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Abstract

I'm trying to understand which Riemannian manifolds can be Lipschitz approximated by polyhedral spaces of the same dimension with curvature bounded below. The necessary conditions I found consist of some special inequality for curvature at each point (the geometric curvature bound). This inequality is also sufficient condition for local approximation. I conjecture that it is also a sufficient condition for global approximation, and I can prove it if the curvature bound is positive. In general I can prove it only with the additional assumption that tangent bundle of the manifold is stably trivial.

0. Introduction

Let P_n be a sequence of m-dimensional polyhedral k-spaces (i.e. spaces with piecewise constant curvature $= k$) with curvature bounded below (see 1.A for precise definition), such that it converges to a Riemannian manifold (M, g) of the same dimension. One has the right to ask the following question:

(i) What one can say about (M, g)?

or even simpler one:

(ii) What one can say about the curvature tensor of (M, g)?

Obviously P_n are Alexandrov spaces with curvature $\geq k$, therefore one immediately gets that (M, g) must have sectional curvature $\geq k$. In fact, it is possible to say much more about curvature of M. The first indication of this phenomenon one can find in Cheeger's generalization of Bochner formulas to metric spaces with cone-like singularities [Ch], which suggests, in particular, that polyhedral spaces must have (in some sense) positive curvature operator.

In fact, the curvature condition on M is even stronger the positive curvature operator. I would like to call it "geometric curvature bound" ($G_p \geq 0$). I say that the curvature operator $R_p \in S^2(\Lambda^2(T_p))$ at point $p \in M$ is geometrically non-negative if it can be

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$$R_p \quad \sum_i \; x_i \wedge y_i \;{}^2,$$

x_i, y_i T_p T_p T_p M p M k $\mathbb{R}$
G_p k

$$R_p - kI \quad \sum_i \; x_i \wedge y_i \;{}^2$$

x_i y_i T_p I
G M k G_p k p M
$x \wedge y$ ^{2} $S^2 \times \mathbb{R}^{m-2}$

$S^2 \times \mathbb{R}^{m-2}$ G

Local Theorem 0.1. *Let P_n be a sequence of m-dimensional polyhedral spaces with curvature k, which Lipschitz converge to a Riemannian manifold M, g of the same dimension, then G M k.*

Moreover if M is a Riemannian manifold with G M k then each point has a neighborhood which is a Lipschitz limit of a sequence of polyhedral spaces with curvature $k - \epsilon$ for arbitrary small $\epsilon >$.

Global Theorem 0.2. *If M, g is Riemannian m-manifold with G M k. Assume that M (or its finite cover) has stably trivial tangent bundle then M can be realized as a Lipschitz limit of a sequence of m-dimensional polyhedral metrics with curvature $k - \epsilon$ for arbitrary $\epsilon >$.*

CP^2
G $-\epsilon$
k $\{-\ ,\ ,\ \}$
M, g
k k G
M
$\widetilde{M}$
$\widetilde{M}$

Corollary 0.3. *An m-manifold M, g can be Lipschitz approximated by m-dimensional polyhedral metrics with curvature if and only if G M .*

PE N N

$C \times \mathbb{R}^{m-2}$ C
C
S^2
$S^2 \times \mathbb{R}^{m-2}$ $x \wedge y$ ^{2}

G

M^m, g $\mathbb{R}^q$
M
$q - m$
$\pi/$
M G M $>$

M T M
M

N M
M T M

1. Notation, Definitions and Preliminaries

1.A. Polyhedral spaces with curvature bounded below or above.

S^0 $\{- \ , \ \}$
k m
k
k
CAT
k $\leq k$

k $\leq k$
$\leq$ π
k
k $\leq k$
k
k $\mathbb{R}^q$ q
k

1.B. Convex submanifolds of higher codimension.

Definition 1.1. $M \subset \mathbb{R}^q$ *locally convex* M
U
F_i U $\cap_i F_i$ U
F_i $> \pi/$

M C^2
x M $\{e_i\} \subset N_x$ M N M
$M \subset \mathbb{R}^q$ s
S^2 T_x M $\to N_x$ M s $\sum_i e_i s_i$ s_i S^2 T

$\mathbb{R}^q$

$\mathbb{R}^q$

1.C. Nice curvature tensor for submanifolds.

