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# Abelian fibred holomorphic symplectic manifolds

*Justin Sawon*

## Abstract

We study holomorphic symplectic manifolds which are fibred by abelian varieties. This structure is a higher dimensional analogue of an elliptic fibration on a K3 surface. We investigate when a holomorphic symplectic manifold is fibred in this way, and are led to several natural conjectures. We then study the geometry of these fibrations. The expectation is that this point of view will prove useful in understanding holomorphic symplectic manifolds, and possibly lead to a classification.

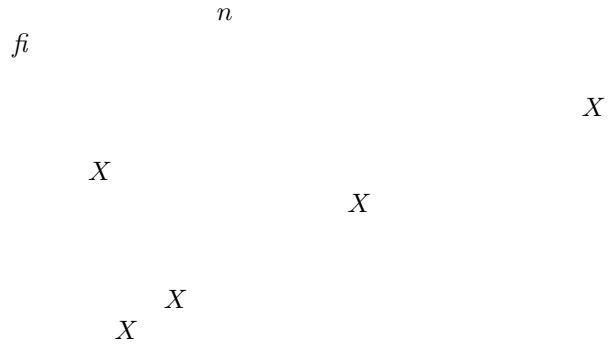
## 1. Introduction

Irreducible holomorphic symplectic manifolds are higher dimensional generalizations of K3 surfaces. It has been roughly twenty years since Fujiki [12] found the first example, the Hilbert scheme  $S^{[2]}$  of two points on a K3 surface  $S$ , and Beauville [1] generalized this to construct two families,  $S^{[n]}$  and the generalized Kummer varieties  $K_n$ . Since then there have been other constructions, but only the two examples of O’Grady [38, 39] have given us manifolds which are not deformations of Beauville’s examples. The purpose of this article is to describe a framework for understanding irreducible holomorphic symplectic manifolds, which hopefully will lead towards some kind of classification. The main results are only conjectural, but we will describe the evidence and motivation behind them, while surveying special cases which have already been proved.

In studying the moduli space of K3 surfaces, one typically looks at Kummer surfaces or quartics in  $\mathbb{P}^3$ , as these are dense but also relatively easy to understand. However, the structure that will generalize to higher dimensions is a fibration by abelian varieties. This suggests that we first review elliptic K3s, which also happen to be dense in the moduli space. We divide this into three main steps. Firstly, we need to know which K3 surfaces are elliptic. Secondly, we describe the family of elliptic K3s which admit a section. Thirdly, we describe the relation between elliptic K3s which don’t admit sections and their *relative Jacobians*, which do. There is nothing new here: for the first step we recall a theorem of Pjateckiĭ-Šapiro and Šafarevič [41] from the 70s (c.f. also Section 5 of Kodaira [27]), while the second and third steps have been well understood for arbitrary elliptic surfaces for a long time.

Elliptic K3s have base  $\mathbb{P}^1$ . There is evidence to suggest that if the  $2n$ -dimensional irreducible holomorphic symplectic manifold  $X$  admits a non-trivial fibration, then the fibres must be abelian varieties and the base must be  $\mathbb{P}^n$  (a large part of this has been proved by Matsushita [32]). So the ‘right’ generalization of an elliptic fibration on a K3

S ON



$S^{[5]}$

**2. Elliptic K3 surfaces**

$$S \xrightarrow{q} {}^2 S, \mathbb{Z} \quad L = H \oplus -E_8,$$

**Definition 2.1.**  $p \quad d \quad S \quad {}^2 S, \mathbb{Z}, q$

$${}^2 S, \mathbb{Z} \otimes \mathbb{C} \quad {}^2 S, \mathbb{C} \quad {}^{2,0} S, \mathbb{C} \oplus {}^{1,1} S, \mathbb{C} \oplus {}^{0,2} S, \mathbb{C} .$$

$$\sigma \quad {}^{0,2} S, \mathbb{C} \quad {}^{1,1} S, \mathbb{C} \quad \sigma \quad S \quad {}^{2,0} S, \mathbb{C} \quad \mathbb{C}\sigma \oplus \mathbb{C}\sigma$$

$$q \quad \sigma \quad \sigma > \quad \sigma \in \begin{matrix} L \otimes \mathbb{C} \\ Q \quad L \otimes \mathbb{C} \end{matrix} \quad {}^2 S, \mathbb{Z}, q \quad \begin{matrix} L \\ q \quad \sigma \end{matrix}$$

**Theorem 2.1.** *If K3  $f \quad S \quad d \quad S' \quad h \quad m \quad ph \quad p \quad d$*

$$\psi \quad {}^2 S, \mathbb{Z} \xrightarrow{\cong} {}^2 S', \mathbb{Z}$$

S ON

$$f: S' \rightarrow S \xrightarrow{\cong} S'$$

$h: h^* \mathcal{K}_S \rightarrow \mathcal{K}_{S'}$

$$f^* \mathcal{K}_S \cap \mathcal{K}_{S'} \cong \mathcal{K}_{S'} \oplus \mathcal{O}_{S'}(1,1)$$

$$L \otimes \mathcal{O}_S(2) \cong \mathcal{O}_S(2)$$

$\mathcal{T}$

$$\mathcal{P} \rightarrow \mathcal{Q} \subset L \otimes \mathbb{C}$$

$\mathcal{T}$

$$\mathcal{M} / L$$

2.1. A categorization of elliptic K3 surfaces

$$E \oplus F \rightarrow E \oplus F \rightarrow E \oplus F \rightarrow E \oplus F$$

**Theorem 2.2.** *A projective elliptic K3 surface  $f: S \rightarrow \mathbb{P}^1$  with a section  $\sigma$  and a divisor  $D$  such that  $D^2 = -2$  is isomorphic to a K3 surface of rank 2.*

$P \rightarrow f$

$$D \otimes L \cong \mathcal{O}_S(D) \oplus \mathcal{O}_S(-D)$$

$$f^* \mathcal{K}_S \cong \mathcal{K}_{S'} \oplus \mathcal{O}_{S'}(1,1)$$

S ON

$$\begin{array}{ccccccc}
 & D & & S & & H & \\
 D.H \in \mathbb{Z} & & & D & & D & \\
 & A & & D.A < & & A & D \\
 A^2 - & A & - & & & & \\
 & & & D_1 & D & D.A & A \\
 & \text{fl} & A & \text{fl} & & & H \\
 & D_1 & & D_1 & & & \\
 & & & D_1.H & D.H & D.A & A.H < D.H \\
 & & & & & & D.H & D_1.H & D_2.H & \dots \\
 & & & & & & D & & & \\
 & & & & & & |D| & & & \\
 & & & & & & & & & C & C^2 \\
 m > & H & & & & & & & & mC \in |D| \\
 & f & S \rightarrow 1 & & C & & f & & & \\
 & & & & C^2 & & & & & \\
 & & & & |C| & & & & & \\
 & & & & & & & & & \square
 \end{array}$$

Remark 2.1.

$$D \quad D^2$$

$$\begin{array}{cccc}
 & & D & D^2 \\
 E & E^2 > & & L
 \end{array}$$

Remark 2.2.

$$\begin{array}{ccc}
 x \in L \setminus \{ \} & x^2 & S \\
 \phi : S, \mathbb{Z} \xrightarrow{\cong} L.
 \end{array}$$

S ON

$$\begin{array}{c}
 S \quad S \\
 \phi_{\mathbb{C}} \sigma \in Q \subset L \otimes \mathbb{C} \quad \phi^{-1} x \\
 , \quad \mathcal{M} \quad x \cdot \phi_{\mathbb{C}} \sigma \quad x \\
 \mathcal{M} \quad x \in L
 \end{array}$$

$$\begin{array}{c}
 A \quad D \quad A \\
 D_1 \quad D_1.A \quad -D.A \quad L
 \end{array}$$

**Proposition 2.3.** *A y K3 f d f m d p K3.*

$$\begin{array}{c}
 P f. \\
 D \quad D^2 \quad S \quad D \\
 T \quad Q \quad D \quad -D \\
 \dots \\
 D \quad -D \quad S \\
 D \\
 {}^2 X, \mathcal{O} D \quad {}^0 X, \mathcal{O} -D \\
 {}^0 X, \mathcal{O} D \geq D \\
 D^2 \quad D \\
 1
 \end{array}$$

□

**Remark 2.3.**

## 2.2. Elliptic K3 surfaces which admit sections

$$\begin{array}{c}
 \mathbb{C} \\
 1 \\
 \mathbb{C}^{-1} \quad \mathbb{C}^{-1} \quad \mathbb{C}^{-1} \quad E \\
 \mathbb{C}^{-1} \quad \mathbb{C}^{-1}
 \end{array}$$

