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Abstract

For every positive integer n, we exhibit a cofinite subgroup Γ_n of the mapping class group of a surface of genus at most two such that Γ_n admits an epimorphism onto a free group of rank n. We conclude that $H^1(\Gamma_n;\mathbb{Z})$ has rank at least n and the dimension of the second bounded cohomology of each of these mapping class groups is the cardinality of the continuum. In the case of genus two, the groups Γ_n can be chosen not to contain the Torelli group. Similarly for hyperelliptic mapping class groups. We also exhibit an automorphism of a subgroup of finite index in the mapping class group of a sphere with four punctures (or a torus) such that it is not the restriction of an endomorphism of the whole group.

1. Introduction

It is well-known that the first homology group of the mapping class group of a closed orientable surface of genus g is trivial for $g \geq 3$ and isomorphic to $\mathbb{Z}_{12}$ and $\mathbb{Z}_{10}$ if $g = 1$ and $g = 2$ respectively. It follows that the first cohomology of this group is trivial. N. V. Ivanov (Problem 2.11(A) in [11]) asked whether $H^1(\Gamma;\mathbb{Z})$ is trivial for any subgroup Γ of finite index in the mapping class group. In the case $g \geq 3$, this question was answered affirmatively by J. D. McCarthy [14] for subgroups Γ containing the Torelli group, the subgroup of the mapping class group consisting of those mapping classes that act trivially on the first homology of the surface. For arbitrary subgroups of finite index, the problem is still open. It was also shown by McCarthy [14] and Taherkhani [18] that the mapping class group of a closed orientable surface of genus 2 contains subgroups of finite index with nontrivial first cohomology. All of the examples of McCarthy contain the Torelli group. More precisely, he shows that if r is an integer divisible by 2 or 3, then the kernel of the action of the mapping class group on the mod r homology of the surface has nontrivial first cohomology. It is not clear whether the examples of Taherkhani contain the Torelli group, because his calculations are carried out by computer.

The purpose of this paper is to give an elementary construction of a sequence Γ_n of subgroups of finite index in the mapping class group of an orientable surface of genus at most 2 and in the hyperelliptic mapping class group such that Γ_n admits a homomorphism onto a finitely generated free group of rank n. In the case of a closed orientable surface of genus 2, we can choose these subgroups in such a way that they do not contain the Torelli group. This shows that for any positive integer n, there is a subgroup of finite index whose first cohomology has rank at least n. Another application is that the dimension of

OR Z

Γ

φ $\Gamma \to \Gamma$ φ

2. Definitions and preliminaries

S g p

q ${}^{q}_{g,p}$ $S \to S$

p q

${}^{q}_{g}$, ${}_{g,p}$ ${}_{g}$ ${}^{q}_{g,0}$, ${}^{0}_{g,p}$ ${}^{0}_{g,0}$

${}^{q}_{g,p}$ ${}^{q}_{g,p}$

${}^{q}_{g,p}$ ${}^{q}_{g,p}$

p

${}_{g}$ S

S xyz

J x, y, z $-x, y, -z$ y

$\jmath$ J

$${}_{g} \quad \{f \in \quad {}_{g} \mid f\jmath \quad \jmath f\}$$

$\jmath$ ${}_{g}$ g

J g S J

π $S \to R$ g R

$$\longrightarrow \quad {}_{2} \longrightarrow \quad {}_{g} \xrightarrow{\pi_*} \quad {}_{0,2g+2} \longrightarrow \quad ,$$

${}_{2}$ ${}_{g}$ $\jmath$

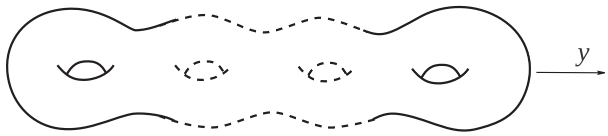

Figure 1. $\mathbb{R}^3$

3. Finite index subgroups with large H^1

Theorem 3.1. *Suppose that $g \leq$. If g , suppose, in addition, that p q . For any positive integer n, there is a subgroup Γ_n of finite index in $_{g,p}^{q}$ such that there is an epimorphism from Γ_n onto a free group F_n of rank n and $\Gamma_{n+1} \subset \Gamma_n$. In the case of g and p q , the group Γ_n can be chosen not to contain the Torelli group.*

