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Adjunction inequality and coverings of Stein surfaces

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Adjunction inequality and coverings of Stein surfaces

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Abstract

A stronger form of the adjunction inequality is proved for immersed real surfaces in non simply-connected Stein surfaces. The result is applied to the geometry of Stein domains and analytic continuation on complex surfaces.

1. Introduction

A complex manifold X is Stein if it admits a strictly plurisubharmonic exhaustion function $\varphi : X \to \mathbb{R}$. (A C^2-smooth function on a complex manifold is called strictly plurisubharmonic if $dd^c\varphi$ is a Kähler form.) Every connected component of a regular sublevel set $\{x \in X \mid \varphi(x) < r\}$ is a strictly pseudoconvex Stein domain, which provides an important example of an exact symplectic manifold with contact boundary.

Applying Morse theory to a perturbation of φ, one shows that X is diffeomorphic to the interior of a (infinite) handlebody without handles of index greater than the complex dimension of X. If X is a Stein complex surface, i. e. $\dim_{\mathbb{C}} X = 2$, then there are further (and subtler) restrictions on the representatives of 2-dimensional homology classes. Namely, if $\Sigma \subset X$ is a closed oriented real surface of genus g embedded in X, then the following *adjunction inequality* holds:

$$[\Sigma] \cdot [\Sigma] + \left|\langle c_1(X), [\Sigma]\rangle\right| \leq 2g - 2 \tag{1}$$

provided that Σ is *not* an embedded 2-sphere with trivial homology class $[\Sigma] \in H_2(X;\mathbb{Z})$.

Inequality (1) was independently derived by Lisca–Matić and the author from similar inequalities for real surfaces in compact Kähler surfaces and the algebraic approximation theorem for Stein manifolds (see [16], [8], and [18] for details and bibliography). Alternatively, one can argue that a strictly pseudoconvex Stein domain has a unique Spin^c-structure with non-zero Seiberg–Witten invariant (in the sense of Kronheimer–Mrowka [14]) whereas a homologically non-trivial embedded surface violating (1) would yield another such structure by the work of Ozsváth–Szabó [21].

The present note explores homotopy theoretic consequences of the adjunction inequality. One result concerns the "exceptional" case of homologically trivial two-spheres.

Theorem 1.1. *A smoothly embedded two-sphere in a Stein complex surface violates adjunction inequality* (1) *if and only if its* ***homotopy*** *class is trivial.*

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Let be an isotopy class of immersed closed real surfaces not necessarily orientable in a complex surface X. Then the following are equivalent

there exists a surface with a Stein neighbourhood base
there exist a surface and a Stein domain U such that the homomorphisms of homotopy groups π_* π_* U are injective
e T e ν c_1 X , *for every surface* .

Here e *denotes the Euler number of a vector bundle, and* c_1 X , *for non-orientable* .

π_* π_* U

^{2} P^2

Any covering of a Stein complex manifold is Stein.

Y Y

$>$ *homeomorphic*

Suppose that Y is an open -manifold such that Y is diffeomorphic to a Stein complex surface. Then every embedded -sphere in Y bounds a homotopy -ball.

Proof. S Y Y S Y

Y Y S
Y

□

M $M^{(n)}$
n M *If the smooth manifold* $M^{(n)}$ *admits a Stein complex structure, then* n *and* M *is a homotopy -sphere*

M $M^{(n)}$
n M

4

2

U 2
U S U

All holomorphic functions in U *can be holomorphically extended to a Riemann domain* $\widetilde{U}$ U *in which* S *becomes homotopically trivial.*

2 V
p_V V 2 V
W W V j V W p_W j p_V
Warning j p_V
V 2 j

Proof. $\widetilde{U}$ U
U $\widetilde{U}$
U
$\widetilde{U}$
S $\widetilde{U}$ S
U 2 $\widetilde{U}$ 2
S
$\widetilde{U}$ □