S ON

II  $S$   $S_1$   $I_1$

$S$   $V$   $1$

$$S \subset \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}$$

$$a \quad b \quad V \quad \mathcal{O} \quad \mathcal{O} \quad \mathcal{O} \quad x \quad y \quad z$$

$$y^2z \quad x^3 - axz^2 - bz^3$$

$$s \quad y^2z - x^3 \quad axz^2 \quad bz^3$$

$$V \quad {}^3V^* \otimes \mathcal{O} \quad S \quad x, y, z \quad , \quad \mathbb{C}^*$$

$$a \mapsto \lambda^4 a \quad b \mapsto \lambda^6 b$$

$$a, b \quad S$$

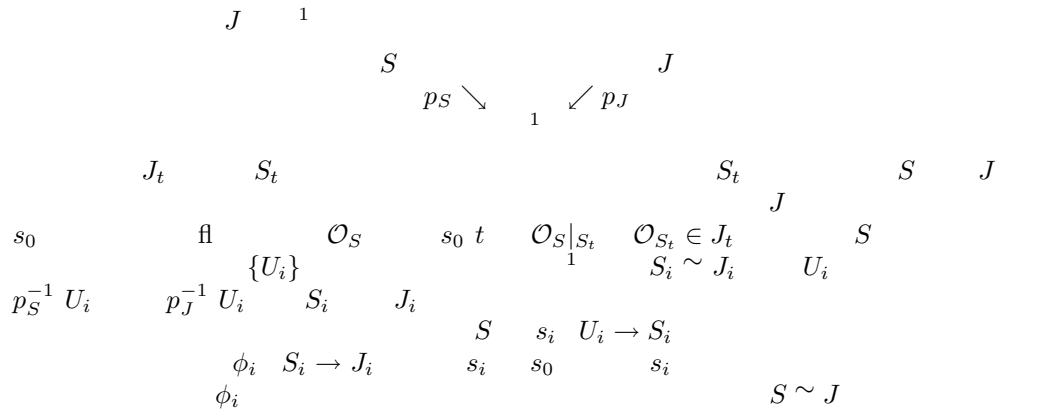
$$a \quad b \quad \mathbb{C}^* \quad , \mathbb{C}$$

$m \quad d$   $W$   $\beta$

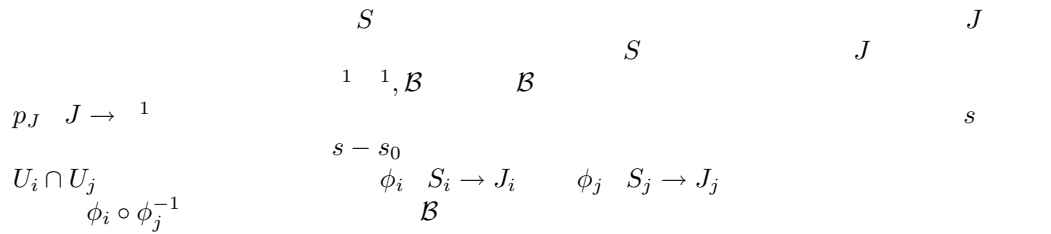
### 2.3. Elliptic K3 surfaces which don't admit sections

$$i \quad S_t \hookrightarrow S \quad L \quad S_t \quad S \quad i_* L \quad J \quad S \quad i_* L$$

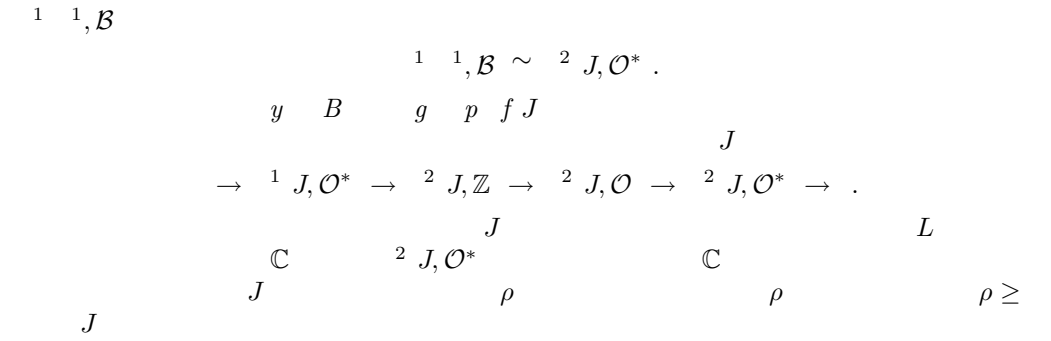
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**Proposition 2.4.** *Th*  $p$   $f_i$   $S$   $m$   $ph$   $J$   $f$   $d$   
 $y$   $f$   $dm$  .



**Definition 2.2.**  $y$   $T$   $- h f$   $S$   $h g$   $p$   $\text{III}^{\text{an}} J$   $J$   
 $\text{III}^{\text{an}} J$   $S$





S ON

**Remark 2.4.**  ${}^2 J, \mathcal{O}^*$   $g$   $J$

$S$   $J$

**Remark 2.5.**  $S$   $S \rightarrow {}^1$   $S$   $S$   $k$   $\alpha$   ${}^2 J, \mathcal{O}^*$   $\alpha$

$k$   $g$   $B$   $g$   $p$   ${}^2_{\text{ét}} J, \mathcal{O}^*$

$d$

### 3. Irreducible holomorphic symplectic manifolds

..

**Definition 3.1.**  $h$   $m$   $ph$   $ymp$   $m$   $f$   $d$   $X$   $\sigma \in {}^0 X, {}^2 T^*$  ..

$h$   $m$   $ph$   $ymp$   $f$   $m$   $X$   $X$   $d$

$\mathcal{K}$   $X$   ${}^2 T^*$   $n$   $c_1 T$   $hyp$   $k$   $h$   $X$  ..  $\text{fl}$

#### 3.1. Examples

S ON

**Example 3.1.**

$$\begin{array}{ccccccc}
H & h & m & f & n & p & S \\
n & & & & & & S \\
& & S & & & & S \\
& & & S^{[2]} & & & S \\
& & & & & & S \times S
\end{array}$$

$$\begin{array}{ccc}
S^{[1]} & & S \\
& n &
\end{array}$$

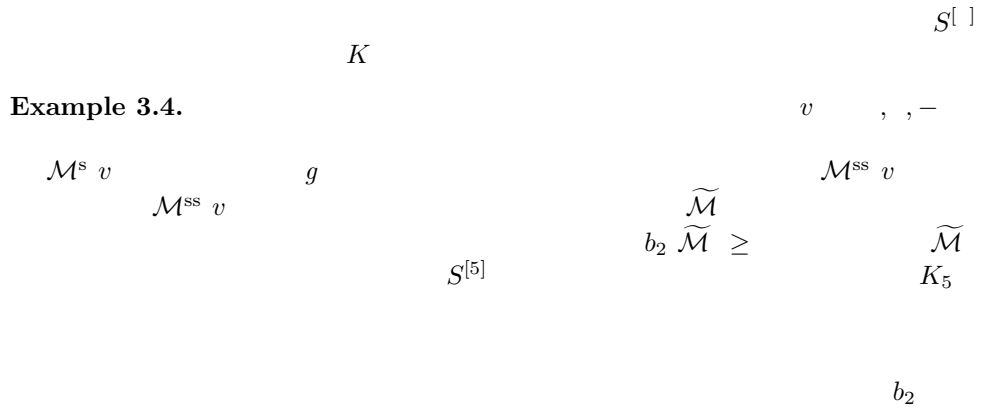
**Example 3.2.**

$$\begin{array}{ccccccc}
& & T & & & & \\
& & & T^{[+1]} & & & \\
& & & \pi & T & & T^{[+1]} \rightarrow \quad +1T \\
& & & & & & g \quad z d \\
K & mm & y & K & n & \pi^{-1} & K_1
\end{array}$$

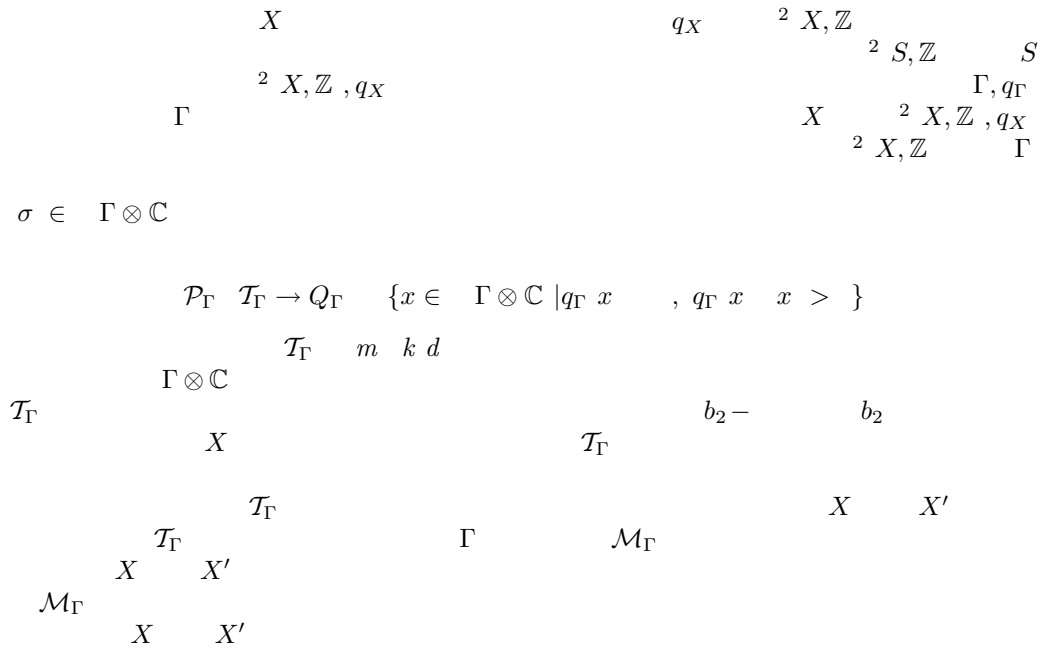
**Example 3.3.**

$$\begin{array}{ccccccc}
S & & & & & & q \\
{}^2 S, \mathbb{Z} & & & & & & \bullet S, \mathbb{Z} \\
& & q & v, w & \int_S -v_0 w_4 & v_2 w_2 - v_4 w_0 & \\
v_i & w_i \in & {}^i S, \mathbb{Z} & & \mathcal{E} & S & M k \\
& & \mathcal{E} & 1/2 & & & \mathcal{E} \\
& & S & & \mathcal{E} & & r \\
& & & & & & v \mathcal{E} \in \bullet S, \mathbb{Z} \\
v \in \bullet S, \mathbb{Z} & & v \mathcal{E} & r, c_1, r & c_1^2 / - c_2 . & & \\
& & \mathcal{M}^s v & & & & \mathcal{E} S \\
& & v & \mathcal{M}^s v & & & n \quad q v, v \\
& & & & & & v \mathcal{M}^s v \\
& & & & S^{[1]} & &
\end{array}$$

$$\begin{array}{ccccccc}
v & & & & & & v \mathcal{E} v \\
& & & \mathcal{M}^s v & & & \mathcal{M}^s v \\
& & & & S^{[1]} & & n \quad q v, v
\end{array}$$



