Turkish Journal of Mathematics

Volume 27 | Number 1 Article 7

1-1-2003

A monopole homology for integral homology 3-spheres

WEIPING LI

Turk J Math
27 (2003) , 125 – 160.

A monopole homology for integral homology 3-spheres

Weiping Li

Abstract

To an integral homology 3-sphere Y, we assign a well-defined $\mathbf{Z}$-graded (monopole) homology $MH_*(Y, I_\eta(\Theta; \eta_0))$ whose construction in principle follows from the instanton Floer theory with the dependence of the spectral flow $I_\eta(\Theta; \eta_0)$, where Θ is the unique $U(1)$-reducible monopole of the Seiberg-Witten equation on Y and η_0 is a reference perturbation datum. The definition uses the moduli space of monopoles on $Y \times \mathbf{R}$ introduced by Seiberg-Witten in studying smooth 4-manifolds. We show that the monopole homology $MH_*(Y, I_\eta(\Theta; \eta_0))$ is invariant among Riemannian metrics with same $I_\eta(\Theta; \eta_0)$. This provides a chamber-like structure for the monopole homology of integral homology 3-spheres. The assigned function $MH_{\text{SWF}} : \{I_\eta(\Theta; \eta_0)\} \to \{MH_*(Y, I_\eta(\Theta; \eta_0))\}$ is a topological invariant (as Seiberg-Witten-Floer Theory).

1. Introduction

Since Donaldson [9] initiated the study of smooth 4-manifolds via the Yang-Mills theory, the gauge theory (Donaldson invariants, relative Donaldson-Floer invariants and Taubes' Casson-invariant interpretation, etc) has proved remarkably fruitful and rich to unfold some of the mysteries in studying smooth 4-manifolds. The topological quantum field theory proposed by Witten [37] stimulates the most exciting developments in low-dimensional topology. In 1994, Seiberg and Witten (see [38]) introduces a new (simpler) kind of differential-geometric equation. In a very short time after the equation was introduced, some long-standing problems were solved, new and unexpected results were discovered. For instance, Kronheimer and Mrowka [15] proved the Thom conjecture affirmatively, several authors proved variants (generalizations) of the Thom conjecture independently in [11, 24, 29], as well as the three-dimensional version of the Thom conjecture [4]. Taubes showed that there are more constraints on symplectic structures in [32, 33] and the beautiful equality $SW = Gr$ in [34, 35]. See [7] for a survey in the Seiberg-Witten theory.

Using the dimension-reduction principle, one expects the Floer-type homology of 3-manifolds via the Seiberg-Witten equation. Indeed Kronheimer and Mrowka [15] analyzed the Seiberg-Witten-Floer theory for $\Sigma \times S^1$, where Σ is a closed oriented surface. Later on Marcolli studied the Seiberg-Witten-Floer homology for 3-manifolds with first Betti number positive in [21]. For a connected compact oriented 3-manifold with positive first Betti number and zero Euler characteristic, Meng and Taubes [23] showed that a (average) version of Seiberg-Witten invariant is the same as the Milnor torsion. The

L

η_0

U
Y

fl

fl η_0

U fl

I_η η_0 fl I_η η_0 μ_η $-\mu_{\eta_0}$

η_0

fl I_η η_0

η g_Y, α I_η η_0

fl I_η η_0

I_η η_0

fl

Y

I_η η_0

Theorem A. *(1) For an integral homology 3-sphere* Y *and any admissible perturbation* η*, there is a well-defined* **Z***-graded monopole homology* $MH_*\ Y, I_\eta\ \ \eta_0$ *constructed by the Seiberg-Witten equation over* Y **R**.

(2) For any two admissible perturbations η_1 *and* η_2*, there is a group homomorphism* Ψ_* *between two monopole homologies* $MH_*\ Y, I_{\eta_1}\ \ \eta_0$ *and* $MH_*\ Y, I_{\eta_2}\ \ \eta_0$ *.*

(3) If $I_{\eta_1}\ \ \eta_0\ \ I_{\eta_2}\ \ \eta_0$ *, then the homomorphism* Ψ_* *is an isomorphism.*

I_η η_0

Y

η_0 η_0' $\mu_{\eta_0'}$ $-$

μ_{η_0}

fl U

L

fl U

I_η η_0

Theorem B. *For a smooth 4-manifold* X X_0 ${}_Y X_1$ *with* b_2^+ X_i $>$ i , *and* Y *an integral homology 3-sphere, the Seiberg-Witten invariant of* X *is given by the Kronecker pairing of* MH_* Y I_η η_0 *with* MH_{-1-*} $-Y$ I_η η_0 *for the relative Seiberg-Witten invariants* $q_{X_0,Y,\eta}$ *and* $q_{X_1,-Y,\eta}$ *(see Definition 8); assume that the moduli space* $\mathcal{M}_X$ *does not split to* $\mathcal{M}_{X_i}$ *through the stretching-neck process,*

$$\langle , \rangle \quad MH_* \; Y \; I_\eta \quad \eta_0 \qquad MH_{-1-*} \; -Y \; I_\eta \quad \eta_0 \; \to \mathbf{Z} \qquad q_{SW} \; X \qquad \langle q_{X_0,Y,\eta}, q_{X_1,-Y,\eta} \rangle .$$

§

§ Y

§

Y $\mathbf{R}$

Y $\mathbf{R}$ §

fl §

§

§ § §

2. Seiberg-Witten equation on 3-manifolds

$Spin$ $\sim$

SU $\sim$ Sp_1 SO $Spin$ $/\{\pm I\}$

g Y g SO P_{SO} Y

Y P_{SO} Y $Spin$

P_{Spin} Y Y

H^1 Y, Z_2

Y

$$Ad \quad Spin \qquad Sp_1 \to Sp_1 \qquad q, \alpha \;\; \mapsto q\alpha q^{-1},$$

W $P_{Spin(3)}$ Y ${}_{Ad}\, C^2$.

L $detW$

$$c \quad T^*Y \otimes W \to W \qquad c \;\; p, \alpha \;\; \otimes \;\; p, v \;\to\; p, \overline{\alpha} v \; .$$

L

c $T^*Y \to Hom\ W, W$ $\tau\ \ W \otimes \overline{W} \to T^*Y$

$$p, v_1 \otimes v_2 \ \to \tau\ -Im\ v_1 i v_2 \ ,$$

τ $P_{Spin(3)}\ Y \ \ Sp_1 \to T^*Y$
a L g_Y Y
W W $W \otimes T^*Y$

$$\partial_a^{g_Y}\ \ \Gamma\ W \ \overset{\nabla_a^{g_Y}}{\to}\ \Gamma\ W \otimes T^*Y \ \overset{c}{\to} \Gamma\ W \ .$$

L θ
$\partial_\theta^g\ \ \Gamma\ W \ \to \Gamma\ W$
Y **trivial**
Y $\mathbf{R}$
Y g_Y dt^2 Y $\mathbf{R}$ $W^{\pm}_{(4)}$
Y $\mathbf{R}$ dt t
$\mathbf{R}$ $W^+_{(4)}$ $W^-_{(4)}$ U
$W \ \to\ Y$ $Y \ \ \mathbf{R} \to\ Y$
σ ${}^2T^*\ Y \ \ \mathbf{R}\ \to Hom\ W^+_{(4)}, W^-_{(4)}$ $\tau^{-1}\ \ T^*Y \to Hom\ W, W$
$\sigma\ \eta$ $\tau^{-1}\ *_g\eta$
${}^2T^*\ Y \ \ \mathbf{R}$ ${}^2T^*Y \oplus\ {}^1T^*Y$ $W^{\pm}_{(4)}$

$$\tau\ \ \overline{W}\ \ \ W \to\ \ {}^1T^*Y.$$

$\gamma\ \ T^*Y \to Hom\ W, W$
$Y \ \ \mathbf{R}$

$$\sigma\ \ T^*\ Y \ \ \mathbf{R}\ \to Hom\ W \oplus W, W \oplus W \qquad \sigma\ v, r \qquad \begin{pmatrix} & \gamma\ v \quad r \\ \gamma\ v \ -r & \end{pmatrix}.$$

$L_{(4)}$ $detW^{\pm}_{(4)}|_{Y \times \mathbf{R}}$ U
$A \ \ a \ \ \phi dt$ D_A^g g_Y dt^2
$Y \ \ \mathbf{R}$

$$D_A^{g_Y} \quad \begin{pmatrix} & -\nabla_t \quad \partial_a^{g_Y} \\ \nabla_t \quad \partial_a^{g_Y} & \end{pmatrix},$$

$\partial_a^{g_Y}$ $\Gamma\ W \ \to \Gamma\ W$ $\nabla_t \ \ \frac{\partial}{\partial t} \ \ \phi$
$\mathbf{R}$
$A \ \ a \ \ \phi dt$ $F_A \ \ F_a \ \ \frac{\partial a}{\partial t} - d_a\phi\ dt$
$\Omega^2\ Y \ \ \mathbf{R}\ \sim \Omega^2\ Y\ \oplus \Omega^1\ Y$ $F_A^+ \ \ *_{g_Y}F_a \ \ \frac{\partial a}{\partial t} - d_a\phi\ \in$
$\Omega^1\ Y$ F_A

$$\begin{cases} \nabla_t \quad \partial_a^{g_Y} \ \psi \\ *_{g_Y} F_a \quad \frac{\partial a}{\partial t} - d_a\phi \qquad i\tau \ \psi, \psi \end{cases}$$

$\psi \in \Gamma \ W$ fl $a \quad \phi dt, \psi$

$$\begin{cases} \frac{\partial \psi}{\partial t} \qquad -\partial_a^{g_Y} \psi - \phi.\psi \\ \frac{\partial (a+\phi dt)}{\partial t} \qquad - *_{g_Y} F_a \quad d_a\phi \quad i\tau \ \psi, \psi \ . \end{cases}$$

$u \in Map \ Y, U$

$a \quad \phi dt, \psi$

$$u \cdot \ a \quad \phi dt, \psi \qquad u^* a \qquad \phi - u^{-1} \frac{du}{dt} \ dt, \psi u^{-1} \ .$$

u

$$u \cdot \ a \quad \phi dt \qquad u^* a \qquad \phi - u^{-1} \tfrac{du}{dt}$$

$$\begin{cases} \frac{\partial \psi}{\partial t} \qquad -\partial_a^{g_Y} \psi \\ \frac{\partial a}{\partial t} \qquad - *_{g_Y} F_a \quad i\tau \ \psi, \psi \ . \end{cases}$$

3. Configuration spaces on Y

$L \quad Y \quad U$ U L_k^p $\mathcal{A}_k^p \quad L_k^p \ \Omega^1 \ Y, i\mathbf{R}$ Y $\Omega^1 \ Y, i\mathbf{R}$ $\theta \quad L$ L $\mathcal{G}_k^p \ Y \quad L_{k+1}^p \ Map \ Y, U$ $\mathcal{A}_k^p \quad L_k^p \ \Gamma \ W$ $k \quad > \ /p$ $\mathcal{G}_Y \quad \mathcal{G}_k^p \ Y$ k p $\mathcal{C}_Y$

$$\mathcal{C}_Y \quad L_k^2 \ \Omega^1 \ Y, i\mathbf{R} \ \oplus \Omega^0 \ Y, i\mathbf{R} \ \oplus \Gamma \ W \ .$$

$\mathcal{B}_Y \quad \mathcal{C}_Y/\mathcal{G}_Y.$ $\mathcal{C}_Y^* \quad \{ \ a, \phi, \psi \ \in \mathcal{C}_Y | \psi \ / \ \ \}$ $a, \phi, \psi \ \in$ $\mathcal{C}_Y^*$ $\Gamma_{(a,\phi,\psi)} \quad \{id\}$ $a, \phi, \psi \ \in \ \mathcal{C}_Y \setminus \mathcal{C}_Y^*$ $\Gamma_{(a,\phi,0)} \quad U$ $\theta, \ ,$ $Y \quad U$ $\mathcal{G}_Y$ $\mathcal{C}_Y^*$ $\mathcal{B}_Y^* \quad \mathcal{C}_Y^*/\mathcal{G}_Y$ $\mathcal{C}_Y/\mathcal{G}_Y$

Proposition 3.1. *$\mathcal{B}_Y^*$ is a Hilbert manifold. For $a_0, \phi_0, \psi_0 \ \in \mathcal{C}_Y^*$, the tangent space of $\mathcal{B}_Y^*$ can be identified with*

$$T_{[(a_0,\phi_0,\psi_0)]}\mathcal{B}_Y^* \quad \{ \ a, \phi, \psi \ \in L_k^2 \ \Omega^1 \ Y, i\mathbf{R} \ \oplus \Omega^0 \ Y, i\mathbf{R} \ \oplus \Gamma \ W \ \ |$$

$$\| \ a, \phi, \psi \ \|_{L_{k-1}^2(Y)} \quad \|a\|_{L_{k-1}^2(Y)} \quad \|\phi\|_{L_{k-1}^2(Y)} \quad \|\psi\|_{L_{k-1}^2(Y)} < \varepsilon, \quad d_{a_0}^{*g_Y}\psi \quad Im \ \psi_0, \psi \qquad \}.$$

Proof:

a_0, ϕ_0, ψ_0

$a_0, \phi_0, \psi_0 \in \mathcal{C}_Y^*$ $\mathcal{G}_Y \to \mathcal{C}_Y^*$

$$g \quad e^{iu} \to \ a_0 - g^{-1}dg, \phi_0, \psi_0 g^{-1} \ .$$

$Id \quad e^0$

$$\delta_0 \quad T_{id}\mathcal{G}_Y \quad \Omega^0 \ Y, i\mathbf{R} \ \to \Omega^1 \ Y, i\mathbf{R} \ \oplus \Omega^0 \ Y, i\mathbf{R} \ \oplus \Gamma \ W$$

$$u \mapsto \ -du, \ \ , -\psi_0 u \ .$$

$\delta_0^* \quad \delta_0$

$$\delta_0^*\psi \quad d_{a_0}^{*_{g_Y}}\psi \quad Im \ \psi_0.\psi \ .$$

$a_0, \phi_0, \psi_0 \ \in \mathcal{B}_Y^*$ $T_{[(a_0,\phi_0,\psi_0)],\varepsilon}\mathcal{B}_Y^*/$
$\Gamma_{(a_0,\phi_0,\psi_0)}$ ε a_0, ϕ_0, ψ_0
$T_{[(a_0,\phi_0,\psi_0)],\varepsilon}\mathcal{B}_Y^*$ a_0, ϕ_0, ψ_0
$u \in \Omega^0 \ Y, i\mathbf{R}$ $e^{iu} \cdot \ a_0 \quad a, \phi_0 \quad \phi, \psi_0 \quad \psi \ \in$
$T_{[(a_0,\phi_0,\psi_0)],\varepsilon}\mathcal{B}_Y^*$ $\psi_0 \ /$