PDR

T T^n
n S^n T ^{n} T
T^n T T^n

A^4 T
^{4} T $^\perp \cap S^2$ ^{2} T
$A^4 \subset S^2$ ^{2} T
S^2 ^{2} T

$$R\ X, Y, Z, W \quad R\ Y, Z, X, W \quad R\ Z, X, Y, W \quad ,$$

T

$M \subset \mathbb{R}^q$ s_x S^2 T_x M $\to N_x$ M
x M T M N M
M *-curvature tensor*

$$X, Y, Z, W \quad \langle s\ X, Y\ , s\ Z, W\ \rangle,$$

S^2 S^2 T M

$$X, Y, Z, W \quad E\ X, Y, Z, W \quad -\Big(R\ X, Z, Y, W \quad R\ X, W, Y, Z\ \Big)$$

E

$$E\ X, Y, Z, W \quad -\Big(\ X, Y, Z, W \quad Y, Z, X, W \quad Z, X, Y, W\Big) \quad S^4\ T$$

$$R\ X, Y, Z, W \quad X, Z, Y, W \ - \ X, W, Y, Z \quad A^4\ T$$

M
E *extrinsic curvature* E S^4 T $\subset S^2$ S^2 T
E
f X E X, X, X, X $|s\ X, X\ |^2,$ f
E
E
T

N s_1, s_2 S^2 T $\to N$
S^2 S^2 T j $N \to N$
$j \circ s_1$ s_2 E
M, g M, g $\to \mathbb{R}^q$
E g

1.D. Positiveness of elements of $S^2 S^2 T$ and convexity of submanifold.

PDR

C

$\mathbb{R}^n$

$$C^* \quad \{r \quad \mathbb{R}^n \ \langle r, r' \rangle > \qquad r' \quad C\}.$$

Definition 1.2. $S^2 \ S^2 \ T$ $>$ $\sum_i s_i^2$ s_i T i $M \to \mathbb{R}^q$ $i >$ $i \ M \ \subset \mathbb{R}^q$ $i \ x$ x M

$S^2 \ S^2 \ T$ $GL \ T$ T $GL \ T$ $S^2 \ S^2 \ T$ $\sum_i s_i^2$ s_i $S^2 \ T$ $S^2 \ S^2 \ T$

s $\sum_i s_i e_i$ s_i $S^2 \ T$ $\sum_i s_i^2$ $>$

$M \subset \mathbb{R}^q$ $>$ M k $M \subset \mathbb{R}^q$ $\mathbb{R}^q \times$ $\subset \mathbb{R}^q \times \mathbb{R}^k$ $\mathbb{R}^q \times \mathbb{R}^k$ *

1.E. Positiveness of curvature tensor and symmetric -tensors.

$S^2 \ S^2 \ T$ $S^4 \ T \ \subset S^2 \ S^2 \ T$ $A^4_+ \ T$ $S^4 \ T^{\perp} \cap S^2 \ S^2 \ T$ $A^4 \ T$ $^4 \ T^{\perp} \cap S^2 \quad ^2 \ T$

$A^4 \ T$ C_+

$$R \ X, Y, Z, T \qquad \sum_i s_i \ X, Z \ s_i \ Y, T \ - s_i \ X, T \ s_i \ Y, Z \ ,$$

s_i $S^2 \ T$ $G >$ $G \ R \ >$ M $G_p >$ p M $G \ M \ >$ M p M

---

*G v $k \in \mathbb{N}$ c c x $M \subset \mathbb{R}^q$ c c v x $M \subset \mathbb{R}^q \times$ $\subset \mathbb{R}^q \times \mathbb{R}^k$ c v x $M \subset \mathbb{R}^q \times$ $\subset \mathbb{R}^q \times \mathbb{R}^{k+1}$