2

n ^{n} n
n n
n
^{n} n
F $/F$
n
n
^{3}

h

$g >$
^{2}
g

X ι $_g \hookrightarrow X$
$_g$ g
ι $_g$ X ι
x ι s_1 ι s_2
x $_g$
$\pm$ $\pm$

c_1 X , g $_+ -$

g $_+$

$_+$ $_+ -$ $-$
^{4}

x ι s_1 ι s_2 g s_1 s_2
x X

$\iota_*\pi_1$ g π_1 X

x ι g *essential*
X $\pm^{\text{ess}}$ $\pm^{\text{ess}}$ $, X$

4

Let ι g X *be an immersed real surface in a Stein complex surface. Then either* *is a homotopically trivial two-sphere or*

$$c_1\ X\ ,\quad g\quad (\ {}_+ -\ {}_-^{\text{ess}}) -\ .$$

Proof. ι g X
p $\widehat{X}$ X
$\iota_*\pi_1$ g π_1 X $\widehat{\iota}$ g $\widehat{X}$ ι g X
$\widehat{\ }$ $\widehat{\iota}$ g
p $\widehat{X}$ X
H_2 $\widehat{X}$ π_2 $\widehat{X}$
$\widehat{\ }$
$\widehat{X}$ $\widehat{\ }$

$$\widehat{\ }\quad \widehat{\ }\quad c_1\ \widehat{X}\ ,\ \widehat{\ }\quad g\quad {}_+\ \widehat{\ }\ -\ .$$

Each lift of *is an immersed surface* $\widehat{\ }$ $\widehat{\iota}$ g $\widehat{X}$ *such that*

$\widehat{\ }_{\pm}$ $\pm$ $-$ $\pm^{\text{ess}}$ $, X$
$\widehat{\ }$ $\widehat{\ }$ $-$ ${}_+^{\text{ess}}$ $, X$ ${}_-^{\text{ess}}$ $, X$
c_1 $\widehat{X}$ $,$ $\widehat{\ }$ c_1 X $,$.

Proof. $\widehat{\ }$ $\widehat{X}$ $\widehat{x}$ $\widehat{\ }$ $\widehat{\ }$ p $\widehat{x}$
p
$\widehat{X}$
$\widehat{\ }$

ν ι^*TX/T g $\nu^{\widehat{\ }}$ ν

$$e\ \nu\quad {}_+\quad -\quad {}_-\quad ,$$

$e\ \nu$

$$c_1\ \widehat{X} \quad p^* c_1\ X \qquad p \quad \widehat{X} \quad X$$

$$c_1\ \widehat{X}\ , \ \widehat{} \quad p^* c_1\ X\ , \ \widehat{} \quad c_1\ X\ , p_* \widehat{} \quad c_1\ X\ , \quad ,$$

□

Y g Y

$$c_1\ Y\ , \qquad g \quad + - \ - \ - \ ,$$

C^0 $'$ Y $'$

$$e\ T \quad e\ \nu \quad c_1\ Y\ ,$$

$$e\ T \quad \chi \ {}_g \quad - \ g \quad e\ \nu \qquad - \quad + - \ -$$

Y $-$

U

Let be an isotopy class of immersions of ${}_g$ *into a complex surface. Suppose that there exist a surface and a Stein domain* U *such that the homomorphisms of homotopy groups* π_* $\pi_*\ U$ *are injective. Then every surface in satisfies inequality and there exists a surface* $'$ *with a Stein neighbourhood base.*

Proof. $\iota\ {}_g$ ${}_g$ $+$ $-$ π_1 $\pi_1\ U$

□

^{2}

$\iota\ {}_g$ ^{2}

$g \quad + - \ -$

g $+-$ $-<$

$--$ $+-g$

g $+$

S ^{2}

the double point loop of the negative double point of S *bounds an immersed disc in the envelope of holomorphy of any domain* U S

P^2 Y

¨ ¨

adjunction formula

$-$ c_1 Y , g $+-$ $-$ $-$.

c_1 Y ,

P^2

P^2

P^2

P^2

P^2

S P^2

S

S' P^2

S' C^0

P^2

d P^1 $>$ U

P^2 d U

P^2

$$\mathfrak{S}_h \quad \underbrace{P^2 \qquad\qquad P^2}_{h+1 \text{ times}}, \qquad h \qquad .$$

χ $\mathfrak{S}_h$ $-h$
π_1 $\mathfrak{S}_h$ π_1^+ $\mathfrak{S}_h$
h $\mathfrak{S}_h$
h
ι $\mathfrak{S}_h$ Y
$\iota^*TY/T\mathfrak{S}_h$

Let ι $\mathfrak{S}_h$ X *be an immersed non-orientable real surface in a Stein complex surface. Suppose that the homomorphisms* π_* π_* U *are injective. Then the following inequality holds*

$$e\ \nu \quad h - \ ,$$

where e ν *is the Euler number of the normal bundle of* .