□

$\mathcal{C}_Y^* \quad {}_{\mathcal{G}_Y} \ \Omega^1 \ Y, i\mathbf{R} \ \oplus \Gamma \ W$ $\mathcal{C}_Y^*$
$\mathcal{G}_Y$ $\mathcal{C}_Y^*$ $f \ \ \mathcal{C}_Y^* \to \mathcal{C}_Y^* \ \ {}_{\mathcal{G}_Y} \ \Omega^1 \ Y, i\mathbf{R} \ \oplus \Gamma \ W$

$$f \ a, \phi, \psi \qquad a, \phi, \psi \ , *_{g_Y} F_a - d_a\phi - i\tau \ \psi, \psi \ , \partial_a^{g_Y}\psi \quad \phi.\psi \ .$$

f $\mathcal{G}_Y$ $f \ g \cdot \ a, \phi, \psi \quad g \cdot f \ a, \phi, \psi$ $\mathcal{B}_Y^*$

$$f \ \ \mathcal{B}_Y^* \to \mathcal{C}_Y^* \ \ {}_{\mathcal{G}_Y} \ \Omega^1 \ Y, i\mathbf{R} \ \oplus \Gamma \ W \ \ .$$

$f \ a, \phi, \psi \ \in T_{[(a,\phi,\psi)],\varepsilon}L_{k-1}^2\mathcal{B}_Y^*$ $\mathcal{L}_{[(a,\phi,\psi)]}$ f
$\mathcal{B}_Y^*$ $\mathcal{B}_Y^*$ f $\mathcal{L}$ $\mathcal{L}_{[(a,\phi,\psi)]}$

Definition 3.2. f $\mathcal{B}_Y^*$

$$f^{-1} \qquad \mathcal{R}_{SW}^* \ Y, g_Y \quad \{ \ a, \phi, \psi \ \in \mathcal{C}_Y^* \qquad \}/\mathcal{G}_Y.$$

$$\begin{cases} \partial_a^{g_Y}\psi \quad \phi.\psi \\ *_{g_Y}F_a - d_a\phi - i\tau \ \psi, \psi \end{cases} .$$

$\mathcal{R}_{SW}^* \ Y, g_Y$

§

f

$$\begin{aligned} f \ a_0 \quad sa, \phi_0 \quad s\phi, \psi_0 \quad s\psi \qquad & *_{g_Y}F_{a_0+sa} - d_{a_0+sa} \ \phi_0 \quad s\phi \ - i\tau \ \psi_0 \quad s\psi, \\ & \psi_0 \quad s\psi \ , \partial_{a_0+sa}^{g_Y} \ \psi_0 \quad s\psi \qquad \phi_0 \quad s\phi \ . \ \psi_0 \quad s\psi \\ & f \ a_0, \phi_0, \psi_0 \qquad sDf \ a_0, \phi_0, \psi_0 \quad a, \phi, \psi \qquad o \ s^2 \ . \end{aligned}$$

$$Df\ a_0,\phi_0,\psi_0 \quad T_{[(a_0,\phi_0,\psi_0)]}\mathcal{B}^*_Y \to \mathcal{L}_{[(a_0,\phi_0,\psi_0)]}$$

$$Df\ a_0,\phi_0,\psi_0 \quad \Omega^1\ Y,i\mathbf{R}\ \oplus\Omega^0\ Y,i\mathbf{R}\ \oplus\Gamma\ W\ \to\Omega^1\ Y,i\mathbf{R}\ \oplus\Gamma\ W\ ,$$

$$a,\phi,\psi\ \longmapsto \begin{pmatrix} *_{g_Y}d_{a_0} & -d_{a_0} & -iIm\ \psi_0,,\cdot \\ c\ \cdot\psi_0 & c\cdot\psi_0 & \partial^{g_Y}_{a_0}\quad \phi_0\cdot \end{pmatrix}\begin{pmatrix} a \\ \phi \\ \psi \end{pmatrix}.$$

δ_0^*

$$MC_\bullet \quad \to\Omega^0\ Y,i\mathbf{R}\ \xrightarrow{\delta_0}\Omega^1\ Y,i\mathbf{R}\ \oplus\Omega^0\ Y,i\mathbf{R}\ \oplus\Gamma\ W\ \xrightarrow{Df}\Omega^1\ Y,i\mathbf{R}\ \oplus\Gamma\ W\ \to\ ,$$

$$\delta_0^*\oplus Df\ a_0,\phi_0,\psi_0 \quad \Omega^1\ Y,i\mathbf{R}\ \oplus\Omega^0\ Y,i\mathbf{R}\ \oplus\Gamma\ W\ \to\Omega^1\ Y,i\mathbf{R}\ \oplus\Omega^0\ Y,i\mathbf{R}\ \oplus\Gamma\ W$$

$$a,\phi,\psi\ \longmapsto \begin{pmatrix} *_{g_Y}d_{a_0} & -d_{a_0} & -iIm\ \psi_0,\cdot \\ -d^{*_{g_Y}}_{a_0} & & Im\ \psi_0,\cdot \\ c\ \cdot\psi_0 & c\cdot\psi_0 & \partial^{g_Y}_{a_0}\quad \phi_0\cdot \end{pmatrix}\begin{pmatrix} a \\ \phi \\ \psi \end{pmatrix},$$

$$\sigma\ \delta_0^*\oplus Df \qquad \sigma\ \delta\ a_0,\ ,\psi_0\ g_Y$$

$$\delta\ a_0,\ ,\psi_0\ g_Y \qquad \begin{pmatrix} *_{g_Y}d_{a_0} & -d_{a_0} & \\ -d^{*_{g_Y}}_{a_0} & & \\ & & \partial^{g_Y}_{a_0} \end{pmatrix}$$

$$Ind\ \delta_0^*\oplus Df \qquad Ind\ \delta\ a_0,\ ,\psi_0\ g_Y$$

$$Ind\begin{pmatrix} *_{g_Y}d_{a_0} & -d_{a_0} \\ -d^{*_{g_Y}}_{a_0} & \end{pmatrix} \quad Ind\partial^{g_Y}_{a_0}$$

.

$$\begin{pmatrix} *_{g_Y}d_{a_0} & -d_{a_0} \\ -d^{*_{g_Y}}_{a_0} & \end{pmatrix}$$

$\delta_0^*\oplus Df$ $\mathcal{R}_{SW}\ Y,g_Y$
$H^0\ MC_\bullet$ $\delta_0\ \ H^1\ MC_\bullet$ $Df/im\delta_0\ \ H^2\ MC_\bullet$ $cokerDf$
$H^1\ MC_\bullet$ $a_0,\phi_0,\psi_0\ \in\mathcal{B}^*_Y$ $a_0,\phi_0,\psi_0\ \in\mathcal{B}^*_Y$
f $\delta_0^*\oplus Df\quad H^1\ MC_\bullet$ $\theta,\ ,$
g_Y $\partial^{g_Y}_\theta$
f Y

4. Admissible Perturbation and Transversality

f

the harmonic spinor may vary or jump as metrics on Y *vary*

$\mathcal{P}_Y$ $_Y$ $\Omega^1(Y, i\mathbf{R})$ Y ^{Y} $\cup_{(g_Y,\alpha)\in\mathcal{P}_Y}\mathcal{R}^*_{SW}(Y, g_Y, \alpha)$

$\mathcal{P}_Y$

$\mathcal{P}_Y$ $\mathcal{R}^*_{SW}(Y, g_Y, \alpha)$

f_η $\mathcal{L} \to \mathcal{B}^*_Y$ $\mathcal{P}_Y$ η

$(g_Y, \alpha) \in \mathcal{P}_Y$ f_η

$$f_\eta : \mathcal{B}^*_Y \to \Omega^1(Y, i\mathbf{R}) \oplus \Gamma(W)$$

$$(a, \phi, \psi) \mapsto (*_{g_Y}F_a - d_a\phi - i\tau(\psi, \psi) \quad \alpha, \partial_a^{\nabla_0+\alpha}\psi \quad \phi.\psi),$$

∇_0 g_Y $f_{1\eta}(a, \phi, \psi)$ $\partial_a^{\nabla_0+\alpha}\psi$ $\phi.\psi$ f_η $\Gamma(W)$ $f_{0\eta}(a, \phi, \psi)$

f_η

Lemma 4.1. *$f_{1\eta}$ is a submersion ($Df_{1\eta}$ is surjective).*

Proof: $Df_{1\eta}$

$$Df_{1\eta}(a, \phi, \psi)(o, \alpha)(\varepsilon a, \varepsilon\phi, \varepsilon\psi, , \varepsilon\alpha) \quad \partial_a^{\nabla_0+\alpha}(\varepsilon\psi) \quad (\varepsilon\alpha \quad \varepsilon a \quad \varepsilon\phi).\psi \quad \phi.\varepsilon\psi,$$

$\{\Omega^1(Y, i\mathbf{R}) \oplus \Omega^0(Y, i\mathbf{R}) \oplus \Gamma(W)\}$ $\{\{\ \}\ \Omega^1(Y, i\mathbf{R})\}$ $T(a, \phi, \psi)\mathcal{B}^*_Y$

$\mathcal{P}_Y$ $Df_{1\eta}$

$\chi \in \Gamma(W)$ $ImDf_{1\eta}$

$$\langle\partial_a^{\nabla_0+\alpha}(\varepsilon\psi), \chi\rangle \quad ,$$

$\varepsilon\psi$ $\chi \in$ $(\partial_a^{\nabla_0+\alpha})^*$ χ

$y \in Y$ $\chi(y) \ /$

$\partial_a^{\nabla_0+\alpha} \cdot (\partial_a^{\nabla_0+\alpha})^*\chi$

U_y y $\chi(y)\ /$ $y \in U_y$ $\varepsilon\alpha$ $\varepsilon a \in \Omega^1(Y, i\mathbf{R})$

$(\varepsilon\alpha \quad \varepsilon a).\psi \quad \lambda\chi$ $\lambda\ /$ U_y $\varepsilon\alpha$ εa

$$\begin{aligned}
&\langle\partial_{a+\varepsilon a}^{\nabla_0+\alpha+\varepsilon\alpha}(\varepsilon\psi), \chi\rangle \\
&\langle\partial_a^{\nabla_0+\alpha}(\varepsilon\psi), \chi\rangle \quad \langle(\varepsilon\alpha \quad \varepsilon a).\varepsilon\psi, \chi\rangle \\
&\langle\lambda\chi, \chi\rangle \quad \lambda\langle\chi, \chi\rangle.
\end{aligned}$$

χ U_y $\chi \equiv$ $f_{1\eta}$ □

Ω^1 $Y, i\mathbf{R}$ $Imd \oplus Imd^{*g_Y}$ Y

$f_{0\eta}$ α, ϕ, ψ $*_{g_Y} F_a - d_a\phi - i\tau$ ψ, ψ α Ω^1 $Y, i\mathbf{R}$

Corollary 4.2. *The spaces* $f_{0\eta}^{-1}$ *and* $f_{1\eta}^{-1}$ *are Banach manifolds.*

□

a_0, ϕ_0, ψ_0 g_0, α $\in \mathcal{C}_Y$ $\mathcal{P}_Y$

f

f a_0, ϕ_0, ψ_0 g_0, α $f_{(g_0,\alpha)}$ a_0, ϕ_0, ψ_0 f_η a_0, ϕ_0, ψ_0

Proposition 4.3. *The differential* Df *is onto at all points of the moduli space* f^{-1} $\subset$ $\mathcal{B}_Y^*$ $\mathcal{P}_Y$.

Proof: Df a_0, ϕ_0, ψ_0 g_0, α $\in \mathcal{C}_Y$ $\mathcal{P}_Y$ Df_0, Df_1

Df_0 $*_{g_0} d_{a_0} a$ g $_* F_{a_0} - d_{a_0}\phi - iIm$ ψ_0, ψ $- a.\phi_0$ α

Df_1 $\partial_{a_0}^{\nabla_0+\alpha_0}\psi$ α a $.\psi_0$ $\phi.\psi_0$ $\phi_0.\psi$ r g

g $*$ g $*$ $\frac{d}{ds}|_{s=0} *_{g_0+sg}$ r g

g_0 sg o s^2 $a.\phi_0$

a $\phi_0 \in \Gamma$ W Df_0

Df_1

$,$ $,\psi,$ $,\alpha$ $\mapsto \partial_{a_0}^{\nabla_0+\alpha_0}\psi$ α $.\psi_0$

$\chi \in \Gamma$ W $\chi \in$

$\partial_{a_0}^{\nabla_0+\alpha_0}$ $*$

$$\langle\partial_{a_0}^{\nabla_0+\alpha_0}\psi \quad \alpha \; .\psi_0, \chi\rangle \quad \langle\alpha.\psi_0, \chi\rangle,$$

$\alpha \in \Omega^1$ $Y, i\mathbf{R}$, W $\alpha.\psi_0$ χ

$\alpha.\psi_0, \chi$ Y ψ_0 χ

$$\partial_{a_0}^{\nabla_0+\alpha_0}\psi_0 \quad \phi_0.\psi_0 \qquad \partial_{a_0}^{\nabla_0+\alpha_0} \; {}^*\chi \quad .$$

ψ_0 χ

$U \subset Y$ ψ_0 χ $\{x_1, x_2, x_3\}$

y α $a_1 dx_1$ $a_2 dx_2$ $a_3 dx_3$

Γ W $\{s_i\}_{i=1,2}$ W $y \in Y$

$$\psi_0 \quad s_1.e_1 \quad s_2.e_2 \qquad \chi \quad s_1^{'}.c_1 \quad s_2^{'}.c_2,$$

$s_i^{'}$ γ^1 s_i γ^1 $\frac{\partial}{\partial x_1}$

$$\alpha.\psi_0 \quad \gamma\ a_1 dx_1 . \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \quad \gamma\ a_2 dx_2 . \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \quad \gamma\ a_3 dx_3 . \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

$$\{\begin{pmatrix} -\ a_1 i & \\ & a_1 i \end{pmatrix} \quad \begin{pmatrix} & -\ a_2 \\ a_2 & \end{pmatrix} \quad \begin{pmatrix} & a_3 i \\ a_3 i & \end{pmatrix}\} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

$$\begin{pmatrix} -\ a_1 i & -\ a_2 & a_3 i \\ a_2 & a_3 i & a_1 i \end{pmatrix} . \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

$$\alpha.\psi_0, \chi \quad \begin{pmatrix} -\ a_1 i & -\ a_2 & a_3 i \\ a_2 & a_3 i & a_1 i \end{pmatrix} . \begin{pmatrix} e_1.c_1 & e_1.c_2 \\ e_2.c_1 & e_2.c_2 \end{pmatrix},$$

a_1, a_2, a_3

$$\begin{pmatrix} e_1.c_1 & e_1.c_2 \\ e_2.c_1 & e_2.c_2 \end{pmatrix} \quad ,$$

$$\begin{pmatrix} -a_1 i & -a_2 & a_3 i \\ a_2 & a_3 i & a_1 i \end{pmatrix}$$

End C^2 $a.Id$ e_2 c_2 e_1, c_1
e_1, c_1 ψ_0 $s_1.e_1$ χ $s_1^{'}.e_1$ e_1, c_1 U
e_1 c_1 e_1, e_1 c_1, c_1
$\partial_{a_0}^{\nabla_0+\alpha_0}\psi_0$ $\phi_0.\psi_0$ $\partial_{a_0}^{\nabla_0+\alpha_0}$ s_1 $\otimes e_1$ $s_1.d_{a_0}e_1$ $\phi_0.s_1.e_1$

