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Abelian fibred holomorphic symplectic manifolds

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Abstract

We study holomorphic symplectic manifolds which are fibred by abelian varieties. This structure is a higher dimensional analogue of an elliptic fibration on a K3 surface. We investigate when a holomorphic symplectic manifold is fibred in this way, and are led to several natural conjectures. We then study the geometry of these fibrations. The expectation is that this point of view will prove useful in understanding holomorphic symplectic manifolds, and possibly lead to a classification.

1. Introduction

Irreducible holomorphic symplectic manifolds are higher dimensional generalizations of K3 surfaces. It has been roughly twenty years since Fujiki [12] found the first example, the Hilbert scheme $S^{[2]}$ of two points on a K3 surface S, and Beauville [1] generalized this to construct two families, $S^{[n]}$ and the generalized Kummer varieties K_n. Since then there have been other constructions, but only the two examples of O'Grady [38, 39] have given us manifolds which are not deformations of Beauville's examples. The purpose of this article is to describe a framework for understanding irreducible holomorphic symplectic manifolds, which hopefully will lead towards some kind of classification. The main results are only conjectural, but we will describe the evidence and motivation behind them, while surveying special cases which have already been proved.

In studying the moduli space of K3 surfaces, one typically looks at Kummer surfaces or quartics in $\mathbb{P}^3$, as these are dense but also relatively easy to understand. However, the structure that will generalize to higher dimensions is a fibration by abelian varieties. This suggests that we first review elliptic K3s, which also happen to be dense in the moduli space. We divide this into three main steps. Firstly, we need to know which K3 surfaces are elliptic. Secondly, we describe the family of elliptic K3s which admit a section. Thirdly, we describe the relation between elliptic K3s which don't admit sections and their *relative Jacobians*, which do. There is nothing new here: for the first step we recall a theorem of Pjateckiĭ-Šapiro and Šafarevič [41] from the 70s (c.f. also Section 5 of Kodaira [27]), while the second and third steps have been well understood for arbitrary elliptic surfaces for a long time.

Elliptic K3s have base $\mathbb{P}^1$. There is evidence to suggest that if the $2n$-dimensional irreducible holomorphic symplectic manifold X admits a non-trivial fibration, then the fibres must be abelian varieties and the base must be $\mathbb{P}^n$ (a large part of this has been proved by Matsushita [32]). So the 'right' generalization of an elliptic fibration on a K3

S ON

n

ﬁ

X

X

X

X

X

$S^{[5]}$

2. Elliptic K3 surfaces

S ^{2} $S, \mathbb{Z}$

q ,

L $H \oplus$ $-E_8$

Definition 2.1. *p d S* ^{2} $S, \mathbb{Z}$, q

$$^2\ S, \mathbb{Z}\ \otimes \mathbb{C}\quad ^2\ S, \mathbb{C}\quad ^{2,0}\ S, \mathbb{C}\ \oplus\ ^{1,1}\ S, \mathbb{C}\ \oplus\ ^{0,2}\ S, \mathbb{C}\ .$$

σ S ^{2,0} $S, \mathbb{C}$

σ ^{0,2} $S, \mathbb{C}$ ^{1,1} $S, \mathbb{C}$ $\mathbb{C}\sigma \oplus \mathbb{C}\sigma$

q ^{2} $S, \mathbb{Z}$, q L

$\sigma\ \in$ $L \otimes \mathbb{C}$ $q\ \sigma$

$q\ \sigma$ $\sigma\ >$ Q $L \otimes \mathbb{C}$

Theorem 2.1. *If K3 f S d* S' *h m ph p d*

$$\psi\quad ^2\ S, \mathbb{Z}\ \xrightarrow{\cong}\ ^2\ S', \mathbb{Z}$$

d m h m g f h K¨h f S - y h K¨h
f S', ψ $\mathcal{K}_S$ $\cap \mathcal{K}_{S'}$ / { }, h h x q m ph m

$$f \quad S \xrightarrow{\cong} S'$$

h h d ψ.

¨ ^{1,1} $S, \mathbb{R}$
¨ ψ $\mathcal{K}_S$ $\cap \mathcal{K}_{S'}$ { } S S'
f ψ ψ ¨

m k d K3 f S
^{2} $S, \mathbb{Z}$, q L m k g

$\mathcal{T}$

$$\mathcal{P} \quad \mathcal{T} \to Q \subset \quad L \otimes \mathbb{C} \ ,$$

σ $\mathcal{T}$

$\mathcal{M}$
$\mathcal{T}$ L

2.1. A categorization of elliptic K3 surfaces

S ^{1}
^{1} F F^2
E $-$ E F ^{2}
E F .E $-$ E F fl E
F

$$E \quad F \mapsto E \quad F \qquad E \quad F \ .E \ E \quad F.$$

Theorem 2.2. *A p j K3 f S p f d y f h x -*
d D h D^2 .