**3.2. Moduli spaces**



3.3. Abelian fibrations

**Definition 3.2.**  $f: X \rightarrow B$  is a fibration if  $f^* \circ f = \text{id}_X$ .

**Theorem 3.1.** Let  $f: X \rightarrow B$  be a fibration. Then  $f^* \circ f = \text{id}_X$  if and only if  $f$  is a fibration.

$$f^* \circ f = \text{id}_X \iff f \text{ is a fibration}$$

2

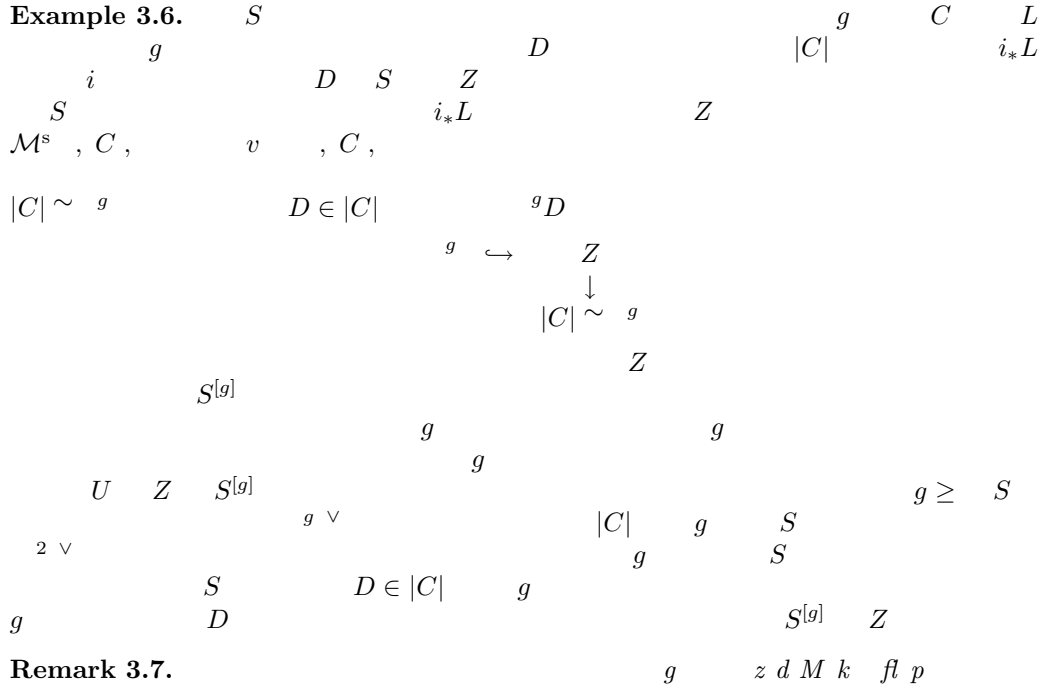
$B$

**Example 3.5.**

$$S \xrightarrow{f} S \rightarrow S^1$$

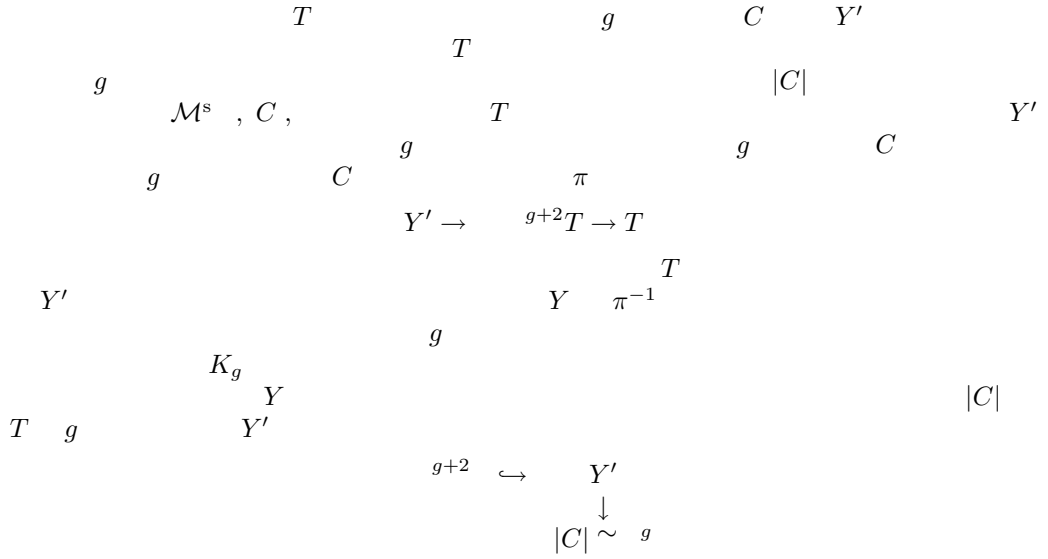
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**Example 3.6.**

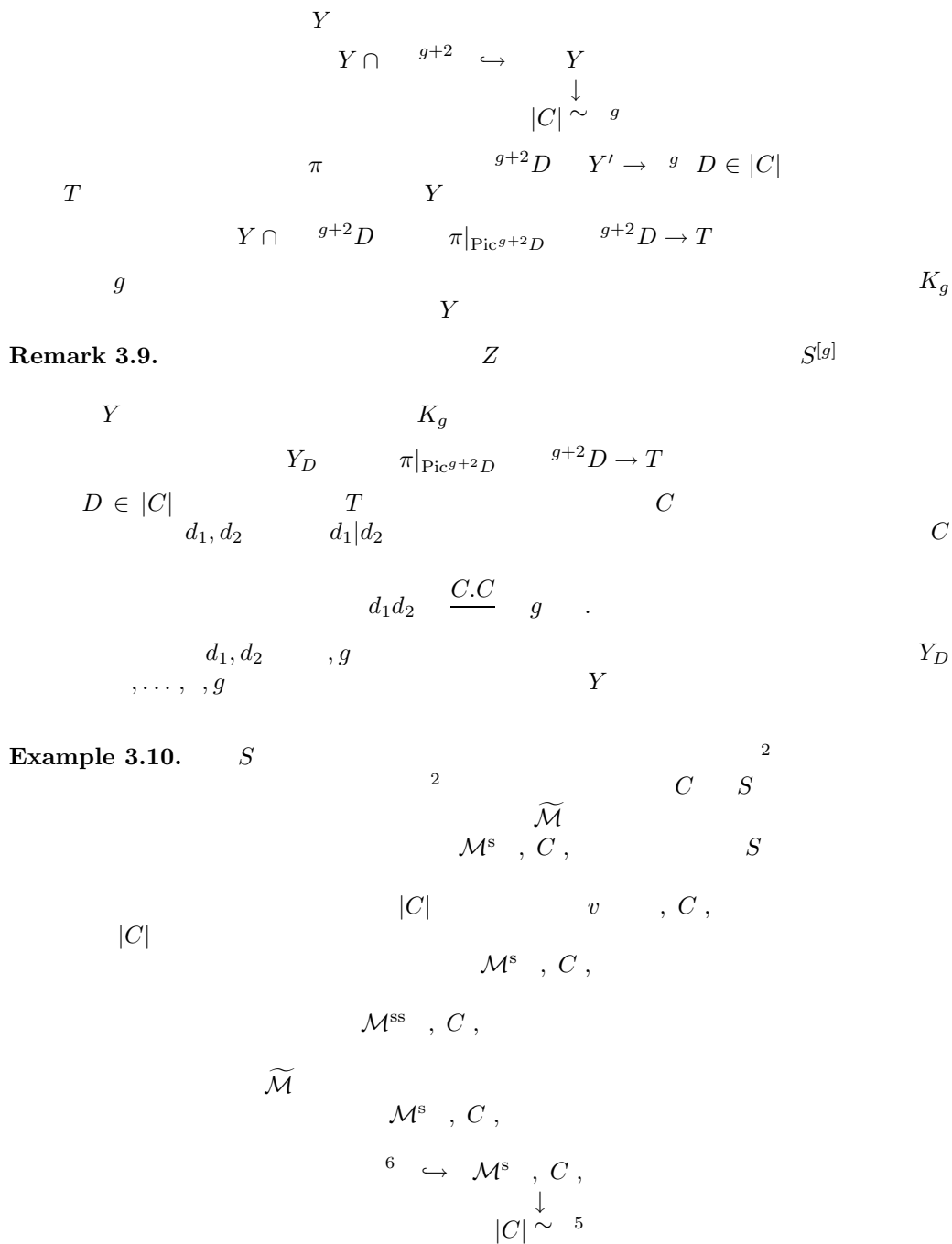


**Remark 3.7.**

**Example 3.8.**



S ON



$\widetilde{\mathcal{M}}$

$y k$

$X$

#### 4. Deforming to abelian fibrations

##### 4.1. Divisors on abelian fibrations

$S$

$F$

$X^2$   $D$   $X$

$D$   $D$   $X$

$D^{+1}$   $D$   $E$   $k$   $E^k$   $X$

$K d$   $m$   $I k d m$   $E$   $m$

$d m$

$D$   $n$

$\bullet X, \mathbb{C}$   $\bullet X, \mathbb{C}$   $X^2$

$\bullet X, \mathbb{C}$   $D$   $X D$   $X, \mathbb{C}$

$D^{+1}$   $q_X D$   $q_X$

$\int_X D^2$   $c_X q_X D$

$D$   $\frac{D^k}{X^2}$   $c_X$   $k \geq n$   $q_X D$   $\frac{D}{D^{+1}}$   $F j k$