Proof. n
n

$$\to F_{n-2} \to \quad {}_{0,n} \to \quad {}_{0,n-1} \to ,$$

F_{n-2} $n -$
$n -$ $_{0,3}$
$_{0,4}$
n
Γ_2 $_{g,p}^{q}$ $\Gamma_2 \to F_2$
g $_{0,p}^{q} \to$ $_{0,4}$

p $q -$ $_{0,p}^{q}$ p $_{0,p}^{q}$
Γ_2 $_{0,p}^{q}$
g
φ $_{1,p}^{q} \to$ $_1$ $_1$
SL , $_{1},$ $_1$ $_1$
$_1$ Γ_2 φ^{-1} $_{1},$ $_1$
g
φ $_{2,p}^{q} \to$ $_2$ $_2$ $_2$
π_* $_2 \to$ $_{0,6}$
$_{0,6}$ $_{0,4}$ Γ_2 φ^{-1} π_*^{-1} $_{0,6}$
Γ_2 $_{2,p}^{q}$
n f $\Gamma_2 \to F_2$ F_n F_2
$n -$ F_n n Γ_n f^{-1} F_n

OR Z

$^{q}_{g,p}$ f Γ_n F_n

g p q Γ_n
S
xyz S
a xz
S a J J a a S
J π $S \to R$ R
J $P_1, P_2, \ldots, P_6$
R π a R
P_1, P_2, P_3
π a
P_3 P_6 R c
π a P_1, P_2, P_4, P_5 δ R
P_2 P_4 c
d δ
t_c t_d $_{0,4}$ F_2
n F_2 $t_d, t_c t_d t_c^{-1}, t_c^2 t_d t_c^{-2}, \ldots, t_c^{n-2} t_d t_c^{-n+2}$ t_c^{n-1}
F_n n F_n F_2 $n-$ n
t_c^2 F_n
a J π a
$a \to \pi$ a π_* t_a $t^2_{\pi(a)}$
$_{0,6}$ ϕ $_{0,6} \to$ $_{0,4}$
P_3 P_6 ϕ $t_{\pi(a)}$ t_c ϕ π_* t_a
ϕ $t^2_{\pi(a)}$ t_c^2
Γ_2 Γ_3 π_*^{-1} ϕ^{-1} F_4 Γ_n π_*^{-1} ϕ^{-1} F_n
n Γ_k k
k $t_a \in$ $_2$ Γ_k $\square$

Remark 3.1. p $q \leq$ $^{q}_{0,p}$

- $p \leq$ $q \leq$
- p, q ,
- p, q ,
- p, q , ,
- $\oplus$ p, q ,
- $\oplus$ $\oplus$ p, q ,

Remark 3.2. n Γ_n $_2$

Corollary 3.2. *Suppose that g . For any positive integer n, there is a subgroup Γ_n of finite index in the hyperelliptic mapping class group $_g$ such that there is an epimorphism from Γ_n onto a free group F_n of rank n.*

Proof. $_g$

$_{0,2g+2}$ □

Corollary 3.3. *Suppose that $g \leq$. If g , suppose, in addition, that p q . For any positive integer n, there is a subgroup Γ_n of finite index in $^q_{g,p}$ such that the rank of H^1 Γ_n is at least n.*

Corollary 3.4. *For any positive integer n, there is a subgroup Γ_n of finite index in the hyperelliptic mapping class group $_g$ such that the rank of H^1 Γ_n is at least n. Moreover, in the case of g the subgroup Γ_n of $_2$ can be chosen so that it does not contain the Torelli group.*