P f.
D / D $-D$
D ^{2} $S, \mathbb{Z}$, q L
L

fl $-$
$-$

D S H
$D.H \in \mathbb{Z}$ D D
A $D.A <$ A D
A^2 $-$ A $-$

$$D_1 \quad D \quad D.A \;\; A$$

ﬂ A ﬂ H
D_1 D_1

$$D_1.H \quad D.H \quad D.A \quad A.H \quad < D.H$$

$D.H \;\; D_1.H \;\; D_2.H \;\; \ldots$
D
$|D|$

C C^2
H $mC \in |D|$
$m >$
C^2 C
f $S \to$ ^{1} C^2 f
$|C|$

□

Remark 2.1.
D D^2

D D^2
L
E $E^2 >$

Remark 2.2. $x \in L \backslash \{ \; \}$ x^2 S

$$\phi \quad ^2 \; S, \mathbb{Z} \; \xrightarrow{\cong} L.$$

S $\phi_{\mathbb{C}}$ σ $\in Q \subset$ $L \otimes \mathbb{C}$ ϕ^{-1} x
S , $x.\phi_{\mathbb{C}}$ σ x
$\mathcal{M}$
$x \in L$
$\mathcal{M}$

A D A L
D_1 $D_1.A$ $-D.A$

Proposition 2.3. *A y K3 f d f m d p K3.*

P f. S
D D^2 D

$\mathcal{T}$ Q D $-D$

..

D $-D$ S

D

^{2} $X, \mathcal{O}$ D ^{0} $X, \mathcal{O}$ $-D$
^{0} $X, \mathcal{O}$ D $\geq$ D
D^2 D
^{1}

□

Remark 2.3.

2.2. Elliptic K3 surfaces which admit sections

$\mathbb{C}$
^{1}

E
$\mathbb{C}$ ^{1} $\mathbb{C}$ ^{1} E
$\mathbb{C}$ ^{1} $\mathbb{C}$ ^{1}

S ON

S S

1 I_1

II

S

V 1

$$S \subset \quad \mathcal{O} \quad \oplus \mathcal{O} \quad \oplus \mathcal{O}$$

$$y^2 z \qquad x^3 - axz^2 - bz^3$$

a b $\mathcal{O}$ $\mathcal{O}$ x y z

V $\mathcal{O}$ $\mathcal{O}$ $\mathcal{O}$

$$s \qquad y^2 z - \quad x^3 \quad axz^2 \quad bz^3$$

${}^3 V^* \otimes \mathcal{O}$

V S x, y, z , , $\mathbb{C}^*$

$$a \mapsto \lambda^4 a \qquad b \mapsto \lambda^6 b$$

a, b S

a b

$\mathbb{C}^*$ $, \mathbb{C}$

1

W ß

m d

2.3. Elliptic K3 surfaces which don't admit sections

S 1

i $S_t \hookrightarrow S$ i_*L

L S_t S J

S i_*L

S

J

J ^{1}

$$S \qquad\qquad J$$
$$p_S \searrow \quad {}^1 \quad \swarrow p_J$$

J_t S_t S_t S J
J
s_0 fl $\mathcal{O}_S$ s_0 t $\mathcal{O}_S|_{S_t}$ $\mathcal{O}_{S_t} \in J_t$ S
$\{U_i\}$ ^{1} $S_i \sim J_i$ U_i
p_S^{-1} U_i p_J^{-1} U_i S_i J_i
S s_i $U_i \to S_i$
ϕ_i $S_i \to J_i$ s_i s_0 s_i
ϕ_i $S \sim J$

Proposition 2.4. *Th p fi* S *m ph* J *f d y f dm .*

S J
S J
^{1} $^1, \mathcal{B}$ $\mathcal{B}$
p_J $J \to$ ^{1} s
$s - s_0$
$U_i \cap U_j$ ϕ_i $S_i \to J_i$ ϕ_j $S_j \to J_j$
$\phi_i \circ \phi_j^{-1}$ $\mathcal{B}$

Definition 2.2. *y T - h f h g p* $Ш^{\mathrm{an}}$ J
S J
$Ш^{\mathrm{an}}$ J S

^{1} $^1, \mathcal{B}$

$$^1 \quad ^1, \mathcal{B} \sim \quad ^2 \; J, \mathcal{O}^* \; .$$

y B g p f J
J

$$\to \quad ^1 \; J, \mathcal{O}^* \to \quad ^2 \; J, \mathbb{Z} \to \quad ^2 \; J, \mathcal{O} \to \quad ^2 \; J, \mathcal{O}^* \to \; .$$

J L
$\mathbb{C}$ ^{2} $J, \mathcal{O}^*$ $\mathbb{C}$
J ρ ρ $\rho \geq$
J

Remark 2.4. ^{2} $J, \mathcal{O}^*$ g J

S J

Remark 2.5. S
$S \to$ ^{1} S α ^{2} $J, \mathcal{O}^*$
S k α
k g B g p
^{2} $J, \mathcal{O}^*$ $^2_{\text{ét}}$ $J, \mathcal{O}^*$

-
d

3. Irreducible holomorphic symplectic manifolds

¨

Definition 3.1. *h m ph ymp m f d* X ¨
$\sigma \in$ ^{0} $X,$ $^2T^*$
h m ph ymp f m X
$Y \times Z$ X d