G R $>$ $\Leftrightarrow R$ X, Y, Z, W X, Z, Y, W $-$ X, W, Y, Z $>$,
M G M $>$
i $M \to \mathbb{R}^q$ i $>$ q
n n / PDR G_+ G_+
O T
$S^2 \times \mathbb{R}^{n-2}$
G_+^*
$K_p >$ K R_p $>$ R_p

G_+ G_+^* GL T
A^4 T GL T G_+ G_+^*

$$Q_+ \quad \{R \quad A^4 \ T \subset S^2 \quad {}^2 \ T \quad R \quad \sum_i \phi_i^2 \quad \phi_i \quad {}^2\},$$

Q_+ / G_+
G_+ G_+^* Q_+ G_+ G_+^*
G_+ Q_+ G_+^* Q_+^* †
$G_+ \subset Q_+ \subset Q_+^* \subset G_+^*$
$Q_+ \subset Q_+^*$ $G_+ \subset Q_+$
$Q_+^* \subset G_+^*$ ‡
S^4 T C_+ *positive forms*
E S^4 T E $\sum s_i^{\circ 2}$ $s_i^{\circ 2}$
s_i $E >$ E $C_+ \subset S^4$ T E
S^2 S^2 T §

† B c v ' c c z c v v c
c v , , M v c v 4 c f M c
$R_x + f(x)\omega \in S^2(\Lambda^2(T_x))$ c v q c $\Lambda^2(T)$ ω v
, c $\Lambda^4(T) \subset S^2(\Lambda^2(T))$, [Z]

‡ cc c [G]$_{SGMC}$ *"The closer of this cone [i.e. $\bar{Q}_+$] (given by $Q \geq$) can be defined as the minimal closed convex $O(n)$-invariant cone which contains the curvature of the product metric on $S^2 \times \mathbb{R}^{n-2}$ [i.e. $\bar{C}_+$]."*

§ B C_+ $GL(T)$ v c $S^4(T)$ c c C_+^*
c c 4 E c

$$E(X, X, X, X) >$$

z $X \in T$ G v [G]PDR 4 c $E \in S^{2k}(T)$
v c q c $E(S^2(T^k)) \to \mathbb{R}$ v fi
c q v c $C_+ = Q_+$, q v $C_+^* = Q_+^*$ c
C_+^* v fi $S^{2k}(T)$, q v v fi
$2k$ T A c Q_+^*

2. Proofs

Proof of the second part of Local Theorem.

M, g G M $>$ M, g
$\mathbb{R}^q$
i $M \to \mathbb{R}^q$ i $>$
M, g G M $>$

i M, g $\to \mathbb{R}^q$
PDR $E \subset S^4$ T E_x $>$
x M C^1 i' M, g $\to \mathbb{R}^q$
E i' E i E E $cg^{\circ 2}$ $c >$ G M $>$
c i' $>$ ¶
$M \subset \mathbb{R}^q$ $U \subset M$
F_i
$> \pi/$ U F_i
F_i^ϵ
U^ϵ F_i^ϵ
U^ϵ □

Proof of Global Theorem. T M
M
$> \pi/$ M

$2k$ T c c x q q v
c *each positively defined polynomial is a sum of squares of polynomials,*
, [ ] O LY
c : ( ) $T \leq 2$ k, ( ) $k =$ T, ( ) $k = 2$ $T =$
ff c , xc (c c) C_+ v
¶ c c i' $E(i') = E(i) + E$ $E = cg \circ g$ c c
c $j : \mathbb{R}^q \to \mathbb{R}^{q'}$, c $E(j) = ch^{\circ 2}$, $h = \sum_{i=1}^q (dx_i)^2$ 2
$\mathbb{R}^q$ $i' = j \circ i$ O c c c j : fi c c c
c $l_i : \mathbb{R}^q \to \mathbb{R}$ c $h^{\circ 2} = \sum_i dl_i^4$, c :
$L : \mathbb{R}^q \to \mathbb{R}^q$ $\tau_i : \mathbb{R}^q \to \mathbb{R}^2$, $\tau_i(x) = (a_i \quad (b_i l_i(x)), a_i$ c $(b_i l_i(x)))$
c L, a_i b_i