4. The second bounded cohomology

$^q_{g,p}$ $g \leq$

$_g$

G

$$C_b^k \; G \; \mathbb{R} \qquad \{f \quad G^k \to \mathbb{R} \mid f \; G^k \qquad \}.$$

$$\delta_b^k \quad C_b^k \; G \; \mathbb{R} \; \to C_b^{k+1} \; G \; \mathbb{R}$$

$$\delta_b^k \; f \quad x_0, \ldots, x_k \qquad f \; x_1, \ldots, x_k \quad \sum_{i=1}^{k} - \; {}^i f \; x_0, \ldots, x_{i-1}x_i, \ldots, x_k$$

$$- \; {}^{k+1} f \; x_0, \ldots, x_{k-1} \; .$$

$\{C_b^k \; G \; \mathbb{R} \; , \delta_b^k\}$ G

$H_b^* \; G \; \mathbb{R}$ $C_b^k \; G \; \mathbb{R}$

$$||f|| \qquad \{ \; |f \; x_1, x_2, \ldots, x_k \; | \; \mid x_i \in G \},$$

$H_b^k \; G \; \mathbb{R}$ $H_b^k \; G \; \mathbb{R}$

k k

$G \to \mathbb{R}$

$C_b^k \; G \; \mathbb{R}$ $C^k \; G \; \mathbb{R}$

$G^k \to \mathbb{R}$

$H^* G \mathbb{R}$ G $C_b^k G \mathbb{R} \hookrightarrow C^k G \mathbb{R}$
$H_b^k G \mathbb{R} \to H^k G \mathbb{R}$ k
$H^2_{b,2} G \mathbb{R}$
G $PX G$
G $PX G$ f $G \to \mathbb{R}$
$|f x$ $f y - f xy| \leq C$ $f x^n$ $nf x$ $x, y \in G$ C
f $X G$ $G \to \mathbb{R}$
$H^2_{b,2} G \mathbb{R}$ $PX G /X G$

.

Lemma 4.1. *Let G be a finitely generated group and let H be a subgroup of finite index in G. The map τ $PX G \to PX H$ induced by the restriction is injective and the quotient space $PX H /\tau PX G$ is finite dimensional.*

Theorem 4.2 . *Let G and F be two groups and let σ $G \to F$ be an epimorphism. Then σ induces an injective linear map $H_b^2 F \mathbb{R} \to H_b^2 G \mathbb{R}$.*

Theorem 4.3 . *Suppose that n . If F_n is a free group of rank n, then the dimension of the space $H_b^2 F_n \mathbb{R}$ is equal to the cardinal of the continuum.*

Theorem 4.4. *Let G be a finitely presented group and let H be a subgroup of finite index in G. Suppose that there is a homomorphism from H onto a free group F_n of rank n . Then the dimension of the space $H_b^2 G \mathbb{R}$ is equal to the cardinal of the continuum.*

Proof. K
$C_b^k K \mathbb{R}$ $H_b^k K \mathbb{R}$
k
F_n H
$H_b^2 H \mathbb{R}$ H $H^2 H \mathbb{R}$
$X H$ $H^2_{b,2} H \mathbb{R}$ $PX H$

$H^2_{b,2} G \mathbb{R}$ $H^2_{b,2} G \mathbb{R}$
$H_b^2 G \mathbb{R}$ □

Theorem 4.5. *Suppose that $g \leq$. If g , suppose, in addition, that p q . Then the dimension of H_b^2 ${}^q_{g,p}$ $\mathbb{R}$ is equal to the cardinal of continuum.*

Proof. ${}^q_{g,p}$ □

Theorem 4.6. *The dimension of the second bounded cohomology group H_b^2 $_g$ $\mathbb{R}$ of the hyperelliptic mapping class group $_g$ is equal to the cardinal of continuum.*

