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U(1)-invariant special Lagrangian 3-folds in $\mathbb{C}^3$ and special Lagrangian fibrations

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Abstract

This is a survey of the author's series of three papers [8, 9, 10] on *special Lagrangian 3-folds* (*SL 3-folds*) in $\mathbb{C}^3$ invariant under the U(1)-action $(z_1, z_2, z_3) \mapsto (e^{i\theta}z_1, e^{-i\theta}z_2, z_3)$, and their sequel [11] on special Lagrangian fibrations and the SYZ Conjecture.

We briefly present the main results of these four long papers, giving some explanation and motivation, but no proofs. The aim is to make the results and ideas accessible to String Theorists and others who have an interest in special Lagrangian 3-folds and fibrations, but have no desire to read pages of technical analysis.

Let N be an SL 3-fold in $\mathbb{C}^3$ invariant under the U(1)-action above. Then $|z_1|^2 - |z_2|^2 = 2a$ on N for some $a \in \mathbb{R}$. Locally, N can be written as a kind of graph of functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ satisfying a nonlinear Cauchy–Riemann equation depending on a, so that $u + iv$ is like a holomorphic function of $x + iy$. When $a = 0$ the equations may have singular points where u, v are not differentiable, which leads to analytic difficulties.

We prove existence and uniqueness results for solutions u, v on domains S in $\mathbb{R}^2$ with boundary conditions, including singular solutions. We study their singularities, giving a rough classification by *multiplicity* and *type*. We prove the existence of large families of *fibrations* of open subsets of $\mathbb{C}^3$ by U(1)-invariant SL 3-folds, including singular fibres. Finally, we use these fibrations as local models to draw conclusions about the *SYZ Conjecture* on Mirror Symmetry of Calabi-Yau 3-folds.

1. Introduction

Special Lagrangian submanifolds (SL m-folds) are a distinguished class of real m-dimensional minimal submanifolds in $\mathbb{C}^m$, which are calibrated with respect to the m-form $\mathrm{Re}(dz_1 \wedge \cdots \wedge dz_m)$. They can also be defined in (almost) Calabi–Yau manifolds, are important in String Theory, and are expected to play a rôle in the eventual explanation of Mirror Symmetry between Calabi–Yau 3-folds.

This paper surveys three papers [8, 9, 10] studying special Lagrangian 3-folds N in $\mathbb{C}^3$ invariant under the U(1)-action

$$e^{i\theta} : (z_1, z_2, z_3) \mapsto (e^{i\theta}z_1, e^{-i\theta}z_2, z_3) \quad \text{for } e^{i\theta} \in \mathrm{U}(1), \tag{1}$$

and also the sequel [11], which applies their results to study the *SYZ Conjecture* about Mirror Symmetry of Calabi–Yau 3-folds [14].

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N

$$N \quad \Big\{ z_1, z_2, z_3 \quad {}^3 \quad z_3 \quad u(\quad z_3 \;, \quad z_1 z_2 \;), \quad z_1 z_2 \quad v(\quad z_3 \;, \quad z_1 z_2 \;), \quad |z_1|^2 - |z_2|^2 \quad a \Big\},$$

a $\mathbb{R}$ u, v $\mathbb{R}^2$ $\mathbb{R}$ N
u, v

$$\frac{\partial u}{\partial x} \quad \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} \quad - \big(v^2 \quad y^2 \quad a^2\big)^{1/2} \frac{\partial u}{\partial y}.$$

a / *elliptic* u, v
N
a $x,$ $v\; x,$ v^2 y^2 a^2 ${}^{1/2}$
a
weak *singular points* $x,$
$v\; x,$ u, v $(\;,\; ,x \quad iu\; x, \;)$
N
nonlinear Cauchy–Riemann equation u, v
$u \quad iv$ $x \quad iy$

N
§

u, v
§
potential f $\frac{\partial f}{\partial y}$ u $\frac{\partial f}{\partial x}$ v f

$$\Big(\Big(\frac{\partial f}{\partial x}\Big)^2 \quad y^2 \quad a^2\Big)^{-1/2} \frac{\partial^2 f}{\partial x^2} \qquad \frac{\partial^2 f}{\partial y^2} \qquad .$$