X n σ
$\mathcal{K}$ ^{2} T^* c_1 T X fl
hyp k¨h ¨

3.1. Examples

Example 3.1. S
H h m f n p S $S^{[\ ]}$
n S n
S
$S^{[2]}$ $S \times S$

$S^{[\ ]}$ S
n

Example 3.2. T
$T^{[\ +1]}$
π T $T^{[\ +1]} \to$ ${}^{+1}T$
T g z d
K mm y K π^{-1}
n n K_1

Example 3.3. S q
^{2} $S, \mathbb{Z}$ ${}^{\bullet}$ $S, \mathbb{Z}$

$$q\ v, w \quad \int_S -v_0 w_4 \quad v_2 w_2 - v_4 w_0$$

v_i $w_i \in$ ^{i} $S, \mathbb{Z}$ $\mathcal{E}$ S M k v $\mathcal{E}$ $\in$ ${}^{\bullet}$ $S, \mathbb{Z}$
$\mathcal{E}$ ${}^{1/2}$ $\mathcal{E}$
S $\mathcal{E}$ r c_1 c_2

$$v\ \mathcal{E} \quad r, c_1, r \quad c_1^2/ \ - c_2\ .$$

$v \in$ ${}^{\bullet}$ $S, \mathbb{Z}$ $\mathcal{M}^{\mathrm{s}}$ v $\mathcal{E}$ S
v $\mathcal{E}$ v $\mathcal{M}^{\mathrm{s}}$ v n q v, v
v $\mathcal{M}^{\mathrm{s}}$ v
$S^{[\ ]}$

v $\mathcal{E}$ v $\mathcal{E}$ v
$\mathcal{M}^{\mathrm{s}}$ v
$\mathcal{M}^{\mathrm{s}}$ v
$S^{[\ ]}$ n q v, v
¨

$S^{[\ ]}$

K

Example 3.4. v , , $-$

$\mathcal{M}^{\mathrm{s}}\ v$ g $\mathcal{M}^{\mathrm{ss}}\ v$

$\mathcal{M}^{\mathrm{ss}}\ v$ $\widetilde{\mathcal{M}}$

$b_2\ \widetilde{\mathcal{M}}\ \geq$ $\widetilde{\mathcal{M}}$

$S^{[5]}$ K_5

b_2

3.2. Moduli spaces

X q_X $^2\ X, \mathbb{Z}$

$^2\ S, \mathbb{Z}$ S

$^2\ X, \mathbb{Z}\ , q_X$ Γ, q_Γ

Γ X $^2\ X, \mathbb{Z}\ , q_X$

$^2\ X, \mathbb{Z}$ Γ

$\sigma\ \in\ \ \Gamma \otimes \mathbb{C}$

$$\mathcal{P}_\Gamma\ \ \mathcal{T}_\Gamma \to Q_\Gamma\ \ \ \{x \in\ \ \Gamma \otimes \mathbb{C}\ |q_\Gamma\ x\ \ \ \ ,\ q_\Gamma\ x\ \ \ x\ >\ \}$$

$\mathcal{T}_\Gamma$ m $k\ d$

$\Gamma \otimes \mathbb{C}$

$\mathcal{T}_\Gamma$ $b_2 -$ b_2

X $\mathcal{T}_\Gamma$

$\mathcal{T}_\Gamma$ X X'

$\mathcal{T}_\Gamma$ Γ $\mathcal{M}_\Gamma$

X X'

$\mathcal{M}_\Gamma$

X X'

3.3. Abelian fibrations

Definition 3.2. fi n
X
n

n
n

Theorem 3.1. *F p j X, f X → B p p j m ph m h h h g fi F d. A m h B m h d < B < X. Th*

F y p fi m fi d ,
B n-d m d h h m H dg m ,
h fi L g g h p h h m ph ymp f m.

X
2

B

Example 3.5.

S T f $S \to$ 1
S

$f^{[\ ]}$ $S^{[\ ]} \to$ 1 $\sim$

$S^{[\ ]}$ n n

Example 3.6. S g C L
g D $|C|$ i_*L
i D S Z
S i_*L Z
$\mathcal{M}^{\mathrm{s}}$, C , v , C ,

$|C| \sim$ ^{g} $D \in |C|$ ${}^g D$

$$\begin{array}{ccc} {}^g & \hookrightarrow & Z \\ & & \downarrow \\ & & |C| \sim {}^g \end{array}$$

Z
$S^{[g]}$

g g
g
U Z $S^{[g]}$ $g \geq$ S
^{g} ${}^\vee$ $|C|$ g S
^{2} ${}^\vee$ g S
S $D \in |C|$ g
g D $S^{[g]}$ Z

Remark 3.7. g z d M k fl p

Example 3.8.
T g C Y'
T
g $|C|$
$\mathcal{M}^{\mathrm{s}}$, C , T Y'
g g C
g C π

$$Y' \to {}^{g+2}T \to T$$

T
Y' Y π^{-1}
g
K_g
Y $|C|$
T g Y'

$$\begin{array}{ccc} {}^{g+2} & \hookrightarrow & Y' \\ & & \downarrow \\ & & |C| \sim {}^g \end{array}$$