$\partial_{a_0}^{\nabla_0+\alpha_0}$ s_1 $\otimes e_1$ $s_1.d_{a_0}e_1$ $\phi_0.s_1.e_1, c_1$ $\partial_{a_0}^{\nabla_0+\alpha_0}$ s_1 $\phi_0.s_1$ e_1, c_1 s_1 $d_{a_0}e_1, c_1$.
s_1 $d_{a_0}e_1, c_1$ s_1 / $d_{a_0}e_1, c_1$ e_1, c_1

$$d_{a_0}e_1, c_1 \quad e_1, d_{a_0}c_1 \quad .$$

$e_1, d_{a_0}c_1$ a_0 a_0
s_1 ψ_0 $s_1.e_1$ $a_0, \phi_0,$ $\in \mathcal{C}_Y \setminus \mathcal{C}_Y^*$
a_0, ϕ_0, ψ_0 $\in \mathcal{C}_Y^*$ χ Df_1 $\square$

$$f_* \quad \mathcal{B}_Y^* \quad \mathcal{P}_Y \to \Omega^1\ Y, i\mathbf{R}\ \oplus \Gamma\ W$$

Corollary 4.4. *There is a dense subset* $\mathcal{P}_Y^{'} \subset \mathcal{P}_Y$ *such that for* $\eta \in \mathcal{P}_Y^{'}$ *the space* f_*^{-1} *is regular (i.e., a smooth Banach manifold).*

Proof: f_* $\mathcal{B}_Y^*$ $\mathcal{P}_Y$ $\mathcal{C}_Y^*$ $\mathcal{G}_Y$ $\Omega^1\ Y, i\mathbf{R}\ \oplus \Gamma\ W$ $\mathcal{P}_Y$
f_*^{-1} $|_{\mathcal{B}_Y^*}$ f_*^{-1} $/\mathcal{G}_Y$

$$\begin{array}{ccc} \mathcal{B}_Y^* \quad \mathcal{P}_Y & \xrightarrow{f} & \Omega^1\ Y, i\mathbf{R}\ \oplus \Gamma\ W \\ \downarrow \pi_2 & & \\ \mathcal{P}_Y & & \end{array}$$

L

π_2

□

Corollary 4.5. *The inverse image* π_2^{-1} g_Y, α *of a generic parameter* η $g_Y, \alpha \in \mathcal{P}_Y^{'}$, *the moduli space* $\mathcal{R}_{SW}$ Y, η *of the 3-dimensional monopole solutions is a zero-dimensional manifold.*

$\mathcal{C}_Y \setminus \mathcal{C}_Y^*$

$\mathcal{B}_Y$

$\mathcal{R}_{SW}$ Y, g_Y

$$\partial_a^{\nabla_0+\alpha}\psi \quad \phi_0.\psi$$
$$- *_{g_Y} F_a \quad d_a\phi \quad ,$$

ψ $g \cdot$ a, ϕ $g^*a,$ g^*a fl

Y U Y Y

U $\theta,$

U $\theta,$

$\delta_0^* \oplus Df$ a_0, ϕ_0, ψ_0 $\partial_{a_0}^{g_Y}$

g_Y $\partial_{a_0}^{g_Y}$ $\partial_{a_0}^{g_t}$

Proposition 4.6. $\mathcal{R}_{SW}^*$ Y, η $\mathcal{R}_{SW}$ Y, η $\setminus \{$ $\}$ *is a zero-dimensional smooth compact manifold for a first category near* η g_Y, α *in* $\mathcal{P}_Y^{'}$.

Proof:

□

a, ϕ, ψ $\in \mathcal{R}_{SW}^*$ Y, η $\delta_0^* \oplus Df_\eta$ a, ϕ, ψ

$\delta_0^* \oplus Df_\eta$ a, ϕ, ψ $\mathcal{L}_{[a,\phi,\psi]}$

L_k^2 $\mathcal{L}_{[a,\phi,\psi]}$

$\mathbf{R} \setminus \{$ $\}$

$\mathcal{R}_{SW}$ Y, η $\eta \in \mathcal{P}_Y^{'}$ $\delta_\eta >$

$\delta_0^* \oplus Df_\eta$ a, ϕ, ψ

a, ϕ, ψ $\in f_\eta^{-1}$ $\mathcal{R}_{SW}$ Y, η

a_0

$$cs_\eta \; a, \psi \quad -- \int_Y \{ \; a - a_0 \; \wedge \; F_a \quad F_{a_0} \quad \alpha \quad \langle \partial_a^{\nabla_0+\alpha}\psi, \psi \rangle \} dvol_{g_Y},$$

$\nabla cs_\eta\ a,\psi \quad f_\eta\ a,\ ,\psi$ $Y\ \mathbf{R}$

$$\frac{\partial}{\partial t}\ a\ t \quad \phi\ t\ dt, \psi\ t \quad -f_\eta\ a\ t\ , \phi\ t\ , \psi\ t\ \ ,$$

$A,\Psi \quad a\ t \quad \phi\ t\ dt, \psi\ t$

$E_\eta\ A,\Psi$

$$-\int_{Y\times\mathbf{R}} \{|\nabla_t \psi\ t\ |^2 \quad |\partial_a^{\nabla_0+\alpha}\psi\ t\ |^2 \quad |\nabla_t a\ t \quad d_a\phi\ t\ |^2 \quad |*_{g_Y} F_a - i\tau\ \psi,\psi \quad \alpha|^2\} dvol_{g_Y} dt,$$

$\nabla_t \quad \frac{\partial}{\partial t} \quad \phi.$

$$E_\eta\ A,\Psi \quad cs_\eta\ A,\Psi\ |_{t=-\infty} - cs_\eta\ A,\Psi\ |_{t=+\infty}$$

$$\int_{Y\times\mathbf{R}} \{|F_A^+ - -\tau\ \Psi,\Psi \quad \alpha\wedge dt \quad *\ \alpha\wedge dt\ |^2 \quad |\partial_A^{\nabla_0+\alpha}\Psi|^2\} dvol_{g_Y} dt,$$

$*$ $Y\ \ \mathbf{R}$ $g_Y \quad dt^2$
A,Ψ fl $\nabla_t\psi\ t \quad \partial_a^{\nabla_0+\alpha}\psi\ t \quad \nabla_t a\ t \quad d_a\phi\ t$
$*_{g_Y} F_a - i\tau\ \psi,\psi \quad \alpha$ L^p $p \geq$

Lemma 4.7. *There is a solution* $a_\infty, \phi_\infty, \psi_\infty \ \in f_\eta^{-1}$ *over* Y *such that* A,Ψ $a\ t \quad \phi\ t\ dt, \psi\ t$ *converges to* $a_\infty \quad \phi_\infty dt, \psi_\infty$ *uniquely up to the gauge equivalence in the sense that* $A,\Psi\ |_{Y\times\{t\}}$ *converges in* C^∞ *over* Y.

Proof: $a\ t\ ,\ \ ,\psi\ t$ Y
, $A,\Psi\ |_{Y\times(t,t+1)}$
$\nabla_t\psi\ t\ , \partial_a^{\nabla_0+\alpha}\psi\ t\ , \nabla_t a\ t \quad d_a\phi\ t\ , *_{g_Y} F_a - i\tau\ \psi,\psi \quad \alpha$
$p \geq \quad t\to \quad \infty$

$$\|\nabla_t\psi\ t\ \|_{L^p(Y\times(0,1))} \to \ , \quad \|\partial_a^{\nabla_0+\alpha}\psi\ t\ \|_{L^p(Y\times(0,1))} \to \ ,$$

$$\|\nabla_t a\ t\ \|_{L^p(Y\times(0,1))} \to \ , \quad \| *_{g_Y} F_a - i\tau\ \psi,\psi \quad \alpha\|_{L^p(Y\times(0,1))} \to \ .$$

$t_n \to \infty$ $t_n' \to \infty$ $A\ \infty\ , \Psi\ \infty$ Y
, $A,\Psi\ |_{Y\times\{t_n'\}} \to \ A\ \infty\ , \Psi\ \infty$
C^∞ Y , $A\ \infty\ , \Psi\ \infty$
$a_\infty,\ \ ,\psi_\infty$ L^p Y ,

$$\nabla_t\psi_\infty \quad \partial_{a_\infty}^{\nabla_0+\alpha}\psi_\infty \quad \nabla_t a_\infty \quad *_{g_Y} F_{a_\infty} - i\tau\ \psi_\infty,\psi_\infty \quad \alpha \quad .$$

$f_\eta\ a_\infty,\ \ ,\psi_\infty$ $a_\infty,\ \ ,\psi_\infty\ \in \mathcal{R}_{SW}\ Y,\eta \quad f_\eta^{-1}$
□

$J\ T \quad E_\eta\ A, \Psi\ |_{Y\times[T,\infty)}$ $\quad Y \quad T, \infty$
A, Ψ $\quad J\ T \quad cs_\eta\ A|_T, \Psi|_T\ - cs_\eta\ A|_\infty, \Psi|_\infty$

$$\frac{dJ\ T}{dT} \quad \nabla cs_\eta\ A|_T, \Psi|_T\ \cdot \frac{\partial\ A|_T, \Psi|_T}{\partial T} \quad -\|f_\eta\ a\ T\ , \phi\ T\ , \psi\ T\ \|^2_{L^2(Y\times\{T\})},$$

$A|_T, \Psi|_T \quad a\ T \quad \phi\ T\ dt, \psi\ T \quad T \quad a\ T \quad \phi\ T\ dt, \psi\ T$
$a_\infty \quad \phi_\infty dt, \psi_\infty \quad a\ t \quad \phi\ t\ dt, \psi\ t \quad f_\eta$

$$f_\eta\ a\ T\ , \phi\ T\ , \psi\ T \quad Df_\eta\ a_\infty, \phi_\infty, \psi_\infty\ \ a\ t\ , \phi\ t\ , \psi\ t \quad N\ a\ t\ , \phi\ t\ , \psi\ t\ \ ,$$

$f_\eta\ a_\infty, \phi_\infty, \psi_\infty \quad N\ a\ t\ , \phi\ t\ , \psi\ t \quad a\ t\ , \phi\ t\ , \psi\ t$

Lemma 4.8. *For* $a, \phi, \psi\ \in \{ \quad \delta_0^* \oplus Df_\eta\ a_\infty, \phi_\infty, \psi_\infty\ \ \}^\perp$ *(the subspace which is perpendicular to* $\delta_0^* \oplus Df_\eta\ a_\infty, \phi_\infty, \psi_\infty$ *), there exist a constant* C_2 *and* T_0 *such that for* $t \geq T \geq T_0$*,*

$$\|\ a, \phi, \psi\ \|_{L^2_k(Y\times\{t\})} \leq C_2 \|f_\eta\ a\ T\ , \phi\ T\ , \psi\ T\ \ \|_{L^2_{k-1}(Y)},$$

where $\|\ a, \phi, \psi\ \|_{L^2_k(Y\times\{t\})} \quad \|\ a \quad \phi dt, \psi\ \|_{L^2_k(Y\times\{t\})} \quad \|a\|_{L^2_k(Y\times\{t\})} \quad \|\phi\|_{L^2_k(Y\times\{t\})}$ $\|\psi\|_{L^2_k(Y\times\{t\})}$.

Proof: $\quad \delta_\eta \|\ a, \phi, \psi\ \|_{L^2_k(Y)} \leq \|\delta_0^* \oplus Df_\eta\ a_\infty, \phi_\infty, \psi_\infty\ \ a, \phi, \psi\ \|_{L^2_{k-1}(Y)}$ $\quad a, \phi, \psi\ \ \in \{ \quad \delta_0^* \oplus Df_\eta\ a_\infty, \phi_\infty, \psi_\infty\ \ \}^\perp$ $\quad a_\infty, \phi_\infty, \psi_\infty\ \ \in f_\eta^{-1} \quad T \quad a\ t \quad \phi\ t\ dt, \psi\ t \quad a\ T$ $\phi\ T\ dt, \psi\ T\ \ - \ a_\infty \quad \phi_\infty dt, \psi_\infty \quad$ ¨ $\quad C^\infty \quad \|\ a\ t\ , \phi\ t\ ,$ $\psi\ t\ \|_{L^2_k(Y)}$

$$\|N\ a\ t\ , \phi\ t\ , \psi\ t\ \ \|_{L^2_{k-1}(Y)} \leq C_1 \|\ a, \phi, \psi\ \|^2_{L^4_{k-1}(Y\times\{t\})} \leq C_1' \|\ a, \phi, \psi\ \|^2_{L^2_k(Y\times\{t\})}.$$

$T_0 \quad C_1' \|\ a, \phi, \psi\ \|^2_{L^2_k(Y\times\{T_0\}} \leq \delta_\eta /$

$T \geq T_0$ □

$J\ T \quad cs_\eta\ A|_T, \Psi|_T\ - cs_\eta\ A|_\infty, \Psi|_\infty$

$$J\ T \quad dcs_\eta\ A|_\infty, \Psi|_\infty\ \ A|_T - A|_\infty, \Psi|_T - \Psi|_\infty \quad N\ a\ t\ , \phi\ t\ , \psi\ t$$
$$dcs_\eta\ A|_\infty, \Psi|_\infty\ \ a\ t \quad \phi\ t\ dt, \psi\ t \quad N\ a\ t\ , \phi\ t\ , \psi\ t\ \ .$$

$T \geq T_0$

$$\begin{aligned} J\ T \quad &\leq \quad C_3 \|\ a, \phi, \psi\ \|^2_{L^2_1(Y\times\{T\})} \\ &\leq \quad C_3 C_2 \|f_\eta\ A|_T, \Psi|_T\ \|^2_{L^2(Y\times\{T\})} \\ &\quad\ \ -C_3 C_2 \frac{\partial J}{\partial T}, \end{aligned}$$

$J\ T\ \ \leq J\ T_0\ e^{-\gamma(T-T_0)} \quad \gamma > \quad T \geq T_0$

Proposition 4.9. *Let* A, Ψ *be the trajectory flow line of (4.7) over* Y $\mathbf{R}$. *For the end* Y T, ∞ *or* Y $-\infty, -T$, *there exist a gauge transformation* $g_{\pm}$, *a constant* C_4 *and* $\gamma_1 >$ *such that* A, Ψ $a\ t$ $\phi\ t\ dt, \psi\ t$ $g_{\pm}^*\ a_{\pm\infty}$ $\phi_{\pm\infty} dt, \psi_{\pm\infty}$ *for* $\pm t \geq T$, *and for* $a\ t\ , \phi\ t\ , \psi\ t$ *satisfying the hypothesis of Lemma 4.8,*

$$_Y \{|a\ t\ |, |\phi\ t\ |, |\psi\ t\ |, |f_\eta\ a\ t\ , \phi\ t\ , \psi\ t\ \ |\} \leq C_4 \cdot e^{-\gamma_1(|t|-T)},$$

for $|t| > T$. *Moreover one can choose* $A|_t, \Psi|_t$ *such that all derivatives decay exponentially:*

$$_Y \{|\nabla^l a\ t\ |, |\nabla^l \phi\ t\ |, |\nabla^l \psi\ t\ |\} \leq C_5 \cdot e^{-\gamma_1(|t|-T)}, \qquad |t| > T,$$

where the constants C_4 *and* C_5 *depends continuously on* A, Ψ .