S O N

$$\begin{matrix} D \\ n \text{ } q_X D \\ n \end{matrix}$$

$$q_X \quad X \quad S$$

**Example 4.1.**  $g: S \rightarrow \mathbb{Z}^2$

$$S^{[2]} \quad \mathcal{M}^s, C, \quad \text{fl} \quad C$$

$$\{\mathcal{K}_D | D \in |C|\} \sim \mathbb{Z}^2 \vee \subset \mathcal{M}^s, C, \quad |C| \quad \mathbb{Z}^2$$

$$\{x, l \in \mathbb{Z}^2 \times \mathbb{Z}^2 \vee | x \in l\}$$

$$G \quad \{g^{-1} x | x \in \mathbb{Z}^2\} \sim \mathbb{Z}^2 \subset S^{[2]}$$

$$G \quad g^{-1} x \quad x \quad \mathcal{M}^s, C, \quad \mathbb{Z}^2 \vee \quad D \quad w \in \mathbb{Z}^2 \quad \mathcal{M}^s, C,$$

$$D' \quad S^{[2]} \quad D' \quad \{\xi \in S^{[2]} | w y \quad z \quad \{y, z\} \quad g \quad \xi \}$$

$$D' \quad G \quad \{g^{-1} x | x \in \mathbb{Z}^2\} \sim \mathbb{Z}^2 \quad w \quad D'^2$$

$$D' \quad D' \quad S^{[2]} \quad D' \quad S^{[2]} \quad D' \quad D \quad D \quad \mathcal{M}^s, C, \rightarrow \mathbb{Z}^2 \vee \quad D' \quad D$$



S ON

$$\begin{array}{ccccccc}
 & & & & X & & D \\
 q_X D & & & & & & \\
 & D & & & & & \\
 & & X & & & & \\
 & & q_X D & & D & & \\
 & & X' & X & & & 
 \end{array}$$

**Conjecture 4.1.**  $A \xrightarrow{d} h \xrightarrow{m} ph \xrightarrow{ymp} m \xrightarrow{f} d \xrightarrow{X} h \xrightarrow{h} p \xrightarrow{fi} -$   
 $B \xrightarrow{-B} g \xrightarrow{m} q \xrightarrow{d} f \xrightarrow{m} z, q_X D \xrightarrow{f} d \xrightarrow{D} h \xrightarrow{q} h \xrightarrow{p} h$

**Remark 4.2.**

**Conjecture 4.2.**  $A \xrightarrow{y} d \xrightarrow{h} m \xrightarrow{ph} ymp \xrightarrow{m} f \xrightarrow{d} X' \xrightarrow{h} h$   
 $\xrightarrow{-} d \xrightarrow{D'} h \xrightarrow{q_X} D' \xrightarrow{fi} X. M$ ,  
 $fD \xrightarrow{h} d \xrightarrow{p} d \xrightarrow{g} D', h \xrightarrow{h} x \xrightarrow{p} d-p \xrightarrow{g} m \xrightarrow{ph} m$   
 $\tau \xrightarrow{f} h \xrightarrow{2} X, \mathbb{Z} \xrightarrow{h} h \xrightarrow{\tau} D \xrightarrow{f}$ .

**Remark 4.3.**  $D \xrightarrow{\tau} D'$

$$\begin{array}{ccccccc}
 D & q_X D & & & n & & \\
 & & & & & & \\
 & & \phi_{mD} & X \rightarrow & {}^0 X, \mathcal{O} mD & & \\
 & m \rightarrow \infty & \mathcal{O} mD & m > & & & \phi_{mD} \\
 & D & n & I \xrightarrow{k} d \xrightarrow{m} & D & & \\
 & & & \phi_{mD} & & & \phi_{mD}
 \end{array}$$

$$\begin{array}{ccccccc}
 D & & & & & & X \\
 & n & & & & & \\
 & & \phi_{mD} & & {}^0 X, \mathcal{O} mD & & \\
 & & n & & & & 
 \end{array}$$

**Proposition 4.3.**  $E \xrightarrow{y} d \xrightarrow{h} m \xrightarrow{ph} ymp \xrightarrow{m} f \xrightarrow{d} X' \xrightarrow{h} d$   
 $B \xrightarrow{m} b_2 X' \geq d \xrightarrow{f} m \xrightarrow{d} h, X, h \xrightarrow{h} X$   
 $\xrightarrow{f} d \xrightarrow{D} h \xrightarrow{q_X} D$ .

$P$   $f$ .  $^2 X', \mathbb{Z}, q_{X'} \sim \Gamma, q_\Gamma$   
 $, b_2 X' -$

$$D \quad q_X D \quad D \quad \begin{matrix} X \\ -D \end{matrix} \quad \dots$$

$$D \quad -D$$

□

**Remark 4.4.**

$$b_2 X'$$

**4.2. Vanishing theorems**

$$a_{2i} \quad \chi \mathcal{O}_D \quad \sum_{i=0}^2 a_{2i} q_X D^i$$

$$D \quad q_X D$$

$$\chi \mathcal{O}_{mD} \quad a_0 \quad \chi \mathcal{O}_X \quad n \quad \dots$$

$$^i mD \quad ^i X, \mathcal{O}_{mD}$$

$$\sum_{i=0}^2 - \quad ^i i mD \quad n \quad \dots$$

$$n \quad m \quad D \quad ^i D \quad i > \quad ^0 D$$

$$i \quad ^0 D \geq n \quad q_X D$$

$$D \quad X$$

$$D \quad ^i D \quad i \quad n \quad i > \quad n \geq$$

$$D$$

$$n$$

$X$

$$\sigma \in {}^0 X, {}^2 T^* \quad {}^{2,0} X .$$

$$\sigma \in {}^{0,2} X \quad {}^2 X, \mathcal{O}_X .$$

$\sigma$

$X$

$\mathcal{F}$

$$L \quad {}^\infty \mathcal{E}^{p,q} \otimes \mathcal{F} \rightarrow {}^\infty \mathcal{E}^{p,q+2} \otimes \mathcal{F}$$

$\sigma$

$$L^* \quad {}^\infty \mathcal{E}^{p,q+2} \otimes \mathcal{F} \rightarrow {}^\infty \mathcal{E}^{p,q} \otimes \mathcal{F} .$$

$\sigma \quad \partial$

**Definition 4.1.**

$$h \quad f \quad L \quad f \quad h \quad z$$

$\sigma$

$$L \quad {}^q X, \Omega^p \otimes \mathcal{F} \rightarrow {}^{q+2} X, \Omega^p \otimes \mathcal{F}$$

$\sigma$

$$L^* \quad {}^{q+2} X, \Omega^p \otimes \mathcal{F} \rightarrow {}^q X, \Omega^p \otimes \mathcal{F} .$$

**Remark 4.5.**

$p$

$\mathcal{F} \quad \mathcal{O} D$   
 $L^*$

$D$

${}^0 D$

$X$

${}^2 D$

$X$

${}^i D$

${}^i D$

${}^i > n$

$$\sum_i h^i D \quad n \quad \sum_i h^i D \geq n$$

${}^2 D$

${}^0 D$

${}^n \phi_D$

$\phi_D$

$D$

$n$

4.3. Some calculations

$$\begin{aligned} \pi : X &\rightarrow D \\ \mathcal{O}_X &\rightarrow \mathcal{O}_D \\ \mathcal{O}_X(mD) &\rightarrow \mathcal{O}_D(mD) \\ R^p \pi_* \mathcal{O}_X(mD) &\rightarrow R^p \pi_* \mathcal{O}_D(mD) \end{aligned}$$

$$R^p \pi_* \pi^* \mathcal{O}_{\mathbb{P}^n}(m) \cong R^p \pi_* \mathcal{O}_{\mathbb{P}^n}(m) \otimes R^q \pi_* \mathcal{O}_X.$$

$q$

$$R^0 \pi_* \mathcal{O}_X \cong \mathcal{O}_{\mathbb{P}^n}.$$

$q$

$$R^1 \pi_* \mathcal{O}_X \cong \begin{cases} \mathcal{O}_{\mathbb{P}^n} & x \in \mathbb{P}^n \\ \mathcal{O}_{\mathbb{P}^n}(-1) & x \in \mathbb{P}^n \end{cases}$$

$$R^1 \pi_* \mathcal{O}_X \cong \begin{cases} \mathcal{O}_{\mathbb{P}^n} & x \in \mathbb{P}^n \\ \mathcal{O}_{\mathbb{P}^n}(-1) & x \in \mathbb{P}^n \end{cases}$$

$$R^1 \pi_* \mathcal{O}_X \cong \begin{cases} \mathcal{O}_{\mathbb{P}^n} & x \in \mathbb{P}^n \\ \mathcal{O}_{\mathbb{P}^n}(-1) & x \in \mathbb{P}^n \end{cases}$$

$$R^1 \pi_* \mathcal{O}_X \cong \begin{cases} \mathcal{O}_{\mathbb{P}^n} & x \in \mathbb{P}^n \\ \mathcal{O}_{\mathbb{P}^n}(-1) & x \in \mathbb{P}^n \end{cases}$$