$\mathbb{R}^2$
N ^{3}
$u_1, v_1 \; - \; u_2, v_2$ u_j, v_j j ,
$u_1, v_1 \; - \; u_2, v_2$ $u_1, v_1 \; \not\equiv \; u_2, v_2$
multiplicity § u, v
a $u\; x, -y \;\equiv u\; x, y$ $v\; x, -y \;\equiv -v\; x, y$
u, v x
multiplicity *types*
§ § *special*
Lagrangian fibrations ^{3} § *SYZ Conjecture*

2. Special Lagrangian geometry

^{m} *almost Calabi–Yau manifolds*

§

2.1. Special Lagrangian submanifolds in ^{m}

calibrated submanifolds

Definition 2.1. M, g *oriented tangent k-plane V* M V T_xM M V k V k M $g|_V$ V $g|_V$ V *volume form* $_V$ V k V φ k M φ *calibration* M k V M $\varphi|_V \leqslant$ $_V$ $\varphi|_V$ α $_V$ α $\mathbb{R}$ $\varphi|_V \leqslant$ $_V$ $\alpha \leqslant$ N M k T_xN x N k N *calibrated submanifold* $\varphi|_{T_xN}$ $_{T_xN}$ x N

minimal submanifolds m ^{m} §

Definition 2.2. ^{m} $z_1, \ldots, z_m$ g ω m Ω ^{m}

$$g \quad | \quad z_1|^2 \qquad | \quad z_m|^2, \quad \omega \quad \frac{i}{} \quad z_1 \quad z_1 \qquad z_m \quad z_m \;,$$

$$\Omega \quad z_1 \qquad z_m.$$

Ω Ω m ^{m} L ^{m} m L *special Lagrangian submanifold* ^{m}, *SL m-fold* L Ω

Proposition 2.1. *Let L be a real m-dimensional submanifold of ^{m}. Then L admits an orientation making it into a special Lagrangian submanifold of ^{m} if and only if $\omega|_L \equiv$ and $\Omega|_L \equiv$.*

m L ^{m} *Lagrangian* $\omega|_L \equiv$

$\Omega|_L \equiv$

2.2. Almost Calabi–Yau m-folds and SL m-folds

almost Calabi–Yau manifolds

Definition 2.3. $m \geqslant$ *almost Calabi–Yau m-fold* X, J, ω, Ω X, J m ω ¨ ¨ g X Ω $m,$ X X, J, ω, Ω *Calabi–Yau m-fold*

$$\omega^m/m \quad - \quad ^{m(m-1)/2}\, i/ \quad ^m\Omega \quad \Omega.$$

x X $T_xX \sim$ ^{m} g_x, ω_x Ω_x fl g, ω, Ω ^{m} g fl m

special Lagrangian submanifolds

Definition 2.4. X, J, ω, Ω m g N m X N *special Lagrangian submanifold* *SL m-fold* $\omega|_N \equiv$ $\Omega|_N \equiv$

m m § m m Ω g $f^2 g$ *compact* m m m m m m m ^{m} m m

3. Background material from analysis

S $\mathbb{R}^n$ *domain* *interior* S° S S $\overline{S^\circ}$ *boundary* ∂S $S \setminus S^\circ$ $\mathbb{R}^n$ S $\mathbb{R}^2$ *strictly convex* S ∂S S $\mathbb{R}^n$ C^k S $k \geqslant$ f S $\mathbb{R}$ k C^∞ S $\bigcap_{k=0}^\infty C^k$ S $k \geqslant$ α , *Hölder space* $C^{k,\alpha}$ S f C^k S

$$\partial^k f \;\alpha \qquad {}_{x \neq y \in S} \; \frac{\left|\partial^k f \;\; x \;\; - \partial^k f \;\; y \;\right|}{|x-y|^\alpha}$$

second-order quasilinear operator Q C^2 S C^0 S

$$(Qu)\ x \quad \sum_{i,j=1}^{n} a^{ij}\ x, u, \partial u\ \frac{\partial^2 u}{\partial x_i \partial x_j}\ x \quad b\ x, u, \partial u\ ,$$

a^{ij} b $S \times \mathbb{R} \times\ \mathbb{R}^n\ ^*$ $\mathbb{R}$ a^{ij} a^{ji} i, j $, \ldots, n$ a^{ij} b *coefficients* Q Q *elliptic* $n \times n$ a^{ij}

