$ $|R|^2 \cdot |\ ^2|$ $^2 \times$ ^{1} $v \times$ ^{2} $\cdot$ $-$ ^{1} $-$

w ^{1} R

$\langle w, iw\rangle$ $\mathcal{O}$ $-$

$\langle jw, jiw$ $-kw\rangle$ R R

H_4 $|\ ^2|, |\ ^1 \times\ ^1|$ $|R|$

$^1 \times$ ^{1} $^1 \times$ ^{2}

$|R|$ $|\ ^1 \times\ ^1|$ $\emptyset$

$$|\ ^1 \times\ ^1| \cdot |\ ^2|^2\ \ \ |\ ^1 \times\ ^1|^2 \cdot |\ ^2|\ \ \ |\ ^1 \times\ ^1| \cdot |R|^2\ \ |\ ^1 \times\ ^1|^2 \cdot |R|\ \ .$$

$|\ ^2|^2 \cdot |R|$ A $|R|^3$ B

A, B A $\pm$ U

R R

1

$\mathcal{O}$ −

$$\Omega_\varepsilon \quad \alpha\,|\ ^2| - \beta\,|\ ^1 \times\ ^1| \pm \varepsilon\,|R|$$

α, β $-\beta$
$-\omega_{\mathbb{P}^2}$ $H_2\ ^2$
$^1 \times\ ^1 \quad |\ ^1 \times\ ^1|^2 \cdot |\ ^2|$

Ω_ε H_4

$$\begin{pmatrix} \pm\varepsilon A & -\beta & \alpha A \mp \varepsilon \\ -\beta & \alpha & \\ \alpha A \mp \varepsilon & & -\alpha \pm B\varepsilon \end{pmatrix} \qquad \begin{pmatrix} & -\beta & \alpha A \\ -\beta & \alpha & \\ \alpha A & & -\alpha \end{pmatrix} \quad O\ \varepsilon\ .$$

$\beta^2\alpha - \alpha^3 A^2 \quad O\ \varepsilon \quad \alpha\ \beta^2 - \alpha^2 \quad O\ \varepsilon$

$\beta > \alpha$ □

¨

¨

X

Pr p siti n 4 3 b , X K¨ w $h^{2,0}$,
b K¨ . b
K¨ .

P . $h^{2,0}$ ¨ $h^{2,0}$ $h^{1,1}$ $h^{0,2}$

$h^{2,0}$ $h^{2,0}$
X j H

X
L $H_3\ X$
1
$h^{2,0}$ D
D
D H
N

$\omega_X \quad \varepsilon\sigma$

ε

□

$h^{2,0}$ $^{1}\times$ ^{2} .. L ..

..

b_3

$h^{2,1}$ h^1 TX

L 4 4 A C b-Y 3- Z

.

P . fl Z

$\int_L \Omega$ L □

L 4 5 L X - w b b

-b X, X L K.. .

P . ^{3} D^4 ^{4}

$\overline{}_2$ H_4 ^{1}

H_4

H_2 ^{4}

..

□

^{6}

.. Q ^{4}

b_3 Q

Qu sti n 4 Q

..

Q

.. j $j \leq$

5 Fibr pr u ts n trip p ints

c_1

R x^3 y^3 z^3 w^3 .

E 3

2 b_2 E $\mathcal{O}_{\mathbb{P}^3}$ $-$ $|_E$ K_E

E

S_i

C $S_1 \times_C S_2$

S_i f_i x_i, y_i t $^2 \times$,

t

$S_1 \times_C S_2$ f_1 x_1, y_1 f_2 x_2, y_2 $^2 \times$ ^{2}.

x_1, y_1, x_2, y_2 b x_i, y_i

t C

f_i t f_i x_i, y_i x_iy_i
x_1y_1 x_2y_2 n
f_i x_i y_i n x_1 y_1 x_2 y_2
n
f_i

3

γ S_1
S_2 1
S_i
γ 2 1 $\times$ 1 1
γ
γ 3

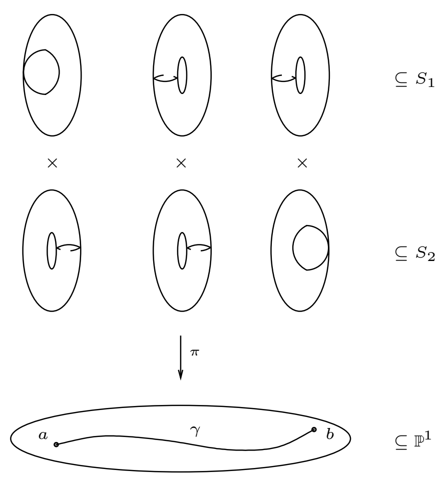
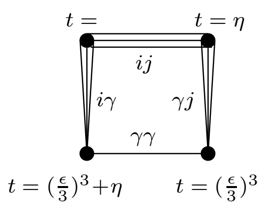