$\Omega_{\mathbb{P}^n}^1(x)$

$$R^1 \pi_* \mathcal{O}_X \cong \Omega_{\mathbb{P}^n}^1.$$

$$R^q \pi_* \mathcal{O}_X \cong \Omega_{\mathbb{P}^n}^q.$$

$q \geq$

Example 4.6.

$$\begin{aligned} \pi : X &\rightarrow \mathbb{P}^n \\ \mathcal{O}_X &\rightarrow \mathcal{O}_{\mathbb{P}^n} \\ \mathcal{O}_X(mD) &\rightarrow \mathcal{O}_{\mathbb{P}^n}(m) \\ R^i \pi_* \mathcal{O}_X(mD) &\cong \begin{cases} \mathcal{O}_{\mathbb{P}^n}(m) & i=0 \\ \mathcal{O}_{\mathbb{P}^n}(m-1) & i=1 \\ \mathcal{O}_{\mathbb{P}^n}(m-i) & i \geq 2 \end{cases} \end{aligned}$$

Example 4.7.

$$\begin{aligned} R^2 \pi_* \mathcal{O}_X &\cong R^2 \pi_* \mathcal{K}_X \otimes \pi^* \mathcal{K}_{\mathbb{P}^2}^\vee \otimes \pi^* \mathcal{O}_{\mathbb{P}^2}(-2) \\ &\cong \mathcal{O}_{\mathbb{P}^2}(-2) \otimes R^2 \pi_* \mathcal{K}_X / \mathbb{P}^2 \\ &\cong \mathcal{O}_{\mathbb{P}^2}(-2) \otimes R^0 \pi_* \mathcal{O}_X^\vee \\ &\cong \mathcal{O}_{\mathbb{P}^2}(-2) \end{aligned}$$

$$p, q, \mathcal{K}_X, \dots$$

$${}^i mD \begin{cases} m & m & / & i \\ m & m- & / & i \\ m- & m- & / & i \end{cases}$$

**Example 4.8.**  $n$

$${}^i mD \begin{cases} m & m & m & / & i \\ m & m & m- & / & i \\ m & m- & m- & / & i \\ m- & m- & m- & / & i \end{cases}$$

**Remark 4.9.**

$$m \quad {}^0 D \quad \chi \mathcal{O}_{n mD} \quad {}^n i D \quad i >$$

**Remark 4.10.**  $n$

$$m \quad {}^2 mD \quad {}^0 mD$$

$$L^* \quad {}^2 X, \mathcal{O}_{n mD} \rightarrow {}^0 X, \mathcal{O}_{mD}$$

$$n \quad {}^2 mD < {}^0 mD \quad m >$$

$$L^*$$

**Conjecture 4.4.** If  $E$

$$f \quad d \quad d \quad h \quad m \quad ph \quad ymp \quad -f \quad d$$

$X, h \quad h \quad m \quad p$

$$L^* \quad {}^2 X, E \rightarrow {}^0 X, E$$

$$h \quad h \quad f \quad L \quad f \quad h \quad z \quad j \quad .$$

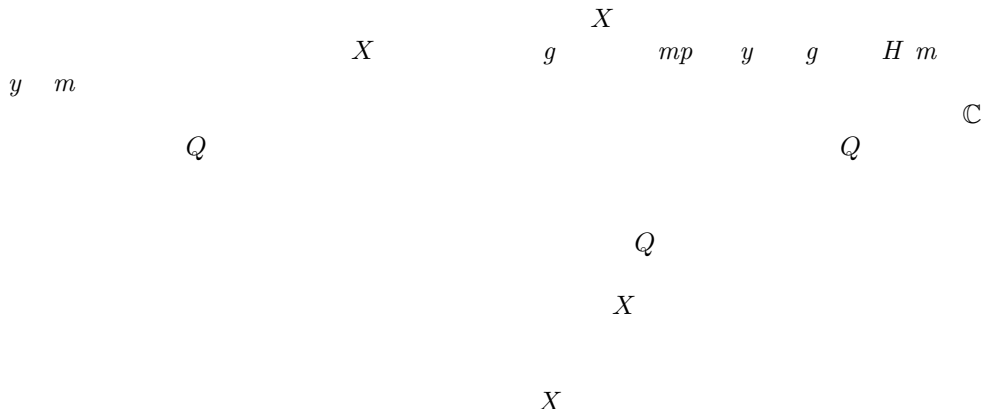
$$L^*$$

### 5. Classification of abelian fibrations which admit sections

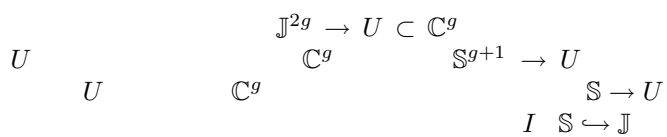
$$X^2$$

$$X$$

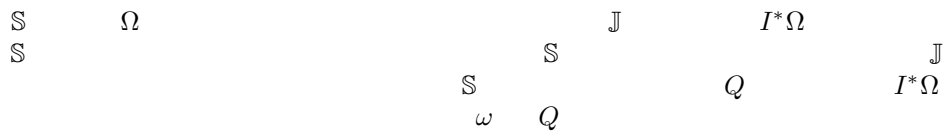
5.1. Integrable systems



fl



$$I^*\Omega \wedge I^*\Omega$$



Theorem 5.1. Th

$m p$

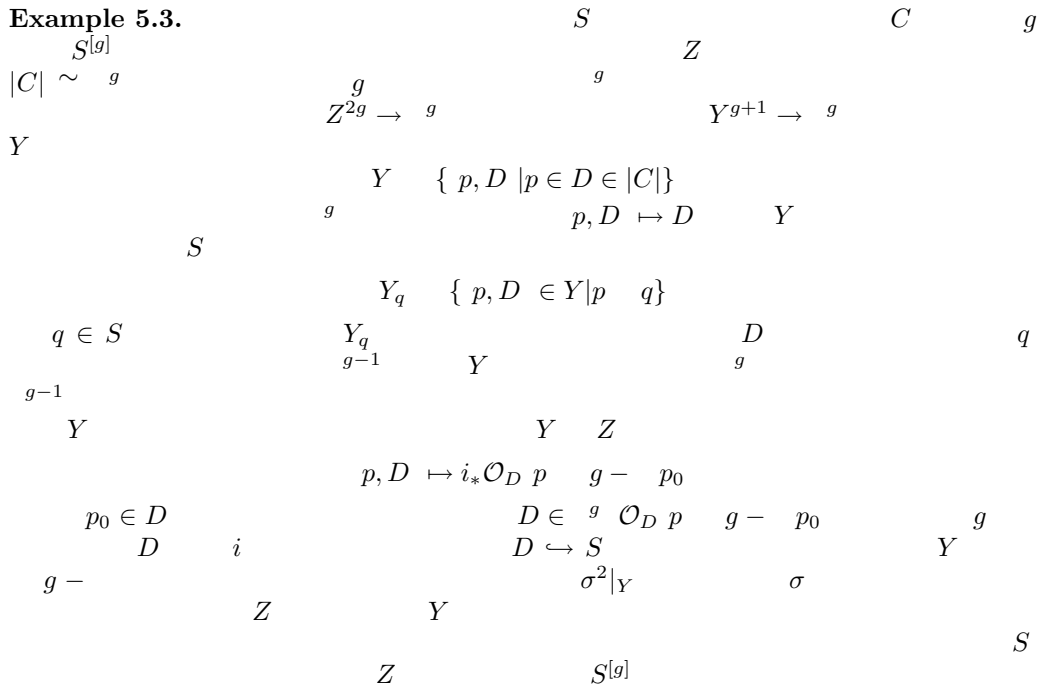
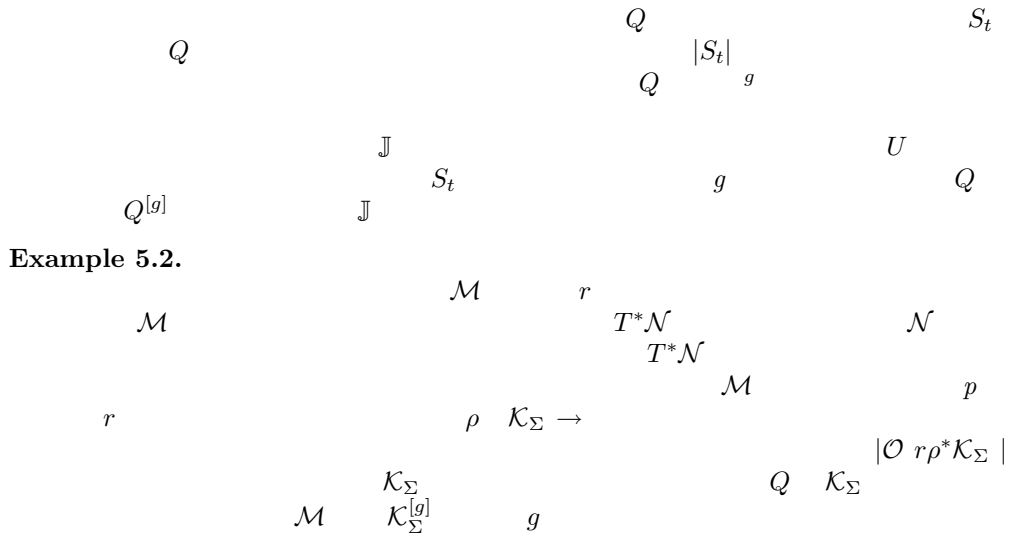
$$Q^{[g]} \rightarrow J$$

$Q^{[g]}$   $d$   $d$   $y$   $\omega$   $\Omega$   $J$   $M$   $, f$   $k$   $h$   $h$   $m$   $ph$   $ymp$   $f$   $m$   
 $H$   $h$   $m$   $S_t^{[g]}$   $^g S_t$   $h$   $J_t$   $f$   $S_t$   $y$   $h$   $A$   $m$   $p$   $(h$   $h$   
 $L$   $g$   $g$   $m$   $f$   $d$ ).