Y

$$\begin{array}{ccc} Y \cap \quad ^{g+2} & \hookrightarrow & Y \\ & & \downarrow \\ & & |C| \sim \quad ^{g} \end{array}$$

π ^{g+2}D $Y' \to$ ^{g} $D \in |C|$

T Y

$Y \cap$ ^{g+2}D $\pi|_{\mathrm{Pic}^{g+2}D}$ $^{g+2}D \to T$

g K_g

Y

Remark 3.9. Z $S^{[g]}$

Y K_g

Y_D $\pi|_{\mathrm{Pic}^{g+2}D}$ $^{g+2}D \to T$

$D \in |C|$ T C

d_1, d_2 $d_1|d_2$ C

$$d_1 d_2 \quad \frac{C.C}{} \quad g \quad .$$

d_1, d_2 $, g$ Y_D

$, \dots ,$ $, g$ Y

Example 3.10. S ^{2}

^{2} C S

$\widetilde{\mathcal{M}}$

$\mathcal{M}^{\mathrm{s}}$ $, C ,$ S

$|C|$ v $, C ,$

$|C|$

$\mathcal{M}^{\mathrm{s}}$ $, C ,$

$\mathcal{M}^{\mathrm{ss}}$ $, C ,$

$\widetilde{\mathcal{M}}$

$\mathcal{M}^{\mathrm{s}}$ $, C ,$

$$\begin{array}{ccc} ^{6} & \hookrightarrow & \mathcal{M}^{\mathrm{s}} \quad , C , \\ & & \downarrow \\ & & |C| \sim \quad ^{5} \end{array}$$

$\widetilde{\mathcal{M}}$

y k

X

4. Deforming to abelian fibrations

4.1. Divisors on abelian fibrations

S

F

X^2

D X

D

D X

D $+1$ E k E^k

m I k d m E m

K d d m

D n

X^2

$\bullet$ ^{2} $X, \mathbb{C}$

$\bullet$ $X, \mathbb{C}$ ^{2} $X, \mathbb{C}$

D X D

D $+1$ q_X D q_X

$$\int_X D^2 \qquad c_X q_X \; D$$

c_X D F j k

D^k $k \geq n$ q_X D D $+1$

D X^2

D
n q_X D
n

X
q_X S

Example 4.1. g $S \to$ ^{2}
^{2} C
$\mathcal{M}^{\mathrm{s}}$ $,$ C $,$
$S^{[2]}$ fl

$$\{\mathcal{K}_D | D \in |C|\} \sim \;\; ^{2\ \vee} \subset \mathcal{M}^{\mathrm{s}} \;\; , \; C \; ,$$

$|C|$ ^{2}
$^{2\ \vee}$

$$\{\; x, l \;\; \in \;\; ^2 \times \;\; ^{2\ \vee} | x \in l\}$$

$$G \quad \{g^{-1}\; x \; | x \in \;\; ^2\} \sim \;\; ^2 \subset S^{[2]}$$

G $g^{-1}\; x$ x

$\mathcal{M}^{\mathrm{s}}$ $,$ C $,$ $^{2\ \vee}$ $w \in$ ^{2}
$^{2\ \vee}$ D $\mathcal{M}^{\mathrm{s}}$ $,$ C $,$

D' $S^{[2]}$
D' $\{\xi \in S^{[2]} | w \;\; y$ z $\{y, z\}$ g ξ $\}$
D' ξ y z
D' ^{2}
D'

$$G \quad \{g^{-1}\; x \; | x \in \;\; ^2\} \sim \;\; ^2$$

w D' ^{2}

D' $S^{[2]}$
D'
D' $S^{[2]}$ D'
G D'
D $\mathcal{M}^{\mathrm{s}}$ $,$ C $,$ $\to$ $^{2\ \vee}$
D $^{2\ \vee}$ D

X D
q_X D X
D q_X D D
X' X

Conjecture 4.1. *A d h m ph ymp m f d X fi -*
f d y f - f d D h q h p h
B -B g m q d f m z , q_X D .

Remark 4.2.

Conjecture 4.2. *A y d h m ph ymp m f d X' h h*
- d D' h q_X D' fi X. M ,
f D h d p d g D', h h x p d-p g m ph m
τ f h ^{2} $X, \mathbb{Z}$ h h τ D f.

Remark 4.3. D D'
τ

D q_X D n

$$\phi_{mD} \quad X \to \quad {}^0 \; X, \mathcal{O} \; mD$$

$\mathcal{O}$ mD $m >$ ϕ_{mD}
$m \to \infty$ I k d m D
D n ϕ_{mD}
ϕ_{mD}

D
X
n ^{0} $X, \mathcal{O}$ mD
ϕ_{mD}
n

Proposition 4.3. *E y d h m ph ymp m f d X' h d*
B m b_2 X' $\geq$ d f m d h , X, h h X -
f d D h q_X D .