Proof. $_g$ □

Remark 4.1. p $q \leq$ $^{q}_{0,p}$

5. Automorphisms of cofinite subgroups of mapping class groups

S g p

S
p g p g g S
$S \to S$
$_{g,p}$

$_{g,p}$ $_{g,p}$
g p g $p \leq$
g, p , g, p ,

$_{g,p}$
$_{g,p}$ g, p
, , , g, p ,

Lemma 5.1. *Let F_n be a nonabelian free group of rank n. If H is a proper subgroup of finite index in F_n, then there exists an automorphism φ $H \to H$ such that φ is not the restriction of any endomorphism of F_n.*

Proof. F_n $\{y, x_1, x_2, \ldots, x_{n-1}\}$
H k H k $n -$ K
F_n k F_n
H K H
$\{y^j x_i y^{-j}, y^k \mid \quad \leq i \leq n - \quad , \quad \leq j \leq k - \quad \}$
φ $H \to H$ φ yx_1y^{-1} $yx_1y^{-1}x_1$
H φ
$F_n \to F_n$ φ φ y^k y^k
φ y y φ x_i F_n φ
∎

Theorem 5.2. *If g, p is equal to , , , or , , then there exists a subgroup Γ of finite index in the mapping class group $_{g,p}$ and an automorphism φ $\Gamma \to \Gamma$ such that φ is not the restriction of any endomorphism of $_{g,p}$.*

Proof. g $_{1,0}$ $_{1,1}$
SL , SL , F_2
SL , Γ
F_2 φ $\Gamma \to \Gamma$

F_2 SL ,
F_2
g p S
P_1, P_2, P_3, P_4 i , , α_i P_i P_{i+1}
a b α_1 α_2
$_{0,4}$ S
t_a t_b w_i α_i
w_i P_i P_{i+1} w_1 ^{2} t_a w_2 ^{2} t_b w_3 ^{2}
α_3 .
$_{0,n}$ n w_1, w_2 w_3
$_{0,4}$ H_1 $_{0,4}$
w_i t_a t_b H_1 $_{0,4}$
φ $_{0,4} \to$ $_{0,4}$ φ t_a t_a φ t_b
$t_a t_b$ φ $_{0,4}$ φ w_1 w_2
t_a φ w_1 ^{2} $t_a t_b$ φ w_2 ^{2} t_a
$t_a t_b$ H_1 $_{0,4}$ t_b w_2 ^{2} H_1 $_{0,4}$
φ $_{0,4}$ $\square$

References

[ ] j *Bounded cohomology of subgroups of mapping class groups*
6 6 8
[ ] *Braids, links and mapping class groups* P
P P N
[ ] H H *On the mapping class groups of closed surfaces as covering spaces* : R 66 P
P P N 8
[4] *Suites exactes en cohomologie bornée réelle des groupes discrets* R
P ´ **320**
[ ] R *Some remarks on bounded cohomology* R :
P 8 N Y
N Y 8 6 P P P N
8
[6] H E D *Bounded cohomology and non-uniform perfection of mapping class groups* **144** 6
[ ] R *Some remarks on bounded cohomology*
L LN **204** 6
[8] N V *Foundations of the theory of bounded cohomology* V Z
N L O LO **143** 8 6 8
[ ] N V *The second bounded cohomology group* Z N L O
LO 6 88 **6**
8 8 4
[ ] N V *Automorphisms of complexes of curves and of Teichmüller spaces*
R N **1997** 6 666

[ ] R *Problems in low-dimensional topology* z z
/ P P
[ ] z *Automorphisms of complexes of curves on punctured spheres and on punctured tori* **95** 8
[ ] *Bounded cohomology of certain groups of homeomorphisms* P **94** 8 44
[4] D *On the first cohomology group of cofinite subgroups in surface mapping class groups* **40** 4 4 8
[ ] Y *Bounded cohomology and* ^{1}*-homology of surfaces* **23** 84 46 4
[6] *Structure of the mapping class groups of surfaces: a survey and a prospect* V P 488
[ ] *Existence of non-Banach bounded cohomology* **37** 8
[8] *The Kazhdan property of the mapping class group of closed surfaces and the first cohomology group of its cofinite subgroups* E **9** 6 4

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