Proof: σ $\sigma < \gamma$ $J\ T\ \leq J\ T_0\ e^{-\gamma(T-T_0)}$
$T \geq T_0$ $J_\sigma\ T$ $\int_T^\infty e^{\sigma t} \|f_\eta\ a\ t\ , \phi\ t\ , \psi\ t\ \ \|^2_{L^2(Y\times\{t\})} dt$
$t \geq T$

$$J_\sigma\ T \qquad \int_T^\infty e^{\sigma t}\ -\frac{dJ\ t}{dt}\ dt \leq J\ T_0\ e^{\gamma T_0} \qquad \frac{\sigma}{\sigma-\gamma}\ e^{-(\gamma-\sigma)T},$$

$J\ T$ $\|f_\eta\ a\ t\ , \phi\ t\ , \psi\ t\ \ \|^2_{L^2(Y\times\{t\})}$
$f_\eta\ a\ t\ , \phi\ t\ , \psi\ t$
$J\ T$ L^2 $f_\eta\ a\ t\ , \phi\ t\ , \psi\ t$
Y T $, T$
Y T $, T$
$< \gamma_1 < \gamma$ $t > T$

$$|f_\eta\ a\ t\ , \phi\ t\ , \psi\ t\ \ |_{L^\infty(Y)} \leq C_6 e^{-\gamma_1(t-T)}.$$

A, Ψ $a\ t$ $a_\infty, \psi\ t$ ψ_∞
fl $\frac{\partial}{\partial t}\ a\ t\ , \psi\ t$ $-f_\eta\ a\ t\ ,\ \ , \psi\ t$

$$_Y \{|\frac{\partial a\ t}{\partial t}|, |\frac{\partial \psi\ t}{\partial t}|\} \leq C_6 e^{-\gamma_1(t-T)}.$$

A, Ψ $a_\infty,\ \ , \psi_\infty$
$f_\eta\ a\ t\ ,\ \ , \psi\ t$ Y $t \in\ T, \infty$ $a_\infty,\ \ , \psi_\infty\ \in C^\infty\ Y$
$\nabla^l a\ t\ , \nabla^l \psi\ t$ $A|_t, \Psi|_t$
$a_\infty,\ \ , \psi_\infty$
$\{g_t\}$ g_∞ C^∞ $g_t^*\ A|_t, \Psi|_t\ \to g_\infty^*\ a_\infty,\ \ , \psi_\infty$
$A|_t, \Psi|_t$ Y T, ∞
$t > T$ Y $-\infty, -T$ □
$a_\infty, \phi_\infty, \psi_\infty\ \in \mathcal{R}^*_{SW}\ Y, \eta$ $\delta_0^* \oplus Df_\eta\ a_\infty, \phi_\infty, \psi_\infty$
$\delta_0^* \oplus Df_\eta\ \theta,\ \ ,$ $\theta,$
$L^p_{k,\delta}$ ξ Y $\mathbf{R}$
ξ $e_\delta \cdot \xi$ L^p_k $e_\delta\ y, t$ $e^{\delta|t|}$ $|t|\ \geq$

$\leq \delta$ $\{\delta_\eta/ \ ,\gamma_1/ \ \}$ A,Ψ Y $\mathbf{R}$

$$D_{A,\Psi} \quad L^p_{k+1,\delta}\ \Omega^1\ Y \quad \mathbf{R}\ \oplus \Gamma\ W^+_{(4)} \quad \to L^p_{k,\delta}\ \ \Omega^0 \oplus \Omega^2_+\ \ Y \quad \mathbf{R}\ \oplus \Gamma\ W^+_{(4)}$$

δ

A,Ψ *regular* $D_{A,\Psi}$ $\mathcal{M}_{Y\times\mathbf{R}}$ *regular*

A,Ψ

$t\to\pm\infty$ $A,\Psi \ \in \mathcal{R}^*\ Y,\eta$

$$D_{A,\Psi} \quad L^p_{k+1}\ \Omega^1\ Y \quad \mathbf{R}\ \oplus \Gamma\ W^+_{(4)} \quad \to L^p_{k}\ \ \Omega^0 \oplus \Omega^2_+\ \ Y \quad \mathbf{R}\ \oplus \Gamma\ W^+_{(4)}$$

τ_Y $\tau_Y\ y,t$ $|t|$ e_δ $e^{\delta\tau_Y(y,t)}$

$|t| > T_0 >$ ∇_0,Ψ_0 Y $\mathbf{R}$

$$\nabla_0|_{Y\times[T_0,\infty)} \quad \frac{d}{dt} \quad a_+ \quad \phi_+ dt\ , \quad \Psi_0|_{Y\times[T_0,\infty)} \quad \psi_+,$$

$$\nabla_0|_{Y\times(-\infty,-T_0]} \quad \frac{d}{dt} \quad a_- \quad \phi_- dt\ , \quad \Psi_0|_{Y\times(-\infty,-T_0]} \quad \psi_-,$$

$a_\pm$ $\phi_\pm dt, \psi_\pm \ \in\ f_\eta^{-1}$ $\Omega^1_c\ Y \quad \mathbf{R}\ \oplus \Gamma\ W$
C^∞ $T^*\ Y \quad \mathbf{R}\ \oplus W$

$$\mathcal{A}^p_{k,\delta}\ Y \quad \mathbf{R} \qquad \nabla_0,\Psi_0 \qquad L^p_{k,\delta}\ \Omega^1_c\ Y \quad \mathbf{R}\ \oplus \Gamma\ W\ \ ,$$

$$\|\ a,\phi,\psi\ \|_{L^p_{k,\delta}(Y\times\mathbf{R})} \quad \|\ a \quad \phi dt,\psi\ \|_{L^p_{k,\delta}(Y\times\mathbf{R})} \quad \|e_\delta\cdot a\|_{L^p_k(Y\times\mathbf{R})} \quad \|e_\delta\cdot\phi\|_{L^p_k(Y\times\mathbf{R})}$$
$\|e_\delta\cdot\psi\|_{L^p_k(Y\times\mathbf{R})}$

$$\mathcal{G}^p_{k+1,\delta} \quad \{u\in L^p_{k+1,\mathrm{loc}}\ Y \quad \mathbf{R}, i\mathbf{R}\ |u \qquad \xi \qquad |t|\geq T_0 \qquad \xi\in L^p_{k+1,\delta}\}.$$

$\mathcal{B}^p_{k,\delta}\ Y \quad \mathbf{R} \quad \mathcal{A}^p_{k,\delta}\ Y \quad \mathbf{R}\ /\mathcal{G}^p_{k+1,\delta}$ c_-
a_-,ϕ_-,ψ_- c_+ a_+,ϕ_+,ψ_+
$\mathcal{M}_{Y\times\mathbf{R}}$
Y $\mathbf{R}$ $\mathcal{M}_{Y\times\mathbf{R}}$
$\coprod_{c,c'\in f_\eta^{-1}(0)}\mathcal{M}_{Y\times\mathbf{R}}\ c,\ c^{'}$ $\mathcal{M}_{Y\times\mathbf{R}}\ c,\ c^{'}$ $A,\ \Psi$
$t\to\infty$ A,Ψ $c^{'}$ $t\to-\infty$ A,Ψ c $\mathbf{R}$ $\mathcal{M}_{Y\times\mathbf{R}}$

$$\mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}\ c,c^{'} \qquad \{\ A,\Psi\ \in \mathcal{M}_{Y\times\mathbf{R}}\ c,c^{'}\ |J \qquad E_\eta\ A,\Psi\ |_{Y\times[0,\infty)} \quad -E_\eta\ A,\Psi\ |_{Y\times\mathbf{R}}\}.$$

fl t
§ $\mathcal{M}_{Y\times\mathbf{R}}\ c,c^{'}$ $\mathcal{M}_{Y\times\mathbf{R}}\ c,c^{'}\ /\mathbf{R}$
$\mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}\ c,c^{'}$
fl $D_{A,\Psi}$

Proposition 4.10. *The set of all perturbations $\eta\in\mathcal{P}^{'}_Y$ of which $\mathcal{M}_{Y\times\mathbf{R}}$ is regular is of Baire's first category.*

Proof: l $\mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}} \to \mathcal{B}_Y$
A, Ψ . t l A, Ψ a ϕ dt, ψ
l

$$F \quad \mathcal{B}^p_{k,\delta}\ Y \quad \mathbf{R} \quad \mathcal{P}_Y \to \mathcal{B}^p_{k,\delta}\ Y \quad \mathbf{R} \quad \mathcal{G}^p_{k+1,\delta} \quad \Omega^1\ Y, i\mathbf{R} \quad \Gamma\ W \quad ,$$

§ F a $\phi dt, \psi, \eta$ $\frac{\partial a}{\partial t} - d_a\phi$ $*_{g_Y} F_a - i\tau$ ψ, ψ
$\alpha, \frac{\partial\psi}{\partial t}$ $\partial_a^{\nabla_0+\alpha}\psi$ $\phi.\psi$ $\frac{\partial}{\partial t}$ a $\phi dt, \psi$ f_η a, ϕ, ψ $\mathcal{B}^p_{k,\delta}$ Y $\mathbf{R}$
$\mathcal{B}^p_k$ Y e_δ t

$$\begin{array}{ccc} \Omega^1\ Y, i\mathbf{R} \quad \Gamma\ W & & \Omega^1\ Y, i\mathbf{R} \quad \Gamma\ W \\ \downarrow\uparrow F & & \downarrow\uparrow f_\eta \\ \mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}} \quad \mathcal{P}_Y & \xrightarrow{l} & \mathcal{B}_Y \quad \mathcal{P}_Y, \end{array}$$

$\mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}} \sim F^{-1}$ $/\mathbf{R} \subset \mathcal{B}^p_{k,\delta}$ Y $\mathbf{R}$ $/\mathbf{R}$ F $\frac{\partial}{\partial t}$ $f_\eta \circ l^*$
l^* $\mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}} \to \mathcal{B}^*_Y$ DF $\frac{\partial}{\partial t}$ Df_η l^* $\cdot$ $\cdot$ Dl^*
$Im\ l^*$ $/$ $\emptyset$ A $, \Psi$ $/$ a $,$ $,$ l $\mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}} \to \mathcal{B}_Y \setminus \mathcal{B}^*_Y$
$l\ t$ A, Ψ $A|_t, \Psi|_t$ $\in \mathcal{B}^*$ A, Ψ $c \in f_\eta^{-1}$
$l\ t$ $\mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}$
$\mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}$ c, c' t_0 $l\ t_0$ $\mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}$ c, c' $\to \mathcal{B}^*_Y$
$\mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}$ c, c'
$\mathcal{P}'_Y$ c, c' $\mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}$ c, c'
$\mathcal{P}'_Y$ $\bigcap_{c,c'\in f_\eta^{-1}(0)} \mathcal{P}'_Y$ c, c'

§ □

η g_Y, α
admissible. $\mathcal{P}'_Y$
Y $\mathbf{R}$

$$F\ A, \Psi \qquad \frac{\partial}{\partial t} \quad Df_\eta\ c \quad a\ t\ , \phi\ t\ , \psi\ t \qquad N\ a\ t\ , \phi\ t\ , \psi\ t \quad ,$$

$c \in f_\eta^{-1}$ A, Ψ $a\ t$
$\phi\ t\ dt, \psi\ t$ N $a\ t\ , \phi\ t\ , \psi\ t$ $a\ t\ , \phi\ t\ , \psi\ t$
$D_{A,\Psi}$ $\frac{\partial}{\partial t}$ δ_0^* Df_η c δ δ
δ_0^* Df_η c Y $D_{A,\Psi}$ δ
δ_0^* Df_η c

5. Spectral flow and dependence on Riemannian metrics

U
fl
U

fl

Proposition 5.1. *For an admissible perturbation* η g_Y, α $\in \mathcal{P}'_Y$ *and a nondegenerate zero* a, ϕ, ψ $\in \mathcal{R}_{SW}$ Y, η f_η^{-1} *, we can associate an integer* μ_η a, ϕ, ψ $\in \mathbf{Z}$ *such that for* $A,$ $\in \mathcal{B}_{Y\times\mathbf{R}}$ a, ϕ, ψ , a', ϕ', ψ'

$$\mu_\eta \; e^{iu} \cdot \; a, \phi, \psi \qquad \mu_\eta \; a, \phi, \psi \; ,$$
$$IndexD_{A,\Phi} \qquad \mu_\eta \; a, \phi, \psi \; - \mu_\eta \; a', \phi', \psi' \; - dim\Gamma_{(a',\phi',\psi')},$$

where $\Gamma_{(a',\phi',\psi')}$ *is the isotropy subgroup of* a', ϕ', ψ' .

Proof: π_1 Y , $\to Y$ $L_{(4)}$ $W_{(4)}$ π_1^* $detW^{\pm}$ $\pi_1^* W^{\pm}$ $A,$ $\in \mathcal{A}_{L_{(4)}}$ $W_{(4)}$ $A,$ $|_{t\leq 0}$ a $\phi dt, \psi$ $A,$ $|_{t\geq 1}$ a' $\phi' dt, \psi'$ $D_{A,\Phi}$ $\frac{\partial}{\partial t}$ δ_t δ_t $\delta_0^* \oplus Df_\eta$ A t , t $D_{A,\Phi}$ fl δ_t

$$SF \; e^{iu} \cdot \; a, \phi, \psi \; , \; a, \phi, \psi \qquad D_{A,\Phi} \; A, \; |_{t=0}, \; A, \; |_{t=0} \; {}_{Y\times S^1}$$
$$- \; c_1 \; L_{(4)} \; {}^2 - \; \chi \; \sigma \; Y \; S^1 \; ,$$

χ σ Y S^1 c_1 $L_{(4)}$ ^{2} Y S^1 Y □

$\eta \in \mathcal{P}'_Y$ μ_η $\eta \in \mathcal{P}'_Y$ μ θ θ

Proposition 5.2. **(Definition)** *Two admissible perturbations* η_0 *and* η_1 *in* $\mathcal{P}'_Y$ *are (called) homotopic to each other through a 1-parameter family* η_t $\leq t \leq$ *in* $\mathcal{P}_Y$ *if and only if* μ_{η_0} μ_{η_1} .