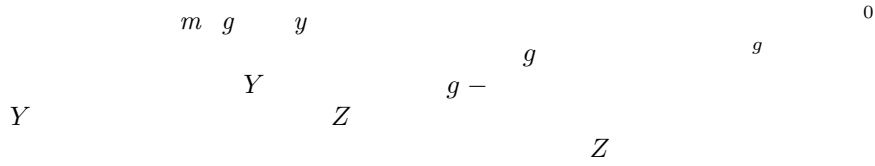
Remark 5.1.

J

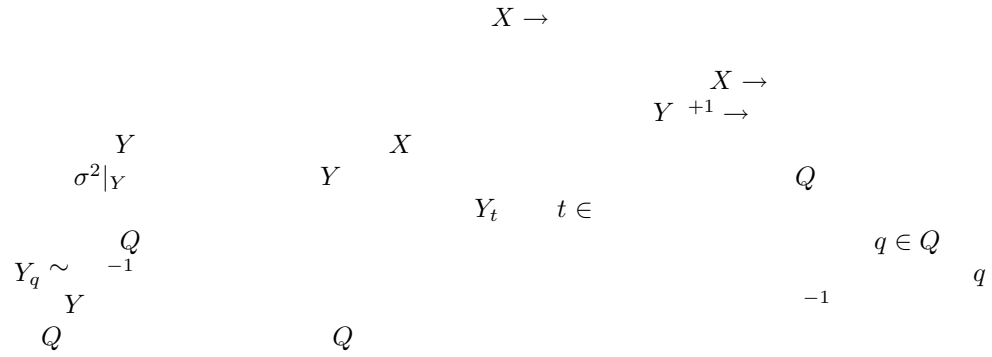
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**Remark 5.4.**  $\mathbb{J} \rightarrow U$

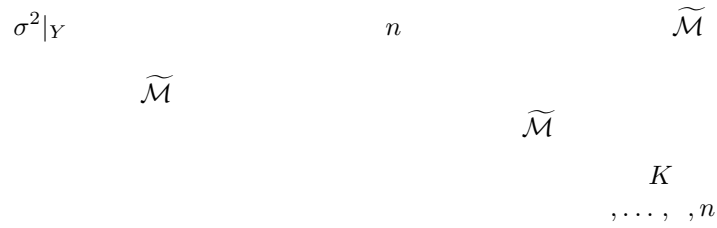


**5.2. Fibrations and foliations**



**Conjecture 5.2.**  $pp$   $h$   $d$   $h$   $m$   $ph$   $ymp$   $m$   $f$   $d$   $X^2$   
 $fi$  ,  $hfi$   $m$   $ph$   $f$  . If  $h$   $fi$   $dm$   $h$   
 $pp$   $p$   $m$  (  $f$   $R$   $m$   $k$   $5.4$  )  $h$   $X$   $h$   $H$   $h$   $m$   $S^{[ ]}$   
 $fn$   $p$   $K3$   $f$   $S$  .

**Remark 5.5.**  $n$



**Proposition 5.3.**  $pp$   $h$   $p$   $j$   $fi$   $X \rightarrow$   $d$   $f$   $m$   $f$   
 $h$   $g$   $z$   $d$   $K$   $mm$   $y$   $K$  . Th  $d$   $d$   $p$   $z$   $f$   $h$   $fi$   $f$   $X$   
 $p$   $p$  .



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$P f. E \quad X \subset N \quad D$   
 $E tD$

$$\int_X E tD^2 \quad c_X q_X E tD$$

$$c_X q_X E \quad tq_X E, D$$

$$\int_X E D \quad \frac{n^2}{n} c_X q_X E, D .$$

$$D \quad c_X \quad F \quad \int_F E|_F$$

$$c_{K_n} \quad \frac{n}{n} \quad K$$

$$q_X E, D \in \mathbb{Z} \quad \int_F E|_F \quad n \quad q_X E, D$$

$$E|_F \quad F$$

□

**Remark 5.6.**

$$S^{[ ]} \quad q_X E, D$$

$$c_{S^{[n]}} \quad \frac{n}{n} \quad q_X E, D$$

**Conjecture 5.4.**  $pp \quad h \quad d \quad h \quad m \quad ph \quad ymp \quad m \quad f \quad d \quad X^2$   
 $f_i, \quad d \quad h \quad f_i \quad h \quad p \quad z, \dots, , n .$  If  $h \quad f_i \quad dm$   
 $h \quad pp \quad p \quad m \quad h \quad X \quad g \quad z \quad d \quad K \quad mm \quad y \quad K .$

T

$$S^{[ ]} \quad K \quad , \dots, \quad , \dots, , n$$

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X

### 5.3. Four-folds fibred by Jacobians

**Theorem 5.5.**  $\pi: P \rightarrow S$  is a fibration with fibers  $P_t \cong \mathbb{P}^3$ . Let  $f: S^{[2]} \rightarrow S$  be the map sending a pair of points to their sum. Then  $\pi^* f^* \mathcal{O}_S(1) \cong \mathcal{O}_P(1)$ .

Let  $\mathcal{O}_P(1)$  be the tautological line bundle on  $P$ . We have a short exact sequence of sheaves on  $S^{[2]}$ :

$$0 \rightarrow \mathcal{O}_{S^{[2]}}(-1) \rightarrow \mathcal{O}_{S^{[2]}} \rightarrow \mathcal{O}_{S^{[2]}}(1) \rightarrow 0$$

where  $\mathcal{O}_{S^{[2]}}(-1)$  is the sheaf of sections of  $\pi^* \mathcal{O}_S(-1)$  and  $\mathcal{O}_{S^{[2]}}(1)$  is the sheaf of sections of  $\pi^* \mathcal{O}_S(1)$ . The map  $\mathcal{O}_{S^{[2]}}(-1) \rightarrow \mathcal{O}_{S^{[2]}}$  is induced by the inclusion  $\pi^* \mathcal{O}_S(-1) \hookrightarrow \mathcal{O}_P(-1)$ . The map  $\mathcal{O}_{S^{[2]}} \rightarrow \mathcal{O}_{S^{[2]}}(1)$  is induced by the map  $\mathcal{O}_P \rightarrow \mathcal{O}_P(1)$ .

Let  $\mathcal{O}_P(1)$  be the tautological line bundle on  $P$ . We have a short exact sequence of sheaves on  $S^{[2]}$ :

$$0 \rightarrow \mathcal{O}_{S^{[2]}}(-1) \rightarrow \mathcal{O}_{S^{[2]}} \rightarrow \mathcal{O}_{S^{[2]}}(1) \rightarrow 0$$

where  $\mathcal{O}_{S^{[2]}}(-1)$  is the sheaf of sections of  $\pi^* \mathcal{O}_S(-1)$  and  $\mathcal{O}_{S^{[2]}}(1)$  is the sheaf of sections of  $\pi^* \mathcal{O}_S(1)$ . The map  $\mathcal{O}_{S^{[2]}}(-1) \rightarrow \mathcal{O}_{S^{[2]}}$  is induced by the inclusion  $\pi^* \mathcal{O}_S(-1) \hookrightarrow \mathcal{O}_P(-1)$ . The map  $\mathcal{O}_{S^{[2]}} \rightarrow \mathcal{O}_{S^{[2]}}(1)$  is induced by the map  $\mathcal{O}_P \rightarrow \mathcal{O}_P(1)$ .

Let  $\mathcal{O}_P(1)$  be the tautological line bundle on  $P$ . We have a short exact sequence of sheaves on  $S^{[2]}$ :

$$0 \rightarrow \mathcal{O}_{S^{[2]}}(-1) \rightarrow \mathcal{O}_{S^{[2]}} \rightarrow \mathcal{O}_{S^{[2]}}(1) \rightarrow 0$$

where  $\mathcal{O}_{S^{[2]}}(-1)$  is the sheaf of sections of  $\pi^* \mathcal{O}_S(-1)$  and  $\mathcal{O}_{S^{[2]}}(1)$  is the sheaf of sections of  $\pi^* \mathcal{O}_S(1)$ . The map  $\mathcal{O}_{S^{[2]}}(-1) \rightarrow \mathcal{O}_{S^{[2]}}$  is induced by the inclusion  $\pi^* \mathcal{O}_S(-1) \hookrightarrow \mathcal{O}_P(-1)$ . The map  $\mathcal{O}_{S^{[2]}} \rightarrow \mathcal{O}_{S^{[2]}}(1)$  is induced by the map  $\mathcal{O}_P \rightarrow \mathcal{O}_P(1)$ .