P f.

^{2} $X', \mathbb{Z}$, $q_{X'}$ $\sim$ Γ, q_Γ

, b_2 X' $-$

X

D q_X D D $-D$

¨

D $-D$ □

Remark 4.4.

, ,

b_2 X'

4.2. Vanishing theorems

$$\chi \; \mathcal{O} \; D \qquad \sum_{i=0} a_{2i} \; q_X \; D \quad ^i$$

a_{2i} D q_X D

χ $\mathcal{O}$ mD a_0 χ $\mathcal{O}_X$ n .

^{i} mD ^{i} $X, \mathcal{O}$ mD

$$\sum_{i=0}^{2} - \quad ^i \; ^i \; mD \qquad n \quad .$$

m D ^{0} D

n ^{i} D $i >$

i ^{0} D $\geq n$

q_X D

D

X

^{i} D i

D n

^{i} D $i >$ $n \geq$

D

n

S ON

X

$$\sigma \in {}^{0} X, {}^{2}T^{*} {}^{2,0} X .$$

$$\sigma \in {}^{0,2} X {}^{2} X, \mathcal{O}_X .$$

σ

,

X

$\mathcal{F}$

$$L {}^{\infty} \mathcal{E}^{p,q} \otimes \mathcal{F} \to {}^{\infty} \mathcal{E}^{p,q+2} \otimes \mathcal{F}$$

σ

$$L^{*} {}^{\infty} \mathcal{E}^{p,q+2} \otimes \mathcal{F} \to {}^{\infty} \mathcal{E}^{p,q} \otimes \mathcal{F} .$$

σ ∂

Definition 4.1. h f L f h z
σ

$$L {}^{q} X, \Omega^{p} \otimes \mathcal{F} \to {}^{q+2} X, \Omega^{p} \otimes \mathcal{F}$$

σ

$$L^{*} {}^{q+2} X, \Omega^{p} \otimes \mathcal{F} \to {}^{q} X, \Omega^{p} \otimes \mathcal{F} .$$

Remark 4.5.

p $\mathcal{F}$ $\mathcal{O}$ D D
L^{*}
^{0} D ^{i} D i
X ^{i} D $i > n$
^{2} D X

$$\sum_i h^i D \quad n \qquad \sum_i h^i D \geq n$$

^{2} D n
^{0} D ϕ_D
ϕ_D
D n

4.3. Some calculations

π $X \to$

D q_X D

^{i} mD

$$ {}^p \quad , R^q\pi_*\mathcal{O}\ mD \quad \Rightarrow \quad {}^{p+q}\ X, \mathcal{O}\ mD \quad . $$

$$ {}^p \quad , R^q\pi_*\pi^*\mathcal{O}_{\mathbb{P}^n}\ m \qquad {}^p \quad , \mathcal{O}_{\mathbb{P}^n}\ m\ \otimes R^q\pi_*\mathcal{O}_X\ . $$

q

$$ R^0\pi_*\mathcal{O}_X \sim \mathcal{O}_{\mathbb{P}^n}. $$

q $R^1\pi_*\mathcal{O}_X$ $x \in$

$$ {}^1\ \pi^{-1}\ x\ , \mathcal{O}_{\pi^{-1}(x)} \sim \overline{{}^0\ \pi^{-1}\ x\ , \Omega^1_{\pi^{-1}(x)}}. $$

X T_X Ω^1_X

$\pi^{-1}\ x$

$\pi^*T_{\mathbb{P}^n} \oplus T_{\pi^{-1}(x)}$ $\pi^*\Omega^1_{\mathbb{P}^n} \oplus \Omega^1_{\pi^{-1}(x)}$

$\Omega^1_{\pi^{-1}(x)}$ $\pi^*T_{\mathbb{P}^n}$

$$ R^1\pi_*\mathcal{O}_X\ {}_x \sim \overline{{}^0\ \pi^{-1}\ x\ , \pi^*T_{\mathbb{P}^n}} \quad \overline{T_{\mathbb{P}^n}}\ {}_x. $$

.. $\Omega^1_{\mathbb{P}^n}\ {}_x$

$$ R^1\pi_*\mathcal{O}_X \sim \Omega^1_{\mathbb{P}_n}. $$

$$ R^q\pi_*\mathcal{O}_X \sim \Omega^q_{\mathbb{P}_n}. $$

$q \geq$ n

Example 4.6. X n

p, q , ,

$$ {}^i\ mD \quad \begin{cases} m & i \\ m - & i \\ & i \end{cases} . $$

Example 4.7. n q

$$ \begin{aligned} R^2\pi_*\mathcal{O}_X &\sim R^2\pi_*\ \mathcal{K}_X \otimes \pi^*\mathcal{K}^\vee_{\mathbb{P}^2} \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}\ - \\ &\sim \mathcal{O}_{\mathbb{P}^2}\ - \quad \otimes R^2\pi_*\mathcal{K}_{X/\mathbb{P}^2} \\ &\sim \mathcal{O}_{\mathbb{P}^2}\ - \quad \otimes\ R^0\pi_*\mathcal{O}_X\ {}^\vee \\ &\sim \mathcal{O}_{\mathbb{P}^2}\ - \end{aligned} $$