Proof: η_0 η_1 § η_t $t \in$, η_t $< t_0 < t_1 \cdots < t_n <$ $\lambda_1, \lambda_2, \cdots, \lambda_n, \lambda_{n+1}$ λ_i δ_t δ_t $\theta,$ $t_{i-1} \leq t \leq t_i$ t_{-1} t_{n+1} n_i $\delta_{t_i} - \lambda Id$ $\lambda \in$ λ_{i+1}, λ_i n_i $-$ $\delta_{t_i} - \lambda Id$ $\lambda \in$ λ_i, λ_{i+1} $\frac{\partial}{\partial t}$ δ_t

DF_{η_t}

$$DF_{\eta_t} \qquad \sum_{i=0}^{n} n_i.$$

DF_{η_t} η_t

η_t

$$DF_{\eta_t} \qquad \mu_{\eta_0} \qquad - \mu_{\eta_1} \qquad .$$

fl

$_Y$ $_Y$ DF_{η_t}

η_0 η_1

fl ☐

$a, \phi, \psi \ \in \mathcal{R}^*_{SW} \ Y, \eta$

$$\mu_\eta \ a, \phi, \psi \qquad DF_\eta \quad , \ a, \phi, \psi \quad \in \mathbf{Z},$$

μ_η μ_η $\mu_\eta \ a, \phi, \psi$

μ_η $\eta_0 \in \mathcal{P}'_Y$

$\eta \in \mathcal{P}'_Y$ η' μ_η $\mu_{\eta'}$

Lemma 5.3. *For an admissible perturbation $\eta \in \mathcal{P}'_Y$, the Seiberg-Witten moduli space $\mathcal{R}_{SW} \ Y, \eta \quad f_\eta^{-1}$ is a compact 0-dimensional oriented manifold. The orientation is well-defined for a fixed homotopy class $\eta \in \mathcal{P}'_Y$.*

Proof: $\mathcal{R}_{SW} \ Y, \eta$ C^∞

$\mathcal{R}_{SW} \ Y, \eta$ fl

η

$\eta \in \mathcal{P}'_Y$

fl ☐

$\mathcal{R}^*_{SW} \ Y, \eta$

$\mathcal{R}^*_{SW} \ Y, \eta$

$\eta \in \mathcal{P}'_Y$

6. Monopole homology of integral homology 3-spheres

$\eta \in \mathcal{P}'_Y$ f_η

§ f_η

$Y \quad _T \mathbf{R}$

fl

Lemma 6.1. *For any* $c, c^{'} \in \mathcal{R}^* \ Y, \eta$ *and* $p \geq$ *, there exists a positive constant* C_p *such that for all* $A, \Psi \ \in \mathcal{M}^{bal}_{Y\times\mathbf{R}} \ c, c^{'}$ *in one component and* $a, \phi, \psi \ \in L^p \ \Omega^1 \ Y, i\mathbf{R} \ \oplus \Omega^0 \ Y, i\mathbf{R} \ \oplus \Gamma \ W$ *, we have*

$$C_p \| \ a, \phi, \psi \ \|_{L^p(Y\times\mathbf{R})} \leq \| \ \frac{\partial}{\partial t} \quad \delta_0^* \quad Df_\eta \ A, \Psi \quad ^* \ a, \phi, \psi \ \|_{L^p(Y\times\mathbf{R})},$$

where * *denotes the adjoint operator with respect to the* L^2*-norm.*

Proof: $D_{A,\Psi} \ ^*$ $A, \Psi \ \in \mathcal{M}^{\text{bal}}_{Y\times\mathbf{R}} \ c, c^{'}$ $D_{A,\Psi}$ $\frac{\partial}{\partial t}$ $\delta_0^* \oplus Df_\eta \ A, \Psi$ $c, c^{'} \in \mathcal{R}^* \ Y, \eta$

$$C_{(A,\Psi),p} \| \ a, \phi, \psi \ \|_{L^p(Y\times\mathbf{R})} \leq \| \ \frac{\partial}{\partial t} \quad \delta_0^* \quad Df_\eta \ A, \Psi \quad ^* \ a, \phi, \psi \ \|_{L^p(Y\times\mathbf{R})},$$

$a, \phi, \psi \ \in L^p \ \Omega^1 \ Y, i\mathbf{R} \ \oplus \Omega^0 \ Y, i\mathbf{R} \ \oplus \Gamma \ W$ $C_{(A,\Psi),p}$ A, Ψ $\mathcal{M}^{\text{bal}}_{Y\times\mathbf{R}} \ c, c^{'} \ \cap \mathcal{B}^p_k \ c, c^{'}$ C_p $C_{(A,\Psi),p}$ □

$\mathcal{M}^{\text{bal}}_{Y\times\mathbf{R}} \ c, c^{'}$ fl $\mu_\eta \ c \ -$ $\mu_\eta \ c^{'}$ $I_\eta \quad , \eta_0$ $I_\eta \quad , \eta_0$ $\mathcal{M}^{\text{bal}}_{Y\times\mathbf{R}} \ c, c^{'}$ $Y \quad \mathbf{R}$

$\chi_- \ t$ $\chi_- \ t$ $t \leq T_1 -$ $\chi_- \ t$ $t \geq T_1$ $|d\chi_-| \leq C_0$ C_0 T_1 $\geq T_0$ $A_-, \Psi_- \ \in \mathcal{M}^{\text{bal}}_{Y\times\mathbf{R}} \ c_-, c^{'}$ $I_\eta \quad , \eta_0$ A_-, Ψ_- $-\chi_- \ c^{'} \quad \chi_- \cdot \ A_-, \Psi_-$ $\Omega^1 \ Y, i\mathbf{R} \ \oplus \Gamma \ W \ \oplus \Gamma \ W$

Lemma 6.2. *There exist* $T_4 \geq T_0$ *and* C_7 *independent of* A_-, Ψ_- *such that for* $T_1 > T_4$ *and* $A_-, \Psi_- \ \in \mathcal{M}^{bal}_{Y\times\mathbf{R}} \ c_-, c^{'}$ *,* $p, q \geq$ *, we have*

$$\| \ A_-, \Psi_- \ - \ A_-, \Psi_- \ \|_{L^q(Y\times\mathbf{R})} \leq C_7 e^{-\delta(T_1-T_4)}, \quad \|F \ A_-, \Psi_- \ \|_{L^p(Y\times\mathbf{R})} \leq C_7 e^{-\delta(T_1-T_4)},$$

where $A^{'}, \Psi^{'} \ - \ A, \Psi \qquad A^{'} - A, \Psi^{'} - \Psi$ *.*

Proof: $A_-, \Psi_- \ - \ A_-, \Psi_-$ $Y \quad T_1 - \ , \infty$

$$_{Y} \ | \ A_-, \Psi_- \ - \ A_-, \Psi_- \ | \qquad _{Y} \ | \quad - \chi_- \quad c^{'} - \ A_-, \Psi_- \ | \leq C_{4,(A_-,\Psi_-)} e^{-\gamma_1(t-T_0)},$$

$t \geq T_0$

$$\| \ A_-, \Psi_- \ - \ A_-, \Psi_- \ \|_{L^q(Y\times\mathbf{R})} \leq C_{4,(A_-,\Psi_-)} \ \frac{Vol \ Y, g_Y}{\gamma_1 q} \ ^{1/q} e^{-\gamma_1(T_1-T_0-1)},$$

$\gamma_1 \ > \ \delta$ $e^{-\gamma_1(T_1-T_0-1)} \ \leq \ e^{-\delta(T_1-T_0-1)}$

A_-, Ψ_- $Y \quad T_1 - \ , T_1$ $C_{4,(A_-,\Psi_-)}$

$\mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}\ c_-, c^{'}$ $T_4 \geq T_0$
$T_1 > T_4$ C_7 A_-, Ψ_- □
$\mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}\ c_-, c^{'}$

$$U_{\varepsilon_-} \quad \{\ B, \quad \in \mathcal{B}_{Y\times\mathbf{R}}\ c_-, c^{'}\ | \qquad A, \Psi\ \in \mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}\ c_-, c^{'}$$

$$\|\ B, \quad -\ A, \Psi\ \|_{L^q(Y\times\mathbf{R})} < \varepsilon_-, \quad \|F\ B, \quad \|_{L^p(Y\times\mathbf{R})} < \varepsilon_-\}.$$

T_1^- $C_7 e^{-\delta(T_1-T_4)}$ $\varepsilon_-/$ $A_-, \Psi_-\ \in U_{\varepsilon_-}$
$T_1 \geq T_1^-$

Lemma 6.3. *There exists* ε_0^- *such that for* $<\varepsilon_- < \varepsilon_0^-$ *there is a constant* C_8^- *independent of* ε_- *with*

$$\|\ a, \phi, \psi\ \|_{L^p_1(Y\times\mathbf{R})} \leq C_8^- \|\ \frac{\partial}{\partial t} \quad \delta_0^* \quad Df_\eta\ B, \quad ^*\ a, \phi, \psi\ \|_{L^p(Y\times\mathbf{R})},$$

for all $B, \quad \in U_{\varepsilon_-}$.

Proof: $B, \quad \in U_{\varepsilon_-}$

$$\|\ \frac{\partial}{\partial t} \quad \delta_0^* \quad Df_\eta\ B, \quad ^*\ a, \phi, \psi\ \|_{L^p(Y\times\mathbf{R})} \geq \|\ \frac{\partial}{\partial t} \quad \delta_0^* \quad Df_\eta\ A_-, \Psi_-\ \ ^*\ a, \phi, \psi\ \|_{L^p(Y\times\mathbf{R})}$$
$$-\|\ \ B, \quad -\ A_-, \Psi_-\ \ \cdot\ a, \phi, \psi\ \|_{L^p(Y\times\mathbf{R})}.$$

$B, \quad -\ A_-, \ddot{\Psi}_-$ $D^*_{B,\Phi} - D^*_{A_-,\Psi_-}$
a, ϕ, ψ

$$\|\ \ B, \quad -\ A_-, \Psi_-\ \ \cdot\ a, \phi, \psi\ \|_{L^p(Y\times\mathbf{R})} \leq \|\ B, \quad -\ A_-, \Psi_-\ \|_{L^q(Y\times\mathbf{R})} \|\ \ a, \phi, \psi\ \|_{L^4(Y\times\mathbf{R})}$$
$$\leq C\varepsilon_- \|\ \ a, \phi, \psi\ \|_{L^p_1(Y\times\mathbf{R})}.$$

∇_0, Ψ_0 A_-, Ψ_- L^p_1
$Y \quad \mathbf{R}$

$$\begin{aligned}
\|\ a, \phi, \psi\ \|_{L^p_1(Y\times\mathbf{R})} & \quad \|\ a, \phi, \psi\ \|_{L^p_1(\nabla_0,\Psi_0)} \\
& \leq \quad C_9 \|\ a, \phi, \psi\ \|_{L^p_1(A_-,\Psi_-)} \\
& \leq \quad C_{10} \|\ \frac{\partial}{\partial t} \quad \delta_0^* \quad Df_\eta\ A_-, \Psi_-\ \ ^*\ a, \phi, \psi\ \|_{L^p} \quad C_9 \|\ a, \phi, \psi\ \|_{L^p} \\
& \leq \quad C_{10} \quad C_9/C_p\ \|\ \frac{\partial}{\partial t} \quad \delta_0^* \quad Df_\eta\ A_-, \Psi_-\ \ ^*\ a, \phi, \psi\ \|_{L^p(Y\times\mathbf{R})}.
\end{aligned}$$

ε_0^- $C\ C_{10} \quad C_9/C_p\ \varepsilon_0^- \leq\ /$ $C_8^- \geq\ C_p^{-1}$
□

$L^p_1 \hookrightarrow L^q$ $/$
$/q \geq\ /p$ $Q_{(B,\Phi)} \quad D^*_{B,\Phi} \circ\ D_{B,\Phi} \circ D^*_{B,\Phi}\ ^{-1}$

$\frac{\partial}{\partial t}$ δ_0^* Df_η $B,$ $D_{B,\Phi}$

$\|Q_{(B,\Phi)}\ a,\phi,\psi\ \|_{L^q(Y\times\mathbf{R})} \le C_{11}\|Q_{(B,\Phi)}\ a,\phi,\psi\ \|_{L_1^p(Y\times\mathbf{R})} \le C_{11}C_8^-\|\ a,\phi,\psi\ \|_{L^p(Y\times\mathbf{R})},$

$B,\ \in U_{\varepsilon_-}$ $C_{11}C_8^-$
$B,\ \in U_{\varepsilon_-}$

$U_{\varepsilon_+}\ \{\ B,\ \in \mathcal{B}_{Y\times\mathbf{R}}\ c',c_+\ |$ $A,\Psi\ \in \mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}\ c',c_+$

$\|\ B,\ -\ A,\Psi\ \|_{L^q(Y\times\mathbf{R})} < \varepsilon_+,\ \ \|F\ B,\ \|_{L^p(Y\times\mathbf{R})} < \varepsilon_+\},$

$c',c_+\ \in\ \mathcal{R}^*\ Y,\eta$ A_+,Ψ_+ $-\ \chi_+\ c'$ $\chi_+\ A_+,\Psi_+$
$A_+,\Psi_+\ \in \mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}\ c',c_+$ χ_+ $t\ \ge\ -T_1$ χ_+ $t\ \le\ -T_1$
$T_1^+ >\ ,\varepsilon_0^+ >$ C_8^+ $T_1 > T_1^+$ $< \varepsilon_+ < \varepsilon_0^+$
$A_+,\Psi_+\ \in U_{\varepsilon_+}$ $Q_{(B,\Phi)}$ $C_{11}C_8^+$
$B,\ \in U_{\varepsilon_+}$
$A_\pm,\Psi_\pm$ ε_+ ε_- $T_1 \ge$ $\{T_1^+,T_1^-\}$
$A_\pm,\Psi_\pm$
Y $\mathbf{R}$ $T_3 > T_2 \ge T_1 \ge$ $\{T_1^\pm\}$ Y T_2,T_3
A_-,Ψ_-
Y $\mathbf{R},g_-$ A_+,Ψ_+ Y $\mathbf{R},g_+$ $T_3 > T_2$
N_- Y T_2,T_3 N_+ Y $-T_3,-T_2$ $g_\pm$ g_Y dt^2 T_3 T_2K
$K >$
θ_- χ_K $-\frac{1}{\ln K}$ $\frac{t}{T_2K}$
Y $T_2,T_2K\ \subset Y$ $\mathbf{R}$ χ_K t T_2K χ_K t T_2 $\|\nabla\chi_K\|_{L^4(Y\times\mathbf{R})}$
$CT_2^{-3/4}\frac{(1-K^{-3})^{1/4}}{\ln K} \to$ T T_2 $\{T_1^\pm\}$ $K\to\infty$ $K_0 >$
$K \ge K_0$ $\|\nabla\chi_K\|_{L^4(Y\times\mathbf{R})}$ K
f_T $N_- \to N_+$ $f_T\ y,t$ $y,-t$ Y
$\{T\}\subset N_-$ Y $\{-T\}\subset N_+$
N_- N_+ f_T Y
$Y\ {}_T\,\mathbf{R}$

$$Y\ {}_T\,\mathbf{R}\quad Y\quad -\infty,TK\ \cup_{f_T} Y\quad -TK,\infty\ ,$$

$N_\pm$ f_T $K\ \ge\ K_0$
$Y\ {}_T\,\mathbf{R}$ g_Y dt^2 f_T

Lemma 6.4. *Let F $E_1 \to E_2$ be a C^1-map between Banach spaces with first order Taylor expansion $F\ \xi$ F DF ξ $N\ \xi$. Assume that DF has a finite dimensional kernel and a right inverse Q such that*

$$\|QN\ \xi_1\ -QN\ \xi_2\ \|_{E_1} \le C\ \|\xi_1\|_{E_1}\ \ \|\xi_2\|_{E_1}\ \|\xi_1-\xi_2\|_{E_1},$$

for some constant C. *Let* $\varepsilon \quad / \quad C\ .$ *If* $\|QF \quad \|_{E_1} \leq \varepsilon/\ $, *then there exists a* C^1-*function* $u \quad K_\varepsilon \to ImQ$ *with* $F\ \xi \quad u\ \xi$ *for all* $\xi \in K_\varepsilon$, *and furthermore we have the estimate*

$$\|u\ \xi\ \|_{E_1} \leq -\|QF \quad \|_{E_1} \quad -\|\xi\|_{E_1},$$

where $K_\varepsilon \quad DF \quad \cap \{\xi \in E_1 \quad \|\xi\|_{E_1} < \varepsilon\}$.