$\{e_i\}$ N M s $\sum_i s_i e_i$
s_i S^2 T
U_k k $\{ , , .., n\}$ M U_k
$\{e_{i,k}\} \subset N$ M T M
N M
M n $\{e_{i,k}\}$ N M
$E_{k,k'}$ O $q-m$ $\{e_{i,k}\}$ $\{e_{i,k'}\}$
$E_{k,k'}$ $M \to O$ $q-m$
u_k $M \to$,
$u_k|_{M \setminus U_k} \equiv$ $\sum_k u_k$ x $\equiv$ x M x $U_k \subset M$
${}_x \equiv \sum_i s_{i,k}^2$ x M ${}_x \equiv \sum_{i,k} u_k$ x $s_{i,k}^2$
nN M N_1 M $\oplus N_2$ M $\oplus ... \oplus N_n$ M n
$\{e_{i,k}\}$ N_k N_Δ M
$\sqrt{u}_1 e_{i,1}, \sqrt{u}_2 E_{12} e_{i,1}, ..., \sqrt{u}_n E_{1n} e_{i,1}$
$\mathbb{R}^{(n-1)(q-m)} \times N$ M
i nN M $\to \mathbb{R}^{(n-1)(q-m)} \times N$ M
N_Δ M N M p_Δ $N^n \to N_\Delta$
$i \circ p_\Delta$ $e_{i,k}$ $\sqrt{u}_k e_{i,k}$
$\{e_{i,k}\} \subset N'$ M $\mathbb{R}^{(n-1)dim(N_x)}$
$\times N$ M N M $M \subset \mathbb{R}^q$ N' M
$M \subset \mathbb{R}^q \times$ $\subset \mathbb{R}^q \times \mathbb{R}^{(n-1)(q-m)}$ i, k
$s_{i,k}$ $\langle s, e_{i,k} \rangle$ ${}_x \equiv \sum_{i,k} u_k$ x $s_{i,k}^2$

$e_{i,k}$ N' M
s $\sum_{i,k} e_{i,k}$ $s'_{i,k}$ $s'_{i,k}$
T M $M \to \mathbb{R}^q$
U M M
M T M N M
N M fl
fl $U' \to U$ M
U' U' fl

∎

Problem section

Conjecture. *Any polyhedral metric with curvature* k *can be smoothed in to a Riemannian orbifold with geometric curvature* $k - \epsilon$.

PE N N

S^3

M G M

X_n M

X_n X_n

M

$\mathbb{C}P^2$,

$-\epsilon$

$\mathbb{C}P^2$,

$\mathbb{C}P^2$ $\mathbb{C}P^2$

S^2 S^2

$\mathbb{C}P^2$

$\mathbb{C}P^2$

$\mathbb{R}^q$

$G >$

Appendix A:

R

SU

G

su $su \to$ $^2 su$
SU R $^2 su \to$
$^2 su$ ad
$Im\ ad$
SU
$/ x$ su ad_x $^2 su$
$/ v \wedge w$
ad_x x

x $diag\{ai, bi, ci\}$ a b c su
A_1 $diag\{i, \ , -i\}, A_2$ $diag\{\ , i, -i\}$
F_1 $e_2 \wedge e_3, F_2$ $e_3 \wedge e_1, F_3$ $e_1 \wedge e_2$ E_1 $ie_2 \circ e_3, E_2$ $ie_3 \circ e_1, E_3$ $ie_1 \circ e_2$
e_1, e_2, e_3 $\mathbb{C}^3$ SU
$_x$ $c - b\ F_1 \wedge E_1$ $a - c\ F_2 \wedge E_2$ $b - a\ F_3 \wedge E_3$
ϕ $^2\ T$ $\phi \wedge \phi$

$$ad_x \wedge ad_x$$

$$c - b\ \ a - c\ F_1 \wedge E_1 \wedge F_2 \wedge E_2 \quad a - c\ \ b - a\ F_2 \wedge E_2 \wedge F_3 \wedge E_3$$

$$b - a\ \ c - b\ F_3 \wedge E_3 \wedge F_1 \wedge E_1$$

^{x} $c - b\ ,\ a - c\ ,\ b - a$
a b c a b c x □

Appendix B:

M m $U \subset M$ F $U \to \mathbb{R}^m$
$F\ p$ $x_1\ p\ , x_2\ p\ , ..., x_m\ p$ F
x_i

B.1 Claim. *Let g be a convex function on U and for some convex chart F $U \to \mathbb{R}^m$ we have $\partial g/\partial x_i <$ then $g \circ F^{-1}$ is a convex function on $F\ U$. Moreover for any p U and v T_p we have*

$$\nabla_v^2 g \le \nabla_{dF(v)}^2 \big(g \circ F^{-1}\big).$$

In particular, if S is a level surface of g and F be a short chart then for any p S

$$I\!I_S\ X, X\ \leq I\!I_{F(S)}\ dF\ X\ , dF\ X\ .$$

B.2 Definition. *Let $X_n \overset{GH}{\to} X$ be a converging sequence of metric spaces and f_n $X_n \to X$ be corresponding sequence of Hausdorff approximations. We say that a sequence of measures μ_n on X_n weakly converge to measure μ on X if for any continuous function α with compact support on X we have $\int_{X_n} \alpha \circ f_n d\mu_n \to \int_X \alpha d\mu$.*

B.3 Theorem. *Let M_n be a sequence of Riemannian m-manifolds with curvature k which Lipschitz converge to a closed Riemannian manifold M. Then scalar curvature on M_n converges weakly to the scalar curvature on M. (i.e. $Sc_{g_n} dv_{g_n}$ converges weakly to $Sc_g dv_g$).*

B.4 Lemma. *Let F_n be a sequence of smooth convex hyper-surfaces in $\mathbb{R}^{m+1}$ which Hausdorff converges to a smooth convex hyper-surface F. Let $Sc\ F$ and $Sc\ F_n$ denote scalar curvatures of F and F_n and $h\ F$, $h\ F_n$ denote m-Hausdorff measure of the corresponding hyper-surface. Then $Sc\ F_n\ dh\ F_n$ converges weakly to $Sc\ F\ dh\ F$*

Proof of Lemma B.4. α $\mathbb{R}^{m+1}$ $C_r\ F$ $\mathbb{R}^{m+1}$ F $< r$ α_F $C_\infty\ F \to \mathbb{R}$ $\alpha_F\ x$ $\alpha\ y$ y F $\int_{C_r(F)} \alpha_F dv$ m r $r^2 \int_F \alpha Sc\ F\ dh\ F$ $F_n \to F$ $C_r\ F_n$ $C_r\ F$ α_{F_n} α_F $\int_{C_r(F_n)} \alpha_{F_n} dv \to \int_{C_r(F)} \alpha_F dv$ r^2 □

Proof of Theorem B.3. M M_n

B.5 Lemma. *Given* p M, v $T_p^* M$ *and* $\epsilon >$ *there is* $\delta >$ *and sequence* $M_n \ni p_n \to p$ M *and sequence of convex functions* f_n B_δ p_n $\subset M_n \to \mathbb{R}$ *which converges to a convex function* f B_δ p $\subset M \to \mathbb{R}$ *such that* $d_p f$ v, $|f''| < \epsilon$ *everywhere on* B_δ p *.*

Proof of Lemma B.5. $\{e_i\}$ T_p M $\sum_i e_i$ cv $r >$ a_i $_p re_i$ f $\sum_i \phi \circ$ $_{a_i}$ ϕ x α $x - \beta x^2$ M ϕ x $\alpha \frac{1}{x^{n-2}} - \beta x^2$ M $>$ f α β B_δ p $a_{i,n} \to a_i$ i f_n $\sum_i \phi \circ$ $_{a_{i,n}}$ $\epsilon >$ n δ p_n □