$\mathcal{K}_X$
p, q , , ,

$$^{i}\; mD \quad \begin{cases} m & m & / & i \\ m & m - & & i \\ m - & m - & / & i \end{cases}$$

Example 4.8. n

$$^{i}\; mD \quad \begin{cases} m & m & m & / & i \\ m & m & m - & / & i \\ m & m - & m - & / & i \\ m - & m - & m - & / & i \end{cases}$$

Remark 4.9. $\chi\ \mathcal{O}\ mD$ n
m $^{0}\ D$ n $^{i}\ D$ $i >$

Remark 4.10. n m $^{2}\ mD$ $^{0}\ mD$

$$L^{*} \quad ^{2}\ X, \mathcal{O}\ mD \quad \to \quad ^{0}\ X, \mathcal{O}\ mD$$

n $^{2}\ mD\ <\ ^{0}\ mD$ $m >$
L^{*}

Conjecture 4.4. *If* E f d d h m ph ymp -f d
$X,$ h h m p

$$L^{*} \quad ^{2}\ X, E \quad \to \quad ^{0}\ X, E$$

h h $f\,L\,f$ h z j .

L^{*}

5. Classification of abelian fibrations which admit sections

X^{2}

X

S ON

5.1. Integrable systems

X

X g mp y g H m
y m

$\mathbb{C}$

Q Q

Q

X

X

ﬂ

$\mathbb{J}^{2g} \to U \subset \mathbb{C}^g$
U $\mathbb{C}^g$ $\mathbb{S}^{g+1} \to U$
U $\mathbb{C}^g$ $\mathbb{S} \to U$
I $\mathbb{S} \hookrightarrow \mathbb{J}$

$I^*\Omega \wedge I^*\Omega$

$\mathbb{S}$ Ω $\mathbb{J}$ $I^*\Omega$
$\mathbb{S}$ $\mathbb{S}$ $\mathbb{J}$
$\mathbb{S}$ Q $I^*\Omega$
ω Q

Theorem 5.1. *Th* *m p*

$$Q^{[g]} \to \mathbb{J}$$

h h p ymp , . k h h m ph ymp f m
$Q^{[g]}$ *d d y* ω Ω $\mathbb{J}$. *M , f g* S_t *h* $t \in U$, *k h*
H *h m* $S_t^{[g]}$ ${}^g S_t$ *h* J_t *f* S_t *y h A m p (h h*
L g g m f d).

Remark 5.1.

$\mathbb{J}$

Q S_t

Q $|S_t|$

Q g

$\mathbb{J}$ U

S_t g Q

$Q^{[g]}$ $\mathbb{J}$

Example 5.2.

$\mathcal{M}$ r

$\mathcal{M}$ $T^*\mathcal{N}$ $\mathcal{N}$

$T^*\mathcal{N}$

$\mathcal{M}$ p

r ρ $\mathcal{K}_\Sigma \to$

$|\mathcal{O}\ r\rho^*\mathcal{K}_\Sigma\ |$

$\mathcal{K}_\Sigma$ Q $\mathcal{K}_\Sigma$

$\mathcal{M}$ $\mathcal{K}_\Sigma^{[g]}$ g

S

Example 5.3. S C g

$S^{[g]}$ Z

$|C| \sim$ g g g

$Z^{2g} \to$ g $Y^{g+1} \to$ g

Y

$$Y \quad \{\ p, D\ |p \in D \in |C|\}$$

g $p, D\ \mapsto D$ Y

S

$$Y_q \quad \{\ p, D\ \in Y|p \quad q\}$$

$q \in S$ Y_q D q

$g-1$ Y g

$g-1$

Y Y Z

$$p, D\ \mapsto i_*\mathcal{O}_D\ p \quad g - \quad p_0$$

$p_0 \in D$ $D \in$ g $\mathcal{O}_D\ p$ $g -$ p_0 g

D i $D \hookrightarrow S$ Y

$g -$ $\sigma^2|_Y$ σ

Z Y

S

Z $S^{[g]}$

Remark 5.4. $\mathbb{J} \to U$

m g y ^{0}

g ^{g}

Y $g -$

Y Z

Z

5.2. Fibrations and foliations

$X \to$

$X \to$

$Y^{+1} \to$

Y X

$\sigma^2|_Y$ Y Q

Y_t $t \in$

Q $q \in Q$

$Y_q \sim$ $^{-1}$ q

Y $^{-1}$

Q Q

Conjecture 5.2. *pp h d h m ph ymp m f d* X^2 *fi , h fi m ph f . If h fi dm h pp p m (f. R m k 5.4) h* X *h* H *h m* $S^{[\,]}$ *f n p K3 f* S.