$F\ A, \Psi$

$Y \quad \mathbf{R} \quad F \quad \frac{\partial}{\partial t} \quad \delta_0^* \quad Df_\eta\ A, \Psi \quad D_{\tilde{A},\tilde{\Psi}}$

$DF \quad N\ a, \phi, \psi$

$L_1^p \cap L^q\ T_{(\tilde{A},\tilde{\Psi})}\mathcal{B}_{Y\times\mathbf{R}} \quad E_1 \quad L^p\ \Omega^1\ Y, i\mathbf{R}\ \oplus \Omega^0\ Y, i\mathbf{R}\ \oplus \Gamma\ W \quad E_2$

$A_+, \Psi_+\ \in \mathcal{M}_{Y\times\mathbf{R}}\ c^{'}, c_+ \quad A_-, \Psi_-\ \in \mathcal{M}_{Y\times\mathbf{R}}\ c_-, c^{'} \quad c_\pm, c^{'} \in \mathcal{R}^*\ Y, \eta \quad A, \Psi$

$$A, \Psi \quad \begin{cases} A_-\ t \quad TK - T\ , \Psi_-\ t \quad TK - T & t \leq -\ TK - T \\ \rho \cdot c^{'} & -\ TK - T\ \leq t \leq\ TK - T \\ A_+\ t - \quad TK - T\ \ , \Psi_+\ t - \quad TK - T & TK - T\ \leq t, \end{cases}$$

$\rho \in \Gamma_{c'} \quad c^{'}$

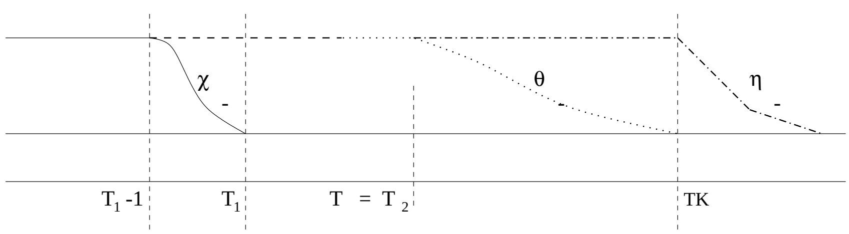

Figure 1. ff

Proposition 6.5. *For* $< \varepsilon < min\{\varepsilon_0^\pm\}$ *as in Lemma 6.3,* T *(fixed) in Lemma 6.2, there is a constant* C *independent of* ε *such that the operator* $D_{\tilde{A},\tilde{\Psi}}$ *has a bounded right inverse* $Q_{(\tilde{A},\tilde{\Psi})}$ *with,* $/ \quad /q \geq \ /p$,

$$\|Q_{(\tilde{A},\tilde{\Psi})}\ a, \phi, \psi\ \|_{L_1^p(Y\times_T\mathbf{R})} \leq C_{13}\|\ a, \phi, \psi\ \|_{L^p(Y\times_T\mathbf{R})},$$

$$\|Q_{(\tilde{A},\tilde{\Psi})}\ a, \phi, \psi\ \|_{L^q(Y\times_T\mathbf{R})} \leq C_{13}\|\ a, \phi, \psi\ \|_{L^p(Y\times_T\mathbf{R})}.$$

Proof: θ_+ θ_- Y $\mathbf{R}, g_+$ θ_+ Y $_T\,\mathbf{R}$ η_- $- \theta_+$ Y $_T\,\mathbf{R} \to \mathbf{R}$ η_- Y $_T\,\mathbf{R}$ Y $\mathbf{R}, g_-$ A_-, Ψ_- η_- θ_- η_- θ_- η_+ $Q_{(\tilde{A}_\pm, \tilde{\Psi}_\pm)}$ $D_{\tilde{A}_\pm, \tilde{\Psi}_\pm}$ A, Ψ

$$Q\ a, \phi, \psi \quad \eta_- Q_{(\tilde{A}_-, \tilde{\Psi}_-)}\ a, \phi, \psi\ _- \quad \eta_+ Q_{(\tilde{A}_+, \tilde{\Psi}_+)}\ a, \phi, \psi\ _+,$$

$a, \phi, \psi\ _-$ Y $-\infty, TK \subset Y$ $\mathbf{R}, g_-$ a, ϕ, ψ $a, \phi, \psi\ _-$ $a, \phi, \psi\ _+$ $\eta_\pm$

$$\eta_-\ a, \phi, \psi\ _- \quad \eta_+\ a, \phi, \psi\ _+ \quad a, \phi, \psi\ .$$

$$D_{\tilde{A}, \tilde{\Psi}} \circ Q\ a, \phi, \psi \quad a, \phi, \psi \quad d\eta_- Q_{(\tilde{A}_-, \tilde{\Psi}_-)}\ a, \phi, \psi\ _- \quad d\eta_+ Q_{(\tilde{A}_+, \tilde{\Psi}_+)}\ a, \phi, \psi\ _+.$$

$Q_{(\tilde{A}_\pm, \tilde{\Psi}_\pm)}$ $D_{\tilde{A}, \tilde{\Psi}} \circ Q - Id$ C^∞ $\|d\eta_\pm\|_{L^4(Y \times \mathbf{R})}$ $Q_{(\tilde{A}, \tilde{\Psi})}$ $Q \circ\ D_{\tilde{A}, \tilde{\Psi}} \circ Q\ ^{-1}$ □

Remark:

Theorem 6.6. *If* $< \varepsilon < min\{\varepsilon_0^\pm\}$ *and* $T \geq$ $\{T_1^\pm\}$ *, then there is a well-defined gluing map*

$$G_T\ \ \mathcal{M}^{bal}_{Y \times \mathbf{R}}\ c_-, c^{'} \quad \mathcal{M}^{bal}_{Y \times \mathbf{R}}\ c^{'}, c_+ \quad TK, \infty\ \to \mathcal{M}_{(Y \times_T \mathbf{R})}\ c_-, c_+\ ,$$

which is a local diffeomorphism with a fixed $K \geq K_0$.

Proof: $A_\pm, \Psi_\pm\ \in U_{\varepsilon_\pm}$ $T >$ $\{T_1^+, T_1^-\}$ ε_+ $\varepsilon_-\ < \varepsilon_0^\pm$ $Q_{(\tilde{A}, \tilde{\Psi})}$

$$\|Q_{(\tilde{A}, \tilde{\Psi})}\ a, \phi, \psi\ \|_{L^q(Y \times_T \mathbf{R})} \leq C_{13} \|\ a, \phi, \psi\ \|_{L^p(Y \times_T \mathbf{R})}.$$

$$\|Q_{(\tilde{A}, \tilde{\Psi})} F\ \ A, \Psi\ \|_{L^q(Y \times_T \mathbf{R})} \leq C_{13}\ \varepsilon_+ \quad \varepsilon_-\ .$$

..

$$\|Q_{(\tilde{A}, \tilde{\Psi})} N\ a, \phi, \psi\ - Q_{(\tilde{A}, \tilde{\Psi})} N\ a^{'}, \phi^{'}, \psi^{'}\ \|_{L^q(Y \times_T \mathbf{R})}$$

$$\leq C_{14}\ \|\ a, \phi, \psi\ \|_{L^q(Y \times_T \mathbf{R})} \quad \|\ a^{'}, \phi^{'}, \psi^{'}\ \|_{L^q(Y \times_T \mathbf{R})}\ \cdot \|\ a, \phi, \psi\ -\ a^{'}, \phi^{'}, \psi^{'}\ \|_{L^q(Y \times_T \mathbf{R})}.$$

C^1 u $D_{\tilde{A},\tilde{\Psi}} \to \mathcal{M}_{Y\times_T\mathbf{R}}$ c_-, c_+ A, Ψ
u A, Ψ ρ $\|u$ A, Ψ $\rho\,\|_{L^q(Y\times_T\mathbf{R})}$ ρ $id \in U$
C^∞

G_T A_-, Ψ_- $, id,$ A_+, Ψ_+ A, Ψ u A, Ψ id .

A, Ψ $\in U_\varepsilon$ G_T A, Ψ A, Ψ u A, Ψ id
$\varepsilon_n \to$ $T_n \to$ ∞ $\varepsilon_n < \varepsilon_\pm$
A_n, Ψ_n $\in U^c_{\varepsilon_n} \cap \mathcal{M}_{Y\times_{T_n}\mathbf{R}}$ c_-, c_+ G_{T_n} $U^c_{\varepsilon_n}$
U_ε G_{T_n}
A_-, Ψ_- $\coprod$ A_+, Ψ_+
$A_\pm, \Psi_\pm$ $\varepsilon_n \to$
$y_0,$
$A_\pm, \Psi_\pm$ Y $\mathbf{R}, g_\pm$ $_{t\to\pm\infty}$ $A_\mp, \Psi_\mp$
$c^{'}$ G_{T_n} $c^{'}$
A_-, Ψ_- $\in \mathcal{M}^{\text{bal}}_{Y\times\mathbf{R}}$ $c_-, c^{'}$ A_+, Ψ_+ $\in \mathcal{M}^{\text{bal}}_{Y\times\mathbf{R}}$ $c^{'}, c_+$
t

G_{T_n} A_-, Ψ_- $, \rho,$ A_+, Ψ_+

$$\|\; A_n, \Psi_n \; - G_{T_n} \; A_-, \Psi_- \;, \rho, \; A_+, \Psi_+ \; \|_{L^q(Y\times_{T_n}\mathbf{R})} < \varepsilon/ \;,$$

A_n, Ψ_n $\in U_{\varepsilon/2} \cap \mathcal{M}_{Y\times_{T_n}\mathbf{R}}$ c_-, c_+ A_n, Ψ_n
U_{ε_n} G_T

□

Remark:
$\mathcal{M}_{Y\times\mathbf{R}}$ c_-, c_+ $\leq$ $c^{'}$ $\mathcal{M}_{Y\times\mathbf{R}}$ c_-, c_+
$\mathcal{M}_{Y\times\mathbf{R}}$ $c_-,$ Γ_Θ $\mathcal{M}_{Y\times\mathbf{R}}$ $, c_+$.
$\mathcal{M}_{Y\times\mathbf{R}}$ $c_-,$ $\mathcal{M}_{Y\times\mathbf{R}}$ $, c_+$ $\mathcal{M}_{Y\times\mathbf{R}}$ c_-, c_+
$\mathcal{M}_{Y\times\mathbf{R}}$ c_-, c_+ $c_\pm$ $\in \mathcal{R}^*_{\text{SW}}$ Y, η
$c^{'}$
§
§

$$G_T \quad \mathcal{M}^{\text{bal}}_{Y\times\mathbf{R}} \; c_-, c^{'} \quad \mathcal{M}^{\text{bal}}_{Y\times\mathbf{R}} \; c^{'}, c_+ \quad TK, \infty \; \to \mathcal{M}^{\text{bal}}_{Y\times_T\mathbf{R}} \; c_-, c_+ \; \sim \mathcal{M}^{\text{bal}}_{Y\times\mathbf{R}} \; c_-, c_+ \;,$$

T $T \geq$ $\{T_1^\pm\}$ $K \geq K_0$

Definition 6.7. a, ϕ, ψ $a^{'}, \phi^{'}, \psi^{'}$ f_η *chain solution*
a, ϕ, ψ $a^{'}, \phi^{'}, \psi^{'}$ $A_1,$ $_1$ $, ...,$ $A_n,$ $_n$ Y $\mathbf{R}$
$c_{i-1}, c_i \in f_\eta^{-1}$ $t \to \mp\infty$ a, ϕ, ψ c_0 c_n $a^{'}, \phi^{'}, \psi^{'}$
$A_i,$ $_i$ $\in \mathcal{M}_{Y\times\mathbf{R}}$ c_{i-1}, c_i $\leq i \leq n$

$\{\; A_\alpha,$ $_\alpha$ $\} \in \mathcal{M}_{Y\times\mathbf{R}}$ a, ϕ, ψ , $a^{'}, \phi^{'}, \psi^{'}$ *(weakly) convergent* $A_1,$ $_1$ $, ...,$ $A_n,$ $_n$

$\{t_{\alpha,1} \leq \cdots \leq t_{\alpha,n}\}_\alpha$ $t_{\alpha,i} - t_{\alpha,i-1} \to \infty$ $\alpha \to \infty$
i $t^*_{\alpha,i}$ $A_\alpha,$ $_\alpha$ $A_\alpha \circ - t_{\alpha,i}$, $_\alpha \circ - t_{\alpha,i}$ $A_i,$ $_i$

Theorem 6.8. *Let* $\{ A_\alpha, \ _\alpha \} \in \mathcal{M}_{Y\times\mathbf{R}} \ a,\phi,\psi \ , \ a^{'},\phi^{'},\psi^{'}$ *be a sequence of Seiberg-Witten solutions with uniformly bounded action over* Y $\mathbf{R}$. *Then there exists a subsequence converging to a chain solution* $A_1, \ _1$ *,...,* $A_n, \ _n$ *such that*

$$Ind\, D_{A_\alpha,\Phi_\alpha} \quad \sum_{i=1}^{n} Ind\, D_{A_i,\Phi_i} \quad \sum_{i=1}^{n} \mu_\eta \ c_i \ - \mu_\eta \ c_{i-1} \ .$$

Proof: G
E_η ﬂ §

□

Proposition 6.9. *The compactification of* $\mathcal{M}_{Y\times\mathbf{R}} \ c_0, c_{n+1}$ *can be described as*

$$\overline{\mathcal{M}_{Y\times\mathbf{R}} \ c_0, c_{n+1}} \quad \cup \quad {}^{n+1}_{i=1}\mathcal{M}_{Y\times\mathbf{R}} \ c_{i-1}, c_i \ ,$$

the union over all sequence $c_0, c_1, \cdots, c_{n+1} \in \mathcal{R}^*_{SW} \ Y, \eta$ *such that* $\mathcal{M}_{Y\times\mathbf{R}} \ c_{i-1}, c_i$ *is nonempty for all* $\leq i \leq n$.

For any sequence $c_0, c_1, \cdots, c_{n+1} \in \mathcal{R}^*_{SW} \ Y, \eta$ *, there is a gluing map*

$$G \quad {}^{n+1}_{i=1}\mathcal{M}^{bal}_{Y\times\mathbf{R}} \ c_{i-1}, c_i \qquad {}^{n+1} \to \overline{\mathcal{M}_{Y\times\mathbf{R}} \ c_0, c_{n+1}} \ ,$$

where $^{n+1}$ $\{ \ \lambda_0, \cdots, \lambda_n \ \in \ -\infty, \infty \ ^{n+1} \quad \lambda_{i-1} < \lambda_i, \ \leq i \leq n\}$.

The image of G *is a neighborhood of* ${}^{n+1}_{i=1}\mathcal{M}^{bal}_{Y\times\mathbf{R}} \ c_{i-1}, c_i$ *in the compactification with chain solutions.*

The restriction of G *to* ${}^{n+1}_{i=1}\mathcal{M}^{bal}_{Y\times\mathbf{R}} \ c_{i-1}, c_i \quad Int \quad ^{n+1}$ *is an orientation-preserving diffeomorphism onto its image.*

G

□

$\mathcal{R}^n_{SW} \ Y, \eta$ a, ϕ, ψ f_η
$\mu_\eta \ a, \phi, \psi \ - \mu_\eta$ n **monopole chain group** $MC_n \ Y, \eta$
$\mathcal{R}^n_{SW} \ Y, \eta$ η
ﬂ μ_η I_η η_0 μ_η $- \mu_{\eta_0}$
$\eta_0 \in \mathcal{P}_Y$ μ_η I_η η_0
§ $\mathcal{R}_{SW} \ Y, \eta$ $\eta \in \mathcal{P}^{'}_Y$
$\mathcal{R}_{SW} \ Y, \eta$ $\eta \in \mathcal{P}^{'}_Y$
I_η η_0

$$\partial \quad MC_n \ Y, \eta \ \to MC_{n-1} \ Y, \eta$$

$$\partial \ a, \phi, \psi \qquad \sum_{(a^{'},\phi^{'},\psi^{'})\in MC_{n-1}(Y,\eta)} \qquad \mathcal{M}^{\mathrm{bal}}_{Y\times\mathbf{R}} \ a, \phi, \psi \ , \ a^{'}, \phi^{'}, \psi^{'} \quad \cdot \ a^{'}, \phi^{'}, \psi^{'} \ .$$

Proposition 6.10. *Let* $\partial \quad MC_n \; Y,\eta \;\to MC_{n-1} \; Y,\eta$ *be defined as above. Then* $\partial \circ \partial$
.