**Remark 5.7.**

$Y_t$

6. Abelian fibrations which don't admit sections

6.1. Gerbes and elliptic fibrations

$$\begin{array}{ccc}
 X & & X \\
 d & &
 \end{array}$$

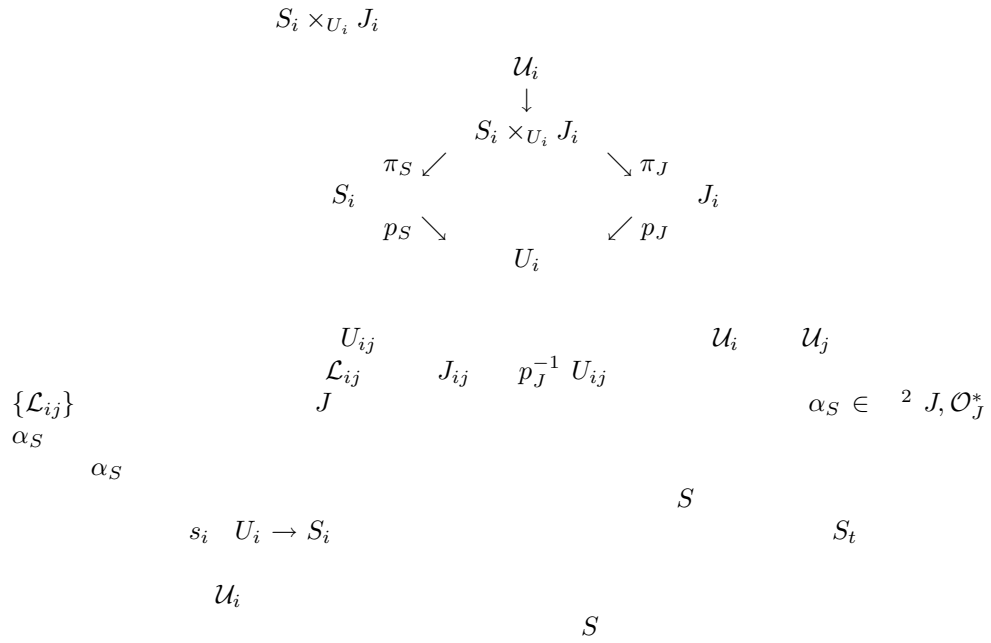
$$\begin{array}{ccc}
 J & & J, \mathcal{O}_J^* \\
 g & J & h \quad m \quad ph
 \end{array}$$

**Definition 6.1.**  $\{U_i\}$   $J$   $\{\mathcal{L}_{ij}\}$

$$\begin{array}{ccc}
 \mathcal{L}_{ji} \sim \mathcal{L}_{ij}^{-1} & i & j \\
 \mathcal{L}_{ijk} & \mathcal{L}_{ij} \otimes \mathcal{L}_{jk} \otimes \mathcal{L}_{ki} & U_{ijk} \\
 \beta_{ijk} & U_{ijk} \rightarrow \mathcal{O}^* & i \quad j \\
 \delta\beta & & \\
 \mathcal{L}_{ijkl} & \mathcal{L}_{jkl} \otimes \mathcal{L}_{ikl}^{-1} \otimes \mathcal{L}_{ijl} \otimes \mathcal{L}_{ijk}^{-1} & U_{ijkl} \\
 \{\mathcal{L}_{ij}\} & h \quad m \quad ph \quad g & J \\
 & \beta \in {}^2 J, \mathcal{O}_J^* &
 \end{array}$$

$$\begin{array}{ccc}
 S & & J \\
 & S & \\
 & S_t \times J_t & J_t \\
 & \{U_i\} & \\
 J_i & p_J^{-1} U_i & {}^1 S_i \quad p_S^{-1} U_i \quad \mathcal{U}_i
 \end{array}$$

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**Proposition 6.1.** *Th*  $f \quad g \quad q \quad :$   
 $S \quad m \quad ph \quad J,$   
 $S \quad dm \quad ,$   
 $h \quad x \quad d \quad S \times_{\mathbb{P}^1} J.$   
*Th*  $f \quad h \quad x \quad p \quad m \quad \alpha_S \in \quad {}^2 J, \mathcal{O}_J^* .$

**Remark 6.1.**

$$\begin{array}{ccc}
 S \rightarrow B & & J \rightarrow B \quad S \\
 \\
 {}^2 J, \mathcal{O}_J^* / {}^2 B, \mathcal{O}_B^* . & & \\
 \\
 {}^2 B, \mathcal{O}_B^* & & J \quad B \\
 B & & \\
 {}^2 B, \mathcal{O}_B^* \hookrightarrow {}^2 J, \mathcal{O}_J^* . & & 
 \end{array}$$

## 6.2. Gerbes and abelian fibrations

**Definition 6.2.**  $X \rightarrow P \rightarrow X$   $P \rightarrow d \rightarrow X$

**Remark 6.2.**  $X_t \hookrightarrow X$   $i_* L$   $i_* L \rightarrow P$

**Remark 6.3.**  $P_t \rightarrow X$   $P_t \rightarrow X_t$   $P_t \rightarrow X_t$   $P$

$P \rightarrow X_t \times P_t$   $X_t \rightarrow P_t$   $P_t$

$\beta_X \in {}^2 P, \mathcal{O}_P^*$   ${}^2, \mathcal{O}_{\mathbb{P}^n}^*$   $P \rightarrow X_0$   $X$   $\mathcal{O}_X$   $\mathcal{O}_{X_0}$   $X \rightarrow X_0$   $\beta_{X_0}$   $X \rightarrow X$   $X_0 \rightarrow X_0$   $\mathcal{B}$   $X$

**Conjecture 6.2.** *Th  $f$   $g$   $q$  :*  
 $X \xrightarrow{m} ph \rightarrow X_0,$   
 $X \xrightarrow{dm} ,$

S O N

$$Th \quad \begin{matrix} h & x & & & & d & & & X \times \mathbb{P}^n & P. \\ & & f & h & x & & p & m & \beta_X \in {}^2 P, \mathcal{O}_P^* . \end{matrix}$$

**Remark 6.4.**  $P$

$$\begin{matrix} \rightarrow & {}^1 P, \mathcal{O}_P^* & \rightarrow & {}^2 P, \mathbb{Z} & \rightarrow & {}^2 P, \mathcal{O}_P & \rightarrow & {}^2 P, \mathcal{O}_P^* & \rightarrow & {}^3 P, \mathbb{Z} & \rightarrow \\ & & & {}^2 P, \mathcal{O}_P \sim \mathbb{C} & & & & {}^2 P, \mathcal{O}_P^* & & {}^3 P, \mathbb{Z} & \\ & & & & & & & & & & \\ & & & & & & & & & & {}^2 P, \mathcal{O}_P^* \end{matrix}$$

**Remark 6.5.**

$$\begin{matrix} J & & & & & & & S \\ S & J & & & & & & \end{matrix}$$

**Remark 6.6.**

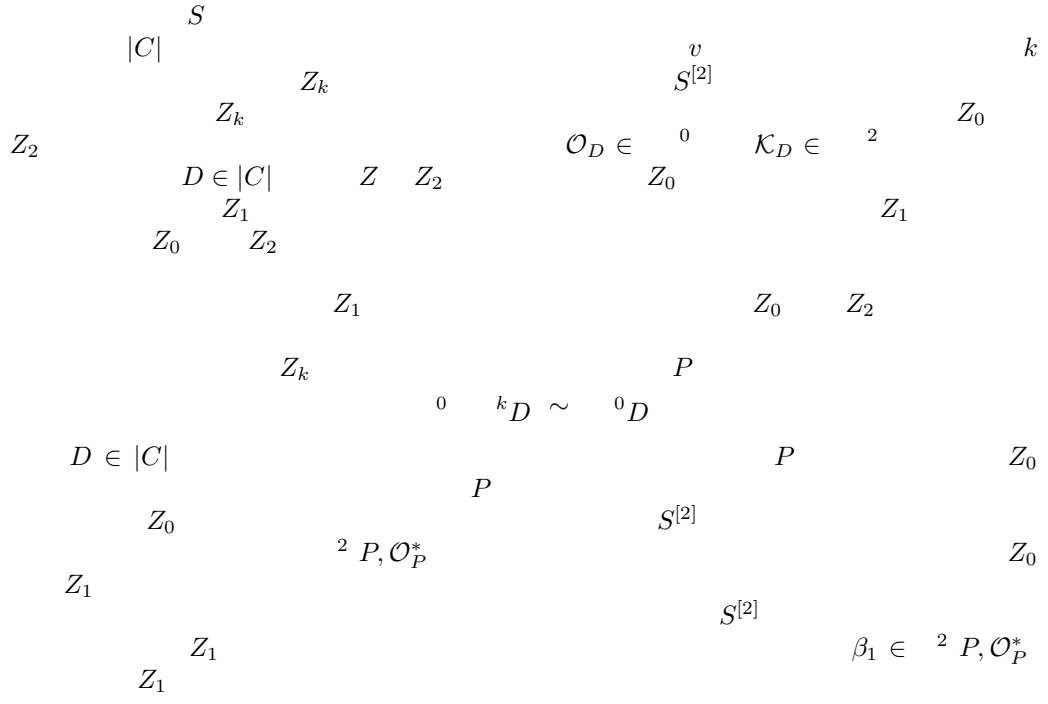
$$\begin{matrix} X_0 & & P & & & & & & X & & P \\ & & & & B & & \beta_X & & & & \\ \dots & & & & & & & & & & \\ f & m & & & & & & & d F & & -M k \end{matrix}$$