Remark 5.5. n

$\sigma^2|_Y$ n $\widetilde{\mathcal{M}}$

$\widetilde{\mathcal{M}}$

$\widetilde{\mathcal{M}}$

K

$, \dots, \ , n$

Proposition 5.3. *pp h p j fi* $X \to$ *d f m f h g z d K mm y K . Th d d p z f h fi f* X *p p .*

P f. E $X \subset {}^N$ D E tD

$$\int_X E \quad tD^{\,2} \qquad c_X q_X \; E \quad tD$$
$$c_X \; q_X \; E \qquad t q_X \; E, D$$

t

$$\int_X E \;\; D \qquad \frac{n \;\; ^2}{n} c_X q_X \; E, D \quad .$$

D F $\int_F E|_F$ c_X K

$$c_{K_n} \quad \frac{n \qquad n}{n}$$

$$\int_F E|_F \qquad n \qquad q_X \; E, D$$

$q_X \; E, D \quad \in \mathbb{Z}$ $E|_F$ F n ∎

Remark 5.6. $q_X \; E, D$ $S^{[\;]}$

$$c_{S^{[n]}} \quad \frac{n}{n}$$

$q_X \; E, D$

Conjecture 5.4. *pp h d h m ph ymp m f d* X^2 *fi , d h fi h p z* $, \dots, \;, n$ *. If h fi dm h pp p m h* X *g z d* K *mm y* K *.*

T

$, \dots ,$ $, \dots , \;, n$ $S^{[\;]}$ K

X

5.3. Four-folds fibred by Jacobians

Theorem 5.5. *pp h d h m ph ymp f -f d* π $X \to$ ^{2} *fi d y f g - , d h h fi dm . Th* X $S^{[2]}$ *f m K3 f* S.

P f.
S $^{2\ \vee}$
B
$Y^3 \to$ ^{2} Y_t $t \in$ ^{2}
$_t^1$ ^{0} $Y_t, \mathcal{K}_{Y_t}$
$_t^1$ $^{2\ \vee}$
X_t Y_t $T_p X_t$
$p \in X_t$ ^{0} $Y_t, \mathcal{K}_{Y_t}$
X $T_p X_t$
$\pi^*\Omega^1_{\mathbb{P}^2}$ $_p$ $\Omega^1_{\mathbb{P}_2}$ $_t$.
^{2}

$$\to \Omega^1_{\mathbb{P}^2} \to \quad ^0 \quad ^2, \mathcal{O} \qquad \otimes\, \mathcal{O} \ - \qquad \to \mathcal{O}_{\mathbb{P}^2} \to$$

$$\Omega^1_{\mathbb{P}^2} \hookrightarrow \qquad ^0 \quad ^2, \mathcal{O} \qquad\qquad ^{2\ \vee}$$

^{2} $^{2\ \vee}$
$t \in$ ^{2}
$_t^1$ t
$^{2\ \vee}$
$_t^1$ $_t^1$
B X X
$\in$ ^{2} $B \in$ $^{2\ \vee}$
B $^{2\ \vee}$ B
S $_t^1$ $^{2\ \vee}$ S
Y_t $_t^1$
^{2}
X Z
$S^{[2]}$ □

Remark 5.7. Y_t
$^{3\ \vee}$

6. Abelian fibrations which don't admit sections

6.1. Gerbes and elliptic fibrations

X

X X

d

J ${}^2\ J, \mathcal{O}_J^*$ h m ph
g J

Definition 6.1. $\{U_i\}$ J $\{\mathcal{L}_{ij}\}$
U_{ij} $U_i \cap U_j$

$\mathcal{L}_{ji} \sim \mathcal{L}_{ij}^{-1}$ i j
$\mathcal{L}_{ijk}$ $\mathcal{L}_{ij} \otimes \mathcal{L}_{jk} \otimes \mathcal{L}_{ki}$ U_{ijk} β_{ijk} $U_{ijk} \to \mathcal{O}^*$ i j
k

$\{\beta_{ijk}\}$ $\mathcal{L}_{ijk}$ $\delta\beta$
$\mathcal{L}_{ijkl}$ $\mathcal{L}_{jkl} \otimes \mathcal{L}_{ikl}^{-1} \otimes \mathcal{L}_{ijl} \otimes \mathcal{L}_{ijk}^{-1}$ U_{ijkl}

$\{\mathcal{L}_{ij}\}$ h m ph g J
$\beta \in {}^2\ J, \mathcal{O}_J^*$

S J
S
J_t
$S_t \times J_t$
$\{U_i\}$ ^{1} S_i $p_S^{-1}\ U_i$
J_i $p_J^{-1}\ U_i$ $\mathcal{U}_i$

$S_i \times_{U_i} J_i$

$$\begin{array}{ccccc} & & \mathcal{U}_i & & \\ & & \downarrow & & \\ & & S_i \times_{U_i} J_i & & \\ & \pi_S \swarrow & & \searrow \pi_J & \\ S_i & & & & J_i \\ & p_S \searrow & & \swarrow p_J & \\ & & U_i & & \end{array}$$

U_{ij} $\mathcal{U}_i$ $\mathcal{U}_j$
$\mathcal{L}_{ij}$ J_{ij} p_J^{-1} U_{ij}
$\{\mathcal{L}_{ij}\}$ J $\alpha_S \in$ ^{2} $J, \mathcal{O}_J^*$
α_S
α_S
S
s_i $U_i \to S_i$ S_t

$\mathcal{U}_i$
S

Proposition 6.1. *Th f g q :*

S *m ph* J,
S *dm* ,
h x d $S \times_{\mathbb{P}^1} J$.