Figure 1. F

L 5 *R b* . *R*
fi *16* *w* *F* *2.*



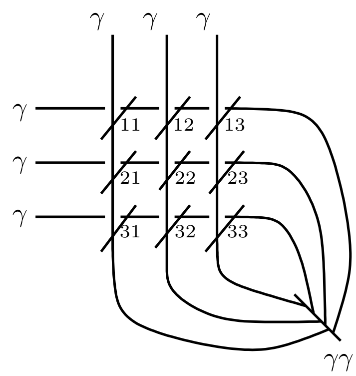


Figure 2. $\mathbb{T}^2$
$\mathbb{P}^1$

P .
E_i 1 xy x y
xy x y t uv u v t
xy x y uv u v
t $t-\eta$
xy x y t xy x y ε t
xy x y ε uv u v ε η

$$x,y,t \quad \Big\{ \quad , \quad , \quad , \quad , -\varepsilon, \quad , \; -\varepsilon, \; , \quad , \Big(-\frac{\varepsilon}{\ }, -\frac{\varepsilon}{\ }, \big(\frac{\varepsilon}{\ }\big)^3\Big)\Big\}.$$

t 1 t $\left(\frac{\varepsilon}{3}\right)^3$
1

ε

I_3
t 1
, , γ 3

2 1

γ , ,

3 $\square$

π E 1 I_3

x^3 y^3 z^3 $txyz$ $\mathbb{P}^1$ I_3
t $e^{i\pi/3}, e^{5i\pi/3}, -\ , \infty$ Z E $\times_\pi E$
N, N N, N π^{-1} t π
Z Z

L 5 2 b b , E b

$$t\ a\ ^3 \cdot t\ b\ ^3 \cdot t\ a\ \ b\ ^3 \cdot t\ \ a\ \ b\ ^3$$

w a, b / ^{2} $t\ C$ D
w $C.$ $(H$
kw b - $.)$

P .
q
I_3

$$t\ a \quad \begin{pmatrix} & \end{pmatrix} \quad t\ b \quad \begin{pmatrix} & \\ - & \end{pmatrix}$$

$$t\ a \quad b \quad \begin{pmatrix} & \\ - & \end{pmatrix} \quad t\ \ a \quad b \quad \begin{pmatrix} - & \\ - & \end{pmatrix}$$

□

a b j
I_3 b I_1 I_2 I_3 a
I_1 b

I_3 b
I_1 I_2 E

c_1 ¨

¨

6 A f ur- i nsi n int r u

–

Pr p siti n 6 X x , X -
.

k .

□

D

t / t

–

x^2 y^2 z^2 t^2 x^2 y^2 z^2 t

$\sum_{j=1}^{3} x_j^2$ ε

d

d ^{2}

d

x_1^3 x_2^3 x_3^3

$-$ $\mathcal{C}_3$

C^2 X^4 X C C J

J X C

..

Pr p siti n 6 2 X b - w

2 q z . w w q . ( ) B w 2

, b b (q

$-$) b $\mathcal{C}_3$. ( ) F b X w E 2

fib .

P . x_1^3 x_2^3 x_3^3 t

t Z

$\sum_{j=1}^3 x_j^3$ tx_4^3 3 Z 2

x^3 y^3 z^3 $Z\setminus$ 2

b_2

2 $-2 \sim E$ 1

$-$

E

$-$ $-$

$\square$

E

$d >$

C X g $d -$ $d -$ / $-d$

d

X g E

7 r - ik nfigur ti ns

A

L 7 k A ff ^{5} n $^2 \times {}^3$ n .