**6.3. Two examples**

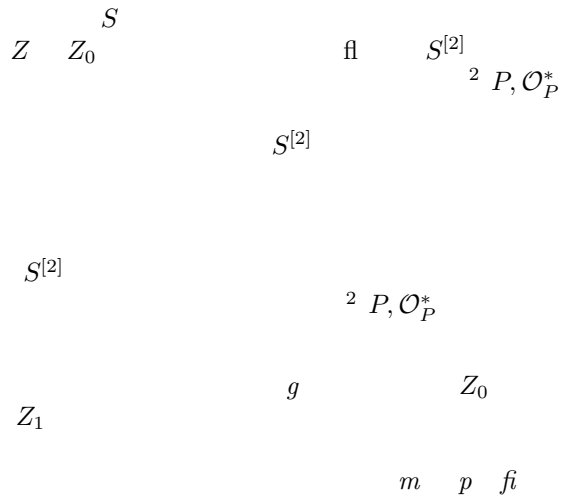
**Example 6.7.**

$$\begin{matrix} & & & & g & & C & & & & \\ & & & & S^{[g]} & & Z & & & & \\ |C| \sim & g & & & {}^g D & Z & D \in |C| & Z & & & \\ Z & & & & & & P & & & & \\ & & g & & S & & & & & & \\ D & & k & & & & & & & & \\ & & & & & & 2 \vee & & & & \\ S & & & & k & & Z_k & & & & |C| \sim {}^2 \\ Z_k & & v & & , C, k- & & & & & & v & \\ & & & & & & k & & & & \end{matrix}$$

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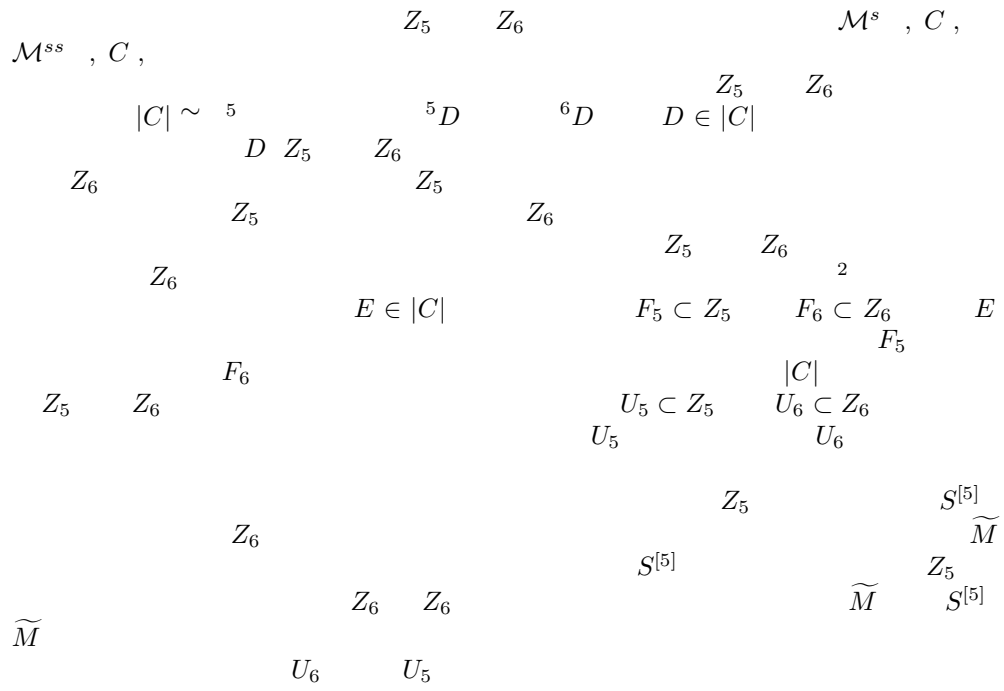
**Remark 6.8.**



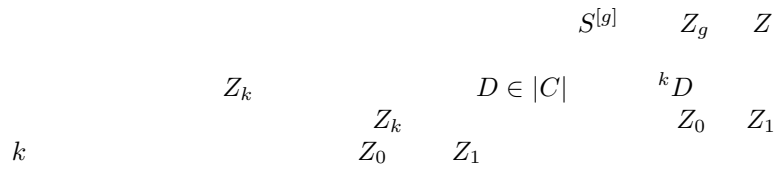
**Example 6.9.**



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**Remark 6.10.**



$Z_5$ 
 $Z_6$

**Acknowledgements:**

.. ..



## References

- [ ] **18** 8 55 8 *Variétés Kählériennes dont le 1<sup>ère</sup> classe de Chern est nulle* D ff
- [ ] **6** *Complex Algebraic Surfaces* S E P
- [ ] **8** *Counting rational curves on K3 surfaces* D **97**
- [4] **84** P V V *Compact complex surfaces* S V
- [5] H L *Complex abelian varieties* S V
- [6] S *Higher dimensional Zariski decompositions* **m h. G/0204336**
- [ ] *Derived categories of twisted sheaves on Calabi-Yau manifolds* P D  
**www.m h.um .edu/~ nd ei /**
- [8] S *Rational quadratic forms* P L 8
- [ ] P D *A Kawamata-Viehweg vanishing theorem on compact Kähler manifolds* **m h. G/0208021**
- [ ] E S *The Borel-Weil theorem for complex projective space* O P 6 45
- [ ] H E E V *Lectures on vanishing theorems* D V S
- [ ] F *On primitively symplectic compact Kähler V-manifolds of dimension four* P **39** 8 5
- [ ] R F *Smooth four-manifolds and complex surfaces* S V
- [4] L **4** D H *Hodge numbers of moduli spaces of stable bundles on K3 surfaces* **7** 6 5
- [5] S D H D *Calabi-Yau manifolds and related geometries*
- [6] N H *Stable bundles and integrable systems* D **54** 8 4
- [ ] N H S *Lectures on special Lagrangian submanifolds* P H  
S P
- [8] H **5** *Integrable systems and algebraic surfaces* D **83** 6 N
- [ ] H P E *Rank 2-integrable systems of Prym varieties* 8 6 6 5
- [ ] D H L *The geometry of moduli spaces of sheaves* **E31** V

[ ] D H		<i>Birational symplectic manifolds and their deformations</i>	D ff
	<b>45</b>	488 5	
[ ] D H		<i>Compact hyperkähler manifolds: basic results</i>	<b>135</b>
	<b>6</b>		
[ ] D H		<i>The Kähler cone of a compact hyperkähler manifold</i>	
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[ 4] D H		<i>Moduli spaces of hyperkähler manifolds and mirror symmetry</i>	L
	S	<b>m h. G/0210219</b>	
[ 5] V		R S <i>Algebraic surfaces</i> Algebraic geometry II E	
		S <b>35</b> S 6	
[ 6]		<i>On compact analytic surfaces II, III</i>	<b>77</b> 6 56 6 6
	<b>78</b> 6	4	
[ ]		<i>On the structure of compact complex analytic surfaces, I</i>	<b>86</b>
	64 5	8	
[ 8] E L		P <i>Torelli theorems for kähler K3 surfaces</i>	<b>42</b> 8
	45 86		
[ ] E		<i>Brill-Noether duality for moduli spaces of sheaves on K3 surfaces</i>	
	<b>10</b>	4 6 6 4	
[ ] D		<i>Completely integrable projective symplectic 4-dimensional varieties</i>	
	z :	<b>59</b> 5 5 8	
[ ] D		<i>Lagrangian families of Jacobians of genus 2 curves</i>	S <b>82</b>
	6	68 84	
[ ] D		<i>On fibre space structures of a projective irreducible symplectic manifold</i>	
	<b>38</b>	N 8 <b>40</b> N 4 4	
[ ] D		<i>Higher direct images of Lagrangian fibrations</i>	<b>m h. G/0010283</b>
[ 4] S		<i>Symplectic structure of the moduli space of simple sheaves on an abelian or K3 surface</i>	<b>77</b> 84 6
[ 5] S		<i>On the moduli space of bundles on K3 surfaces I</i> V	
	V	F O P 8 4 4	
[ 6] Y N		<i>Counter-example to global Torelli problem for irreducible symplectic manifolds</i>	
		<b>m h. G/0110114</b>	
[ ] O'		<i>The weight-two Hodge structure of moduli spaces of sheaves on a K3 surface</i>	
		<b>6</b> 4 5 644	
[ 8] O'		<i>Desingularized moduli spaces of sheaves on a K3</i> R	<b>512</b>
	4		
[ ] O'		<i>A new six dimensional irreducible symplectic variety</i>	
	<b>m h. G/0010187</b>		
[ 4] O'		<i>Periods of irreducible symplectic varieties I</i> H ..	
		<b>www.m .uni om 1.i /people/og dy/</b>	
[ 4] P	ř Š	R Š ~ <i>A Torelli theorem for algebraic surfaces of type K3</i>	
	SSR z	<b>5</b> N 54 588	
[ 4] S		<i>Twisted Fourier-Mukai transforms for abelian fibred holomorphic symplectic manifolds</i>	
[ 4] S	S Y E Z	<i>Mirror symmetry is T-duality</i> N P	<b>B 479</b>
	6 4 5		

- [44] V *Action of the Lie algebra of  $SO(5)$  on the cohomology of a hyper-Kähler manifold* E F **24**
- [45] V *Mirror symmetry for hyperkähler manifolds* S/ P  
S **10** P R 5 56
- [46] V *Cohomology of compact hyperkähler manifolds and its applications* F **6**  
6 6 6
- [4 ] Y *Irreducibility of moduli spaces of vector bundles on K3 surfaces*  
**m h. G/9907001**
- [48] Y *Moduli spaces of stable sheaves on abelian surfaces* **321**  
4 8 884

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