Th f h x p m $\alpha_S \in$ ^{2} $J, \mathcal{O}_J^*$.

Remark 6.1.
$S \to B$ $J \to B$ S

$$^2 \; J, \mathcal{O}_J^* \; / \; ^2 \; B, \mathcal{O}_B^* \; .$$

^{1}
^{2} $B, \mathcal{O}_B^*$ B
B J

$$^2 \; B, \mathcal{O}_B^* \hookrightarrow \; ^2 \; J, \mathcal{O}_J^* \; .$$

6.2. Gerbes and abelian fibrations

Definition 6.2. $X \to$ P d X
P
X

Remark 6.2. i $X_t \hookrightarrow X$
i_*L
X_t P
X i_*L P

Remark 6.3. P X_t
P_t ^{0} X_t X_t
P_t
X P
P

P X
X_t P_t
$X_t \times P_t$

$\beta_X \in$ ^{2} $P, \mathcal{O}_P^*$

^{2} $, \mathcal{O}_{\mathbb{P}^n}^*$
P X_0
β_{X_0} ^{2} $P, \mathcal{O}_P^*$ X
X_0
$P \to$ fl $\mathcal{O}_X$
$\mathcal{O}_{X_t}$ X X_0
s_0 P
$X \to$ X_0
s_0 X X_0 ^{1} $, \mathcal{B}$
$\mathcal{B}$ $X_0 \to$
X

Conjecture 6.2. *Th f g q :*
X *m ph* $X_0,$
X *dm* ,

h x d $X \times_{\mathbb{P}^n} P$.

Th f h x p m $\beta_X \in {}^2\, P, \mathcal{O}_P^*$.

Remark 6.4. P

$$\to {}^1\, P, \mathcal{O}_P^* \to {}^2\, P, \mathbb{Z} \to {}^2\, P, \mathcal{O}_P \to {}^2\, P, \mathcal{O}_P^* \to {}^3\, P, \mathbb{Z} \to$$

$${}^2\, P, \mathcal{O}_P \sim \mathbb{C} \qquad {}^2\, P, \mathcal{O}_P^*$$

$${}^3\, P, \mathbb{Z}$$

$${}^2\, P, \mathcal{O}_P^*$$

Remark 6.5.
J S

S J

Remark 6.6. X_0 P X P
B β_X
¨

d F -M k
f m

6.3. Two examples

Example 6.7. g C
$S^{[g]}$ Z
$|C| \sim$ g ${}^g D$ Z $D \in |C|$
Z P

g S $2 \vee$
D $2 \vee$ Z_k $|C| \sim$ 2
k k Z_k
S v , C , $k-$ v
Z_k k

S
$|C|$ v k
Z_k $S^{[2]}$
Z_k Z_0
Z_2 $\mathcal{O}_D \in$ ^{0} $\mathcal{K}_D \in$ ^{2}
$D \in |C|$ Z Z_2 Z_0
Z_1 Z_1
Z_0 Z_2

Z_1 Z_0 Z_2

Z_k P
^{0} $^kD \sim$ ^0D
$D \in |C|$ P Z_0
P
Z_0 $S^{[2]}$
^{2} $P, \mathcal{O}_P^*$ Z_0
Z_1
$S^{[2]}$
Z_1 $\beta_1 \in$ ^{2} $P, \mathcal{O}_P^*$
Z_1

Remark 6.8. S
Z Z_0 fl $S^{[2]}$
^{2} $P, \mathcal{O}_P^*$
$S^{[2]}$

$S^{[2]}$
^{2} $P, \mathcal{O}_P^*$

g Z_0
β_1 Z_1
Z_0
m p fi

Example 6.9. S ^{2}
C ^{2} C

Z_5 Z_6 $\mathcal{M}^s$, C ,
$\mathcal{M}^{ss}$, C ,
Z_5 Z_6
$|C| \sim$ ^{5} ^5D ^6D $D \in |C|$
D Z_5 Z_6
Z_6 Z_5
Z_5 Z_6
Z_5 Z_6
^{2}
Z_6
$E \in |C|$ $F_5 \subset Z_5$ $F_6 \subset Z_6$ E
F_5
F_6 $|C|$
Z_5 Z_6 $U_5 \subset Z_5$ $U_6 \subset Z_6$
U_5 U_6

Z_5 $S^{[5]}$
Z_6 $\widetilde{M}$
$S^{[5]}$ Z_5
Z_6 Z_6 $\widetilde{M}$ $S^{[5]}$
$\widetilde{M}$
U_6 U_5

Remark 6.10.
$S^{[g]}$ Z_g Z
Z_k $D \in |C|$ ^kD
Z_k Z_0 Z_1
k Z_0 Z_1

Z_5 Z_6

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