$v_i \subset T_p^*$ M f_i B_δ p $f_{i,n}$ B_δ p_n δ $\pi/$ $\pm \epsilon$

$< m$ n M_n p M $S_1, S_2, ..., S_m$ f_i c Sc_i S_i Sc_i $Sc - Ricc$ u_i u_i S_i $|$ $| < c_1$ $c_2 Sc$ α $m -$ Sc Sc_1 Sc_2 ... Sc_m α $\alpha \to$ $\pi/$ $\epsilon \to$ Sc S_i

Sc $S_{i,n}$ dv_{g_n} Sc S_i dv_g Sc_i G S_i Sc S_i G S_i $\sum_{i \neq j} k_i k_j$ k_i S_i S_i

$$Sc_i \leq Sc\ S_i \quad G\ S_i\ .$$

$Sc_{i,n}$ $Sc_i \leq$ Sc S_i $\leq Sc_i$ n $n -$ ϵ^2 Sc_i $Sc_i - C\epsilon^2$ C C m

$G\ S_{i,n}$

$G\ S_i$ Sc_i $Sc\ S_i\ -G\ S_i$ $Sc_i - G\ S_i$

$G\ S_i\ \leq C\epsilon^2$ C

H

$h_1, h_2, ..., h_m$ H_n $h_{1,n}, h_{2,n}, ..., h_{m,n}$ $\partial f/\partial h_i <$

$-\ /\ \ m$ $G\ S_n\ \leq cG\ H\ S_n$ $G\ S\ \leq$

$cG\ H\ S\ \ \leq C\epsilon^2$

$H\ S_n$ $H\ S$ $\mathbb{R}^m$

$G\ H\ S_n$ $G\ H\ S$

$\epsilon >$ M

□

Smooth Proposition. *Let M_n, g_n be a sequence of Riemannian m-manifolds with curvature κ which GH-converges to a Riemannian manifold M, g of the same dimension m. Then there is a sequence of reparametrizations (diffeomorphisms) $f_n\ \ M \to M_n$, such that curvature tensor of $df_n^*\ g_n$ weakly converges to the curvature tensor of g on M.*

Corollary. *Let R be an $SO\ T$ invariant convex set in $A^4\ T$. Assume that there is a lower bound $\kappa > -\infty$ for sectional curvature in R. Let M_n be a sequence of Riemannian manifolds with curvature tensor from R at each point which converges to a Riemannian manifold M of the same dimension. Then the curvature tensor at any point of M is from R.*

$SO\ T$ $A^4\ T$

$R\ \ \{r\ \ A^4\ \ Ricci\ r\ \leq c\}$

$R'\ \ \{r\ \ A^4\ \ c \leq Sc\ r\ \ \leq c\ \ \epsilon\}$

R

$\pm\ \epsilon$ P

$f\ \ M \to P$ f^{-1} P M, d

d d

$n -$ P $n -$

$h_{n-2}\ \ \pi - \omega\ \alpha$ h_{n-2} ω

α $dx \wedge dy$

$|\alpha|$ $\alpha|_{f^{-1}(\Delta)}$

Singular Proposition. *Let P_n be a polyhedral m-spaces with curvature κ which GH-converges to a Riemannian manifold M, g of the same dimension m. Then there is a sequence of smooth parametrizations f_n $M \to P_n$, such that the described singular curvature tensor weakly converges to the curvature tensor of g on M.*

M, d_n

M, g

References

[ ] D V [P
5 4
[ ] ff O P N
S S 6 5 6
[ ]PDR P ff E
[ ] S V
N Y 6 + 6 / “ ” 5 6
[ ]SGMC S E E
S F 6 4
[W] ; W F
4
[H] H D ¨ D fi F S F
S 4 5 H D
Z P
N Y 65 +45
[ ]*GLC* W ON
5 F 4 S P 6
[ ]*SCH* S N
5 4
[ ] ; D
[ ] D
S V
S N 6 6 5
[P] D P P
[P] P D S S
[PP] P Y ; P E
5 4 56;
S P 5 4 5
[Z] Z S N : D ff
4 5

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