e ν
K $\mathfrak{S}_1$ 2

Proof of the theorem. p $\widehat{X}$ X
$\iota_*\pi_1^+$ $\mathfrak{S}_h$ $\widehat{X}$ p
ι $\widehat{\iota}$ h $\widehat{X}$
$\mathfrak{S}_h$ $\widehat{\iota}$ π_1
π_1 X $\widehat{\ }$ $\widehat{\iota}$ h

$$\widehat{\ } \quad \widehat{\ } \quad e\ \nu\widehat{\ } \quad e\ \nu$$

$$c_1\ \widehat{X}\ , \widehat{\ } \quad .$$

$\widehat{\ }$
ι_* π_2 π_2 X

$\widehat{\ }$ $\widehat{X}$ □

ι $\mathfrak{S}_h$ Y

$$e\ \nu \quad h - \ .$$

$h -$ $-\chi$ $\mathfrak{S}_h$ $-e$ T

2

2

$-$ χ , χ , χ , ..., $-$ χ .

P^2

X ^{2} π_1^+

π_1 X

$e\ \nu$ $-\chi$ P^2 $e\ \nu$

^{2}

$e\ \nu$ $-\chi$

$e\ \nu$ $> -\chi$ U

$\widetilde{U}$

P^2 ^{2}

$-$

X n

φ X X_r x X φ x $< r$

φ r

For any covering p V X_r, the manifold V with the induced complex structure is Stein, that is, admits a strictly plurisubharmonic exhaustion function.

X ¨ g , $dd\ \varphi$, J

J X

C^2 β X

$$\lambda\ \beta, x \qquad {}_{\xi\in T_xX,\ \|\xi\|_g=1}\ dd\ \beta\ \xi, J\xi \qquad x \quad X.$$

$\lambda\ \beta, x$ $dd\ \beta$, J

¨ g x X $\lambda\ \varphi, x$

g β

x $\lambda\ \beta, x$

There exists an exhaustion function X_r *such that*

$\lambda \; , x \quad \lambda \; \varphi, x$ *for all* $x \; X_r$.

Proof. $f \; - \; , r$

$$dd \; f \; \varphi \quad f' \; \varphi \; dd \; \varphi \quad f'' \; \varphi \; d\varphi \quad d \; \varphi.$$

$d \; \varphi \; \xi \quad d\varphi \; -J\xi$

$$d\varphi \quad d \; \varphi \; \xi, J\xi \quad \left(d\varphi \; \xi \;\right)^2 \quad \left(d\varphi \; J\xi \;\right)^2 \qquad \xi \quad TX.$$

f *convex* $f' \; t$ $t \; - \; , r$

$\lim_{t\to r-} f \; t$ $f \; \varphi$ X_r □

$p \; V \; X_r$ v

$p \; v$ V [1]

$r' > r \; \varphi$ r, r'

$p \; V \; X_r$ $p' \; V' \; X_{r'}$ $G \; p' \; {}^*g$

$V' \; V$

V $v_0 \; V$ $v \; V'$

$\rho \; v$ $v \; v_0 \; V'$ G

$v \; \rho \; v$ $p \; v$ V

X_r ρ

There exists a C^2*-smooth function* $\rho \; V$ *with the following properties*

$\rho \; v \; > \rho \; v \; -$ *for all* $v \; V$

$\lambda \; \rho, v \; > -K$ *for all* $v \; V$ *and a constant* $K >$ *independent of* $v \; V$.

$\lambda \; \rho, v$ G

X λ

Sketch proof of Lemma $\varepsilon >$ A_ε

V'

$$A_\varepsilon f \; v \quad \frac{}{B \; v, \varepsilon} \int_{B(v,\varepsilon)} f \, d \quad ,$$

$B \; v, \varepsilon$ ε v

$A_\varepsilon f$ $v \; V'$ $B \; v, \varepsilon$

V' $A_\varepsilon f - f$

f $A_\varepsilon f$ C^1

f ε

---

[1] X_r

$\varepsilon >$ ρ A_ϵ ${}^2\rho$

$\varepsilon <$ / x X_r
B $x,$ ε ε x
$X_{r'}$

V' ρ A_ε ${}^2\rho$
V
ρ C^2
ρ v $- \rho$ v $\varepsilon <$ λ ρ, v
ε G λ ρ, v
V p^{-1} X_r G $X_{r'}$ X_r
$X_{r'}$ $\square$

C^2

$$\Psi\ v \quad \rho\ v \quad K \quad p\ v \qquad v \quad V.$$

Ψ ρ v K p v
Ψ

$$\lambda\ \Psi, v \quad \lambda\ \rho, v \quad K\lambda \quad , p\ v \quad >$$

¨

¨

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