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Comparing open book and Heegaard decompositions of 3-manifolds

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Comparing open book and Heegaard decompositions of 3-manifolds

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Abstract

We study the maximal value of the Euler characteristic of the pages of all open book decompositions of closed orientable 3-manifolds. In particular, we describe some examples where the minimal genus Heegaard splittings of such 3-manifolds give rise to open book decompositions and other examples where the simplest open book decomposition has larger maximal Euler characteristic of pages than the smallest genus Heegaard splittings. Also, special properties of the Heegaard splitting associated to an open book decomposition are given. Techniques of minimal surface theory and hyperbolic geometry are shown to be useful for such problems.

1. Introduction

It is well-known that open book decompositions of 3-manifolds give a natural way of constructing contact structures. This was first observed by Thurston and Winkelnkemper [18]. This correspondence and its converse (going from contact structures to open book decompositions) has been studied recently by Giroux [4]. Some useful basic references on the very active area of contact structures are [2], [3]. On the other hand, open book decompositions were constructed by Alexander [1] but have not attracted much attention in recent years by 3-manifold topologists. (See [10] for a neat way of constructing open book decompositions with connected binding). However, many interesting results have been proven about Heegaard splittings (cf [16]). Our aim in this paper is to sketch some comparisons between open book decompositions and Heegaard splittings, noting that open book decompositions can be viewed as special types of Heegaard splittings. This paper is closely related to [14] (see also [6]) and a more detailed version of applications of minimal surfaces in the study of 3-manifolds is in preparation by the author.

In particular, it would be good to understand how the complexity (maximal Euler characteristic) of open book decompositions of closed orientable 3-manifolds compare with other natural measures of complexity, such as minimal spines, (cf [7], [8]) smallest triangulations ([5]) and minimal Heegaard splittings ([16]).

Our first result is that in the Heegaard genus 2 case, open book decompositions are nearly always more complicated than minimal Heegaard splittings. A natural measure of the complexity of an open book decomposition is to take the Heegaard genus of the

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R NS N

$|\chi|$ χ

genus

M

n

n

Definition

Heegaard splitting M M

Definition

open book decomposition M *binding*

$M \setminus$

pages

..

2. Genus 2 Heegaard splittings

Theorem 2.1. *Most 3-manifolds M of Heegaard genus 2 have open book genus > .*

Proof. g
M Γ S^3 g

Γ S^3 M/g S^3

M
g' Γ' S^3

M M $S^1 \times S^2$

M M
$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
M g
M g

M
Γ^* θ Γ^*
θ

Γ^*
Γ Γ
σ S^3
C Γ $C \cap \Gamma$

Γ

□

3. Open book decompositions and minimal Heegaard splittings

M n
M S^3
M $M_{(p_1,q_1),\ldots,(p_n,q_n)}$
p_1, q_1 , ..., p_n, q_n

S^3

Theorem 3.1. *Open book decompositions of the surgered manifolds $M_{(p_1^k,q_1^k),\ldots,(p_n^k,q_n^k)}$ either have unbounded genus or have associated Heegaard splittings constructed by adding trivial handles to one of a finite number of splittings of the cusped manifold M, as $lim_k|p_i^k| \quad |q_i^k| \to \infty$, for all i, where any sequence of vectors V_k of Dehn surgery coefficients $V_k \quad p_1^k, q_1^k, \ldots, p_n^k, q_n^k$ is chosen, which avoids finitely many lower dimensional lattices in the lattice $\mathbb{Z}^{2n}$.*

Proof.

$H_1\ M, Z_2$

$H_1\ M, Z_2\ \ /$

M

M

M

R NS N

M

M

□

Remark

M

Theorem 3.2. *Consider the sequence of open book decompositions given by the n-fold cyclic covering M_n of $\quad, n$ -surgery on a simple knot or link which is the binding of an open book decomposition of an irreducible atoroidal manifold M. For n sufficiently large, the minimal Heegaard genus of M_n is the open book genus. In fact, the only small genus irreducible Heegaard splitting of such a manifold is a regular neighbourhood of the page of the open book decomposition.*

Proof.

π_1

M

$, n$

$, n \qquad\qquad n$

$n \to \infty$

R NS N

M

n

n M_n

M **R**

M_n

M_n

M_n
n

□

4. Conclusion

Remarks

S^3

Theorem 4.1. *Suppose that M is a closed orientable 3-manifold and S is a Heegaard surface for some Heegaard splitting of M. If S is associated with an open book decomposition of M, then S satisfies the following two conditions;*

- there is an isotopy of M taking S back to itself which interchanges the two handlebodies on either side of S.

- let G be the subgroup of $\pi_0\ Homeo\ S$ consisting of isotopy classes of homeomorphisms of S extending to homeomorphisms of M. Then the mapping from G into $\pi_0\ Homeo\ M$ has a kernel K of infinite order, unless M is a Seifert fibered space or a connected sum of lens spaces and copies of $S^1 \times S^2$.

Proof.

S

S

K

M M

$S^1 \times S^2$

□

Question

S

S S

S M

S M

S G

$\pi_0\ Homeo\ M$

M

T M $\pi_1\ T\ \rightarrow \pi_1\ M$

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