Proof:

$$\partial^2 \; c_0 \qquad \sum_{c_1 \in \mathcal{R}_{SW}^{n-1}(Y,\eta)} \sum_{c_2 \in \mathcal{R}_{SW}^{n-2}(Y,\eta)} \mathcal{M}^{\text{bal}}_{Y\times\mathbf{R}} \; c_0, c_1 \; \cdot \; \mathcal{M}^{\text{bal}}_{Y\times\mathbf{R}} \; c_1, c_2 \; c_2,$$

c_i $a_i, \phi_i, \psi_i \; \in \; \mathcal{R}^*_{SW} \; Y, \eta \; i$, ,
$c_2 \in \mathcal{R}^{n-2}_{SW} \; Y, \eta$ c_0, c_2
$\mathcal{M}^2_{Y\times\mathbf{R}} \; c_0, c_2$ $\mathcal{M}^{\text{bal},2}_{Y\times\mathbf{R}} \; c_0, c_2$
$\mathcal{M}^{\text{bal},1}_{Y\times\mathbf{R}} \; c_0, c_1$ $\mathcal{M}^{\text{bal},1}_{Y\times\mathbf{R}} \; c_1, c_2$ $c_1 \in \mathcal{R}^{n-1}_{SW} \; Y, \eta$
c_1 U f_η Γ_{c_1}

I_η η_0 I_η η_0
ﬂ

$$\sum_{c_1 \in \mathcal{R}_{SW}^{n-1}(Y,\eta)} \mathcal{M}^{\text{bal}}_{Y\times\mathbf{R}} \; c_0, c_1 \; \cdot \; \mathcal{M}^{\text{bal}}_{Y\times\mathbf{R}} \; c_1, c_2 \qquad \partial \mathcal{M}^{\text{bal},2}_{Y\times\mathbf{R}} \; c_0, c_2 \qquad .$$

□

Y
$\eta \in \mathcal{P}'_Y$ I_η η_0
Monopole Homology

$$MH_* \; Y \; \eta \qquad \partial_* / \quad \partial_{*+1}, \qquad * \in \mathbf{Z}.$$

$MH_* \; Y \; \eta$ I_η η_0
$MH_* \; Y \; \eta$ $\mathcal{R}^*_{SW} \; Y, \eta$

7. Homomorphisms induced by cobordisms

Y

ﬂ I_η η_0 μ_η $- \mu_{\eta_0}$
I_η η_0
X Y_1 $\mathbf{R}_+$ Y_2 $\mathbf{R}_-$
Y_1 Y_2 τ_X $X \to$ $, \infty$
$\tau_X \; x$ x Y_1 $\mathbf{R}_+ \cup Y_2$ $\mathbf{R}_-$
$\tau_X \; y, t$ $|t|$ $y, t \; \in Y_1$ $\mathbf{R}_+ \cup Y_2$ $\mathbf{R}_-$ $|t| > t_0 >$ e_δ $e^{\delta \tau_X(x)}$

τ_X $\frac{d}{dt}$ α $\frac{d}{dt}$ β
∇_0 X

$$\nabla_0|_{Y_1\times[T_5,\infty)} \quad \frac{d}{dt} \quad \alpha, \quad \nabla_0|_{Y_2\times(-\infty,-T_5]} \quad \frac{d}{dt} \quad \beta.$$

$W_X^\pm$ Ω^1 $X, AdP \oplus\Gamma$ $W_X^\pm$
C^∞ $T^*X\otimes AdP \oplus\Gamma W_X^\pm$

$$\mathcal{A}^p_{k,\delta}\ X \quad \nabla_0, \quad L^p_{k,\delta}\ \Omega^1\ X, AdP \ \oplus\Gamma\ W_X^\pm \ ,$$

$\|c\|_{L^p_{k,\delta}}$ $\|e_\delta\cdot c\|_{L^p_k}$ $c\in\Omega^1$ $X, AdP \oplus\Gamma$ $W_X^\pm$
$\mathcal{G}^p_{k+1,\delta}$ $L^p_{k+1,\delta}$ $W_X^\pm$ $\mathcal{B}^p_{k,\delta}\ X$
$\mathcal{A}^p_{k,\delta}\ X\ /\mathcal{G}^p_{k+1,\delta}$ η_1 g_{Y_1},α_1 η_2 g_{Y_2},α_2
f_{η_1} f_{η_2} f_{η_1} Y_1 f_{η_2}
Y_2 η_1 η_2
Y_1 $\mathbf{R}_+$ Y_2 $\mathbf{R}_-$
$\mathcal{B}^p_{k,\delta}\ Y_1$ $\mathbf{R}_+$ $\mathcal{B}^p_{k,\delta}\ Y_2$ $\mathbf{R}_-$ δ

$\mathcal{M}et\ X$ $_X$ g_X α_X $\mathcal{M}_{\eta_X}\ c,c'$
η_X g_X,α_X

$$\mathcal{M}_{\eta_X}\ c,c' \quad \mu_{\eta_1}\ c\ -\mu_{\eta_2}\ c' \quad - \quad \chi \quad \sigma\ X\ .$$

$\mathcal{M}_{\eta_X}\ c,c'$
$H^1\ X,\mathbf{R}$ $H^0\ X,\mathbf{R}\ \oplus H^2_+\ X,\mathbf{R}$
Ψ_* $\Psi_*\ X\ \eta_X$ $MC_*\ Y_1\ \eta_1\ \to MC_*\ Y_2\ \eta_2$

$$\Psi_*\ c \qquad \sum_{c'\in\mathcal{R}^*_{SW}(Y_2,\eta_2)} \mathcal{M}^0_{\eta_X}\ c,c'\ \cdot c', \quad c\in\mathcal{R}^*_{SW}\ Y_1,\eta_1\ ,$$

$\mathcal{M}^0_{\eta_X}\ c,c'$ c c' X
$\mu_{\eta_1}\ c\ -\mu_{\eta_2}\ c'$ $-\frac{1}{2}$ χ $\sigma\ X$

Proposition 7.1. *Given a cobordism X and perturbation data $\eta_X\in\mathcal{M}et\ X$ $_X$ as before, the homomorphism Ψ_* is a chain map shifting the degree by $\frac{1}{2}$ χ $\sigma\ X$. Furthermore the induced homomorphism*

$$\Psi_* \quad \Psi_*\ X\ \eta_X \quad MH_*\ Y_1\ \eta_1\ \to MH_*\ Y_2\ \eta_2$$

on the monopole homologies depends only on the cobordism X and the data I_{η_1} $_{Y_1}$ η_{0,Y_1} and I_{η_2} $_{Y_2}$ η_{0,Y_2} .

Proof: fl I_{η_1} $_{Y_1}$ η_{0,Y_1} I_{η_2} $_{Y_2}$ η_{0,Y_2} Y_1 Y_2 § $\partial_{Y_2} \circ \Psi_* \ c$

$$\sum_{c' \in \mathcal{R}^*_{SW}(Y_2,\eta_2)} \mathcal{M}^0_{\eta_X} \ c, c' \ \cdot \sum_{c_1 \in \mathcal{R}^*_{SW}(Y_2,\eta_2)} \mathcal{M}^{\mathrm{bal}}_{Y_2 \times \mathbf{R}} \ c', c_1 \ \cdot c_1,$$

$$\mathcal{M}^1_{\eta_X} \ c, c_1$$

$$\sum_{d' \in \mathcal{R}^*_{SW}(Y_1,\eta_1)} \mathcal{M}^{\mathrm{bal}}_{Y_1 \times \mathbf{R}} \ c, d \ \cdot \sum_{d \in \mathcal{R}^*_{SW}(Y_1,\eta_1)} \mathcal{M}^0_{\eta_X} \ d, c_1 \ .$$

c' d $_{Y_2}$ $_{Y_1}$ I_{η_1} $_{Y_1}$ η_{0,Y_1} I_{η_2} $_{Y_2}$ η_{0,Y_2} fl

$\partial_{Y_2} \circ \Psi_* \ c$ $\Psi_* \circ \partial_{Y_1}$ □

$\Psi_* \ X \ \eta_X$ $U \ \eta_U$ Y_1 Y_2 $V \ \eta_V$ Y_2 Y_3 η_U η_V Y_2 $W \ \eta_W$ Y_1 Y_3

$$\Psi_* \ W \ \eta_W \quad \Psi_* \ V \ \eta_V \ \circ \Psi_* \ U \ \eta_U \ .$$

$MH_* \ Y, \eta$ η g_Y, α I_η η_0

g_Y^{-1} g_Y^1 α_{-1} α_1 $I_{\eta_{-1}}$ η_0 I_{η_1} η_0 η_t g^t_Y, α_t $\{\eta_t \quad g^t_Y, \alpha_t \mid -\infty \le t \le \infty\}$ η_t g_Y^{-1}, α_{-1} $t \le -$ η_t g^1_Y, α_1 $t \ge$ Y $\mathbf{R}$

$$\frac{\partial \psi}{\partial t} \quad \partial_{a_t}^{\nabla_{g^t_Y} + \alpha_t} \psi \quad , \quad \frac{\partial a_t}{\partial t} \quad *_{g^t_Y} F \ a_t \quad \alpha_t \quad i\tau_{g^t_Y} \ \psi, \psi \ .$$

$c \in \mathcal{R}^*_{SW} \ Y, \eta_{-1}$ $c' \in \mathcal{R}^*_{SW} \ Y, \eta_1$ $\mathcal{M}_\Lambda \ c, c'$ $\mathcal{B}^p_{k,\delta} \ c, c'$

$$\Psi_\Lambda \quad MC_n \ Y \ \eta_{-1} \ \to MC_n \ Y \ \eta_1$$

$$\Psi_\Lambda \ c \quad \sum_{c' \in \mathcal{R}^n_{SW}(Y,\eta_1)} \mathcal{M}^0_\Lambda \ c, c' \ \cdot c', \quad c \in \mathcal{R}^n_{SW} \ Y, \eta_{-1} \ .$$

Proposition 7.2. *Let* $\{\eta_t \quad g^t_Y, \alpha_t \ |t \in \mathbf{R}\}$ *be an family of admissible perturbations as defined above such that* $Ind\, DF_{\eta_t}$ *(the same homotopy class). Then*

If is a constant family of admissible perturbations (g_Y^t g_Y, α_t α)*, then* Ψ_Λ *id.*

Ψ_Λ *is a chain map:* $\partial\Psi_\Lambda$ $\Psi_\Lambda\partial$.

Given two families and $'$ *of admissible perturbations joining* g_Y^{-1}, α_{-1} *to* g_Y^0, α_0 *and from* g_Y^0, α_0 *to* g_Y^1, α_1 *, we have* $\Psi_{\Lambda\circ\Lambda'}$ $\Psi_\Lambda \circ \Psi_{\Lambda'}$.

If a family $_0$ *of admissible perturbations connecting* g_Y^{-1}, α_{-1} *and* g_Y^1, α_1 *can be deformed into another* $_1$ *by admissible families* $_\lambda$ $\leq \lambda \leq$ *, then the two monopole chain maps* Ψ_{Λ_0} *and* Ψ_{Λ_1} *are chain homotopic to each other.*

Proof: η_t g_Y, α $\mathcal{M}_\Lambda^0$ c, c'
$\mathcal{M}_{Y\times\mathbf{R}}^0$ c, c' $\mathcal{M}_\Lambda^0$ c, c'
c_t Y $\mathbf{R}$ c_t c c' $\mathcal{M}_\Lambda^0$ c, c' $\delta_{cc'}$
Ψ_Λ id

$\mathcal{M}_\Lambda$ c, c'
$\mathcal{M}_\Lambda$ α, β

$$\cup_{c_{-1}}\mathcal{M}_{Y\times\mathbf{R}}^{\mathrm{bal}}\ c, c_{-1}\quad {}_{c_{-1}}\ \mathcal{M}_\Lambda\ c_{-1}, c' \coprod \cup_{c_1}\mathcal{M}_\Lambda\ c, c_1\quad {}_{c_1}\ \mathcal{M}_{Y\times\mathbf{R}}^{\mathrm{bal}}\ c_1, c' .$$

$c_{\pm1} \in \mathcal{R}_{SW}$ $Y, \eta_{\pm1}$ $\mathcal{M}_{Y\times\mathbf{R}}$ c, c_{-1} Y
$-\infty, -$ η_{-1} $\mathcal{M}_{Y\times\mathbf{R}}^{\mathrm{bal}}$ c_1, c'
η_1 $\mathcal{M}_\Lambda^1$ c, c' $\mathcal{M}_\Lambda$ c, c'
$\partial\Psi_\Lambda$ $\Psi_\Lambda\partial$
$c_{\pm1}$
U

$I_{\eta_{-1}}$ η_0 I_{η_1} η_0

$\mathcal{M}_{\Lambda*\Lambda'}$ T_6 α, β Y $\mathbf{R}$
$*_{T_6}$ $'$

$$*_{T_6}\ ' \quad \begin{cases} \eta_{-1} \quad g_Y^{-1}, \alpha_{-1} & -\infty < t \leq -T_6 - \\ \quad g_Y^{t+T_6}, \alpha_{t+T_6} & -T_6 - \quad \leq t \leq -T_6 \\ \eta_0 & -T_6 \leq t \leq T_6 \\ ' \quad g_Y^{t-T_6}, \alpha_{t-T_6} & T_6 \leq t \leq T_6 \\ \eta_1 & T_6 \quad \leq t < \quad \infty. \end{cases}$$

T_6 $\mathcal{M}_{\Lambda*\Lambda'}$ T_6 c, c' $T_6 \geq T_7$

$$\cup_{c_0}\overline{\mathcal{M}}_\Lambda\ c, c_0\quad {}_{c_0}\ \overline{\mathcal{M}}_{\Lambda'}\ c_0, c' .$$

$\overline{\mathcal{M}}_\Lambda\ c, c_0$ $\mathcal{M}_\Lambda\ c, c_0\ /\ \Gamma_c$ Γ_{c_0}
$\overline{\mathcal{M}}_\Lambda\ c, c_0$ $_{c_0}\ \overline{\mathcal{M}}_{\Lambda'}\ c_0, c^{'}$ $c^{'}$

$$\Psi_{\Lambda'} \circ \Psi_\Lambda\ c \quad \sum_{c_0} \quad \overline{\mathcal{M}}^0_\Lambda\ c, c_0 \ \cdot \ \overline{\mathcal{M}}^0_{\Lambda'}\ c_0, c^{'} \ \cdot c^{'}.$$