P . $^3 \times {}^2$ $^2 \times {}^2$ ^{3} ^{2} $_2$

$$^3 \times {}^2 \quad \mathbb{S}^2 \ \mathbb{S}^2 \quad {}^2 \times {}^3 \sim {}^5.$$

$^2 \times {}^2$ ^{3} ^{2} $^3 \times {}^2$ $_2$

$$D^3 \times {}^2 \cup_{\mathbb{S}^2} \ \mathbb{S}^2 \quad {}^2 \times D^3 ,$$

^{3} D^3 $^2 \times {}^2$ ^{5} ^{5} ^{5} $D^3 \times {}^2$ ^{5} ^{3} $^2 \times {}^2$ ^{5} z_1, z_2, z_3 $^3 \,|\, |z_1|^2 \quad |z_2|^2$ $\frac{1}{2}$ $|z_2|^2 \quad |z_3|^2$ □

A n k k $\sum_1 x_i^2 \quad t^{2k}$ $\sum x_i^2 \quad t^2$ t ı t^k k A

L 7 2 X A - $(n$ k $)$ ff - b b .

P . $\sum_{i=1}^3 x_i^2 \quad p_k \ t^2$ $\sum x_i^2 \quad t^{2k}$ p_k k A $^2 \times$,

A p_k t^2 t^{2k} k

1 k 1

A

k $^2 \times {}^3$

$^1 \times {}^1$ $^2 \times {}^3$

H_2 □

5 $^2 \times {}^3$

n

..

Pr p siti n 7 3 $L_1, \ldots, L$ b X w

. (

). b q

.

P .

ϕ L X

T L X ω_X $dp \wedge dq$

L_i L_j L_i L_j

P ϕ^{-1} P L_i L_j

$D \times D$ $_{i,j}$ $\amalg_{j=1} T$ L_j

i T L_i j

T L_j

$D \times t$ $_i$ $t \times D$ $_j$

ω_X, ω_X L_j

T

$\amalg L_i$ $\amalg L_i$

S H HO S

ϕ $\amalg L_i$ X

□

`

A kn w g nts:

R f r n s

[ ] F *Geometry of Yang-Mills fields.* S N S P
[ ] P V V *Compact complex surfaces.* S 84
[ ] E *The Milnor lattices of the elliptic hypersurface singularities.* P S **53** 86 8
[4] F *Symplectic structures of algebraic surfaces and deformation.* P / 4
[ ] S D *Polynomials, Vanishing Cycles and Floer homology.* *Mathematics: Frontiers and Perspectives* S
[6] Y E *On symplectic manifolds with some contact properties.* D **33** 8
[ ] E *Symplectic convexity in low-dimensional topology.* **88** 8
[8] R F R S *Knots, links and 4-manifolds.* **134** 8 6 4
[ ] P ffi H *Principles of algebraic geometry.* N Y 8
[ ] P *Some non-trivial families of symplectic structures.* P 8
[ ] *Some new applications of general wall crossing formula.* R **3** 6 86
[ ] D D D S *Introduction to symplectic topology.* O P 8
[ ] R *Persson's list of singular fibres for rational elliptic surfaces.* Z **205**
[4] S S *Classification of Fano 3-folds with* $_2 \geq$. **36** 8 4 6

[ ] P *Une structure symplectique sur $\mathbb{R}^6$ avec une sphère Lagrangienne plongée et un champ Liouville complet.* H **65** 6 66

[6] P *Configurations of Kodaira fibres on rational elliptic surfaces.* Z **205** 4

[ ] S *On the geometry of a special determinantal hypersurface associated to the Mumford-Horrocks vector bundle.* R **364** 86 8

[8] S *On fiber products of rational elliptic surfaces with section.* Z **197** 88

[ ] P S *Lagrangian two-spheres can be symplectically knotted.* D **52** 4

[ ] P S *Graded Lagrangian submanifolds.* S F **128** 4

[ ] P S *A long exact sequence for symplectic Floer cohomology.* P S / 86

[ ] S R P S Y *Symplectic conifold transitions.* P S /

[ ] D S *A quintic hypersurface in $\mathbb{P}^4$ with 130 nodes.* **32** 8 64

[4] *Symplectic manifolds and their Lagrangian submanifolds.* **6** 46

[ ] *Small resolutions of special three-dimensional varieties* D R F ¨ 8 S 86

Ecole Polytechnique, Palaiseau, France
E-mail address: `smith@math.polytechnique.fr`

Imperial College, London, UK
E-mail address: `rpwt@ic.ac.uk`