$T_6 \ \to$
$\mathcal{M}_{\Lambda * \Lambda'}\ T_6\ c, c^{'}$ $c^{'}$ $\Psi_{\Lambda * \Lambda'}\ c$ $\sum \mathcal{M}^0_{\Lambda * \Lambda'}\ c, c^{'} \ \cdot c^{'}$
$\cup_{0 \le T_6 \le T_7} \mathcal{M}^0_{\Lambda * \Lambda'}\ T_6\ c, c^{'}$ $\mathcal{M}^0_{\Lambda * \Lambda'}$ $c, c^{'}$ $\mathcal{M}^0_{\Lambda * \Lambda'}\ T_7\ c, c^{'}$

$$\overline{\mathcal{M}}_\Lambda\ c, c_0 \quad \mu_{\eta_{-1}}\ c\ - \ _{\eta_t \in \Lambda, \eta_t \to \eta_0} \mu_{\eta_t}\ c_0\ - \quad \Gamma_{c_0}$$

$$\overline{\mathcal{M}}_{\Lambda'}\ c_0, c^{'} \quad _{\eta_t \in \Lambda', \eta_t \to \eta_0} \mu_{\eta_t}\ c_0\ - \mu_{\eta_1}\ c^{'}\ .$$

$$_{\eta_t \in \Lambda, \eta_t \to \eta_0} \mu_{\eta_t}\ c_0 \quad _{\eta_t \in \Lambda', \eta_t \to \eta_0} \mu_{\eta_t}\ c_0 \quad \mu\ c_0\ .$$

$\mu_{\eta_{-1}}\ c\ - \mu\ c_0$ c_0 $\mu\ c_0\ - \mu_{\eta_1}\ c^{'}$
$\mu_{\eta_{-1}}\ c$ $\mu_{\eta_1}\ c^{'}$ *If these spectral flows* $I_{\eta_{\pm 1}}$ η_0 *are not fixed to be same, then the above argument becomes invalid.*

$_i\ i$,
η_{-1} η_1 ^{0} ^{1}
$_s$ $\{\eta^s_t \quad g^{s,t}_Y, \alpha^s_t\ , \quad \le s \le \ , \ - \quad \le t \le \ \}$
$_s$ ^{0} $\le s \le \frac{1}{4}$ $_s$ ^{1}
$\frac{3}{4} \le s \le$

$\mathcal{HM}\ c, c^{'}$ $\cup_{0 \le s \le 1} \mathcal{M}_{\Lambda_s}\ c, c^{'}$

$\mathcal{HM}\ c, c^{'}$ $\{ \quad , s\ | \quad \in \mathcal{M}_{\Lambda_s}\ c, c^{'}\ , \quad \le s \le \ \} \subset \mathcal{B}^p_{k,\delta}\ c, c^{'}$, ,

$\mathcal{HM}$ η^s_t
$\mu_{\eta_{-1}}\ c\ - \mu_{\eta_1}\ c^{'}$

$$\mathcal{M}_{\Lambda_1}\ c, c^{'} \quad \{\ \} \coprod \mathcal{M}_{\Lambda_0}\ c, c^{'} \quad \{\ \},$$

$$\cup_{(s,c_0)} \mathcal{M}_{\Lambda_s}\ c, c_0 \quad \mathcal{M}_{\eta_1}\ c_0, c^{'} \coprod \cup_{(s,\gamma)} \mathcal{M}_{\eta_{-1}}\ c, c_0 \quad \mathcal{M}_{\Lambda_s}\ c_0, c^{'}\ .$$

$\mathcal{M}_{\Lambda_s}\ c, c_0$ $\mathcal{M}_{\Lambda_s}\ c_0, c^{'}$
$-$ $< \ s \ <$ H
$MC_*\ Y\ \eta_{-1}\ \to MC_*\ Y\ \eta_1$

$$H\ c \quad \sum_{c_0} \sum_s \quad \mathcal{M}^0_{\Lambda_s}\ c, c_0\ \cdot c_0, \qquad c \in \mathcal{R}^n_{SW}\ Y, \eta_{-1}\ , c_0 \in \mathcal{R}^{n+1}_{SW}\ Y, \eta_1\ .$$

c_0 U $\mathcal{M}_{\eta_1}\ c_0, c^{'}$
$\mathcal{M}_{\eta_{-1}}\ c, c_0$ I_{η_1} η_0 $I_{\eta_{-1}}$ η_0 $c^{'} \in \mathcal{R}^n_{SW}\ Y, \eta_1$

$$\Psi_{\Lambda_0}\ c\ - \Psi_{\Lambda_1}\ c \quad H \circ \partial_{\eta_{-1}}\ c \quad \partial_{\eta_1} \circ H\ c\ .$$

Ψ_{Λ_0} Ψ_{Λ_1} □

$MH_*\ Y\ \ \eta_{\pm 1}$

I_{η_1} η_0 $I_{\eta_{-1}}$ η_0
U Y $MH_*\ Y\ \ \eta$
$MH_*\ Y\ \ I_\eta$ η_0 Y

$$MH_{SWF}\quad \{I_\eta \quad \eta_0 \quad \eta \in \mathcal{P}^{'}_Y\} \to \{MH_*\ Y, I_\eta \quad \eta_0 \quad \eta \in \mathcal{P}^{'}_Y\}.$$

η_0 $I_{\eta^{'}_0}$ η_0
MH_{SWF}
Y MH_{SWF}

$\{I_\eta \quad \eta_0 \quad \eta \in \mathcal{P}^{'}_Y\}$

Y

8. Relative Seiberg-Witten invariants

X $b_1\ X\ >$ Y
X Y $-$,
Y X § *Fixing* I_η η_0 *should be understood through this section.*

Definition 8.1. X Y

$$q_{X,Y,\eta} \quad \sum_{c \in \mathcal{R}^*_{SW}(Y,\eta)} \mathcal{M}^0_X\ c\ \cdot c,$$

$\mathcal{R}^*_{SW}\ Y, \eta$ f_η I_η η_0

$$\mu_\eta\ c \quad SF\ c,$$

$\mathcal{M}^0_X\ c$ $\mu_\eta\ c$ $\mathcal{M}_X$ $-\ c_1\ \pi^*\ L\ \ ^2 -\ \ \chi\ \ \sigma\ \ X\ \ --\ \ \chi\ \ \sigma\ \ X\ ,$

c_1 L Y $q_{X,Y,\eta}$
$-\frac{1}{4}$ χ σ X

Proposition 8.2. *For* $q_{X,Y,\eta} \in MC_{\mu_X}$ Y, η *with* μ_X $-\frac{1}{4}$ χ σ X *and a fixed class* I_η η_0 *, we have* $\partial_Y \circ q_{X,Y,\eta}$ *, i.e.,* $q_{X,Y,\eta}$ *is a monopole cycle in the sense of §6.*

Proof:

$$\partial_Y \circ q_{X,Y,\eta} \; c \quad \sum_{c \in \mathcal{R}^{\mu}_{SW}(Y,\eta)} \sum_{c' \in \mathcal{R}^{\mu-1}_{SW}(Y,\eta)} \mathcal{M}^0_X \; c \; \cdot \; \mathcal{M}^1_{Y\times\mathbf{R}} \; c, c' \; \cdot c'.$$

c c' f_η
$\mathcal{M}^1_X$ c' c'

$\mathcal{M}_X$ $\mathcal{M}_{Y\times\mathbf{R}}$ $, c'$

$\{\mathcal{M}_X$ $\mathcal{M}_{Y\times\mathbf{R}}$ $, c'\}$ $\mathcal{M}_X$ Γ_Θ $\mathcal{M}_{Y\times\mathbf{R}}$ $, c' \geq$.
c $\partial_Y \circ q_{X,Y,\eta}$ $q_{X,Y,\eta}$

$\square$

$q_{X,Y,\eta}$ g_X g_X
$\eta \in \mathcal{P}'_Y$
$q_{X,Y,\eta}$ g_X g_X $g_X|_Y$
I_η η_0

Proposition 8.3. *Let* g^i_X i , *be two generic metrics on* X *with induced metric* g^i_Y *generic such that* I_{η_1} η_0 I_{η_2} η_0 *and* η_i g^i_Y, α_i *. Then there exist* $c' \in MC_{\mu_X+1}$ *with* μ_X $-\frac{1}{4}$ χ σ X *such that we have*

$$q_{X,Y,\eta_2} \; g^2_X \; - q_{X,Y,\eta_1} \; g^1_X \quad \partial \; c' \; .$$

In particular, q_{X,Y,η_2} g^2_X q_{X,Y,η_1} g^1_X *as the monopole homology class in* MH_{μ_X} Y, I_{η_i} η_0 .

Proof: $\{g^{t+1}_X\}_{0\leq t\leq 1}$ X $I_{\eta_{t+1}}$ η_0
t η_{t+1} $g^{t+1}_X|_Y, \alpha_{t+1}$ $\mathcal{M}^0_X$ g^{t+1}_X c
c $\{\mathcal{M}^0_X \; g^{t+1}_X \; c\}_{0\leq t\leq 1}$
X ,
$\mathcal{B}_X$ g^{t+1}_X c

$$\partial \; \{\mathcal{M}^0_X \; g^{t+1}_X \; c\}_{0\leq t\leq 1}$$

$$\{ \; \} \; \mathcal{M}^0_X \; g^1_X \; c \coprod -\{ \; \} \; \mathcal{M}^0_X \; g^2_X \; c \coprod \partial \sum_{\mu_{\eta_{t+1}}(c)-\mu_{\eta_{t+1}}(c')=-1} \; , \quad \mathcal{M}^{-1}_X \; g^{t+1}_X \; c' \; .$$

$\langle\partial_Y c^{'}, c\rangle$, $\mathcal{M}_X^{-1}$ g_X^{t+1} $c^{'}$ $c^{'}$

I_{η_1} η_0

$$q_{X,Y,\eta_2}\ g_X^2\ \ c\ - q_{X,Y,\eta_1}\ g_X^1\ \ c\ \ \ \langle\partial_Y c^{'}, c\rangle.$$

q_{X,Y,η_i} g_X^i i ,

□

Y $-Y$ μ_η c

$-$ $-\mu_\eta$ c

MC_{μ_η} Y, η $CF_{-1-\mu_\eta}$ $-Y, \eta$

Theorem 8.4. *For a smooth 4-manifold* X X_0 ${}_Y X_1$ *with* b_2^+ X_i $>$ i , *and* Y *an integral homology 3-sphere, the Seiberg-Witten invariant of the 4-manifold* X *is given by the Kronecker pairing of* MH_* Y I_η η_0 *with* MH_{-1-*} $-Y$ I_η η_0 *for* $q_{X_0,Y,\eta}$ *and* $q_{X_1,-Y,\eta}$*; assume that the moduli space* $\mathcal{M}_X$ *does not split to* $\mathcal{M}_{X_i}$ *through the stretching-neck process,*

$$\langle,\rangle\ \ MH_*\ Y\ I_\eta\ \ \eta_0\ \ \ MH_{-1-*}\ -Y\ I_\eta\ \ \eta_0\ \to \mathbf{Z}\ \ q_{SW}\ X\ \ \ \langle q_{X_0,Y,\eta}, q_{X_1,-Y,\eta}\rangle.$$

More precisely, q_{SW} X_0 ${}_Y X_1$ $\sum_c$ $\mathcal{M}^0_{X_0,Y,\eta}$ c $\cdot$ $\mathcal{M}^0_{X_1,-Y}$ $-c$, *where* I_η η_0 *is fixed. The invariant* q_{SW} X *is independent of the choice of* I_η η_0 .

Proof: Y

I_η η_0 b_2^+ X $>$

X

$\mathcal{M}_{X_0}$ c Γ_c $\mathcal{M}_{X_1}$ c $\mathcal{M}_X.$

$\mathcal{M}^0_{X_0,Y,\eta}$ c $\cdot$ $\mathcal{M}^0_{X_1,-Y,\eta}$

$-c$ c / X

X_0, Y $X_1, -Y$ $\mathcal{R}^*_{SW}$ Y, η

q_{SW} X $\langle q_{X_0,Y,\eta}, q_{X_1,-Y,\eta}\rangle$

q_{SW} X

I_η η_0 □

Remark:

$\mathcal{M}_{X_0,Y}$ $\mathcal{M}_{X_1,-Y}$ b_2^+ X_i $>$ i ,

I_η , η_0 U Y

X X_0 ${}_Y X_1$

§

$\mathcal{M}_{X_i}$ i ,

U

Y S^3

L

fl

MH_{SWF}

S^3 SU U

fl

Remark:

fl U

Acknowledgement

fl

.. ..

References

[ ] *Casson's invariant for oriented homology 3-spheres, an exposition,* N V **36** P P

[ ] N z j *A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of the second order* P **36**

[ ] V P *Spectral asymmetry and Riemannian geometry* P P **77** **78** **79**

[ ] D *The Thurston Norm and three-dimensional Seiberg-Witten Theory* O **33**

[ ] L E *Self adjoint elliptic operators and manifold decompositions, Part I, II, III* P **49** N 8 8 8 N 8 **52** N

[ ] L *Seiberg-Witten-Floer homology and gluing formula*

[ ] D *The Seiberg-Witten Equations and 4-Manifolds Topology* V **33** N

[8] D *Polynomial invariants for smooth 4-manifolds* V **29** N

[ ] D P Ox Ox P

[ ] D D *Quasiconformal 4-manifolds* 8 8

[ ] *Immersed spheres in 4-manifolds and the immersed Thom conjecture* **19** N

[ ] *An instanton invariant for 3-manifolds* P **118** 88

[ ] D P V **1** N Y 8

[ ] N H *Harmonic Spinors* **14**

[ ] P *The genus of embedded surfaces in the projective plane* L **1** 8 8

[ ] L L *Floer homologies of rational homology 3-spheres*

[ ] L *Casson-Lin's invariant and Floer homology* V **6** N 8 8

[8] L *Floer homology for connected sums of homology 3-spheres* D ff **40**

[ ] L *Cap-product structures on the Fintushel-Stern spectral sequence* P P **131** 8

[ ] L O *Elliptic operators on noncompact manifolds* N P **12** 8

[ ] *Seiberg-Witten-Floer homology and Heegaard splitting* V **7** N

[ ] L *Equivariant Seiberg-Witten Floer homology* **9**

[ ] *SW = Milnor torsion* L **3**

[ ] Z z *A product formula for the Seiberg-Witten invariants and the generalized Thom conjecture* D ff **44** N 88

[ ] *A local Mayer-Vietoris principle for Yang-Mills moduli spaces* P D 8

[ ] P Oz Y *Seiberg-Witten monopoles on Seifert fibered spaces* **5** 8

[ ] L N N 8

[8] V P *Invariants of the smooth structure on an algebraic surface arising from the Dirac operator* z V **40** N

[ ] D *The minimal genus of an embedded surface of non-negative square in a rational surface* **20** N

[ ] D

[ ] D *Removable singularities and a vanishing theorem for Seiberg-Witten invariants* P ¨

[ ] *The Seiberg-Witten invariants and symplectic forms* L **1** 8 8

[ ] *More constraints on symplectic forms from Seiberg-Witten equations* L **2**

[ ] *SW = Gr: From Seiberg-Witten equations to pseudo-holomorphic curves* **9** 8 8

[ ] *Gr = SW: From pseudo-holomorphic curves to Seiberg-Witten solutions* **51**

[ ] Y *Equivariant and Bott-type Seiberg-Witten Floer homology*
/ 8
[ ] E *Topological Quantum Field Theory* P **117** 88 8
[8] E *Monopoles and 4-manifolds* L **1**

Department of Mathematics, Oklahoma State University
Stillwater, Oklahoma 74078-0613
E-mail address: wli@@math.okstate.edu