Q *divergence form*

$$(Qu)\ x \quad \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\big(a^j\ x, u, \partial u\ \big) \quad b\ x, u, \partial u$$

a^j $C^1\big(S \times \mathbb{R} \times\ \mathbb{R}^n\ ^*\big)$ j $, \ldots, n$ b $C^0\big(S \times \mathbb{R} \times\ \mathbb{R}^n\ ^*\big)$ Q u, f *weak solution* Qu f u ∂u $a^j\ x, u, \partial u\ , b\ x, u, \partial u$

$$-\sum_{j=1}^{n} \int_S \frac{\partial \psi}{\partial x_j}\ a^j\ x, u, \partial u\ \ \mathbf{x} \quad \int_S \psi\ \ b\ x, u, \partial u\ \ \mathbf{x} \quad \int_S \psi\ \ f\ \ \mathbf{x}$$

ψ C^1 S $\psi|_{\partial S} \equiv$ Q Qu f

- Qu f u C^2 S f C^0 S Qu f C^0 S
- Qu f *holds with weak derivatives* u Qu f Qu
- Qu f *holds weakly* Q u Qu f u *once* u

Q *elliptic* a^j, b, f Qu f

4. Finding the equations

N ^{3}

$$^{i\theta} \quad z_1, z_2, z_3 \qquad ^{i\theta} z_1, \ ^{-i\theta} z_2, z_3 \qquad ^{i\theta}$$

N

$$N \quad \big\{\ z_1, z_2, z_3 \qquad ^3\ \ z_1 z_2 \quad v\ x, y \quad iy, \quad z_3 \quad x \quad iu\ x, y\ , \quad |z_1|^2 - |z_2|^2 \quad a, \quad x, y \quad S\big\},$$

S $\mathbb{R}^2$ a $\mathbb{R}$ u, v S $\mathbb{R}$
$|z_1|^2 - |z_2|^2$ *moment map* $|z_1|^2 - |z_2|^2$
^{3}
a x z_3 y $z_1 z_2$
$N/$

Proposition 4.1. *Let S, a, u, v and N be as above. Then*

If a , then N is a (possibly singular) special Lagrangian -fold in ^{3} if u, v are differentiable and satisfy

$$\frac{\partial u}{\partial x} \quad \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} \quad - \big(v^2 \quad y^2\big)^{1/2} \frac{\partial u}{\partial y},$$

except at points $x,$ in S with $v\, x,$, where u, v need not be differentiable. The singular points of N are those of the form $,\ , z_3$, where z_3 x $iu\, x,$ for $x,$ S with $v\, x,$.

If $a\ /$, then N is a nonsingular special Lagrangian -fold in ^{3} if and only if u, v are differentiable in S and satisfy

$$\frac{\partial u}{\partial x} \quad \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} \quad - \big(v^2 \quad y^2 \quad a^2\big)^{1/2} \frac{\partial u}{\partial y}.$$

$\mathbf{z}$ N $T_{\mathbf{z}}N$
$\partial u, \partial v$ $T_{\mathbf{z}}N$
$\mathbb{R}^3$ ^{3} $\mathbf{z}$ $,\ , z_3$ $\big(|z_1|^2 - |z_2|^2\big)$ $\mathbf{z}$ $\mathbf{z}$
N $T_{\mathbf{z}}N$
$\frac{\partial}{\partial y}\big(\frac{\partial u}{\partial x}\big)$ $\frac{\partial}{\partial x}\big(\frac{\partial u}{\partial y}\big)$ v $\frac{\partial^2 u}{\partial y \partial x}$ $\frac{\partial^2 u}{\partial x \partial y}$

Proposition 4.2. *Let S be a domain in $\mathbb{R}^2$ and u, v $C^2\ S$ satisfy for $a\ /$. Then*

$$\frac{\partial}{\partial x}\Big[\big(v^2 \quad y^2 \quad a^2\big)^{-1/2} \frac{\partial v}{\partial x}\Big] \quad \frac{\partial^2 v}{\partial y^2} \quad .$$

Conversely, if v $C^2\ S$ satisfies then there exists u $C^2\ S$, unique up to addition of a constant u u c, such that u, v satisfy .

weak solutions a

N *singular solutions*

Definition 4.1. S $\mathbb{R}^2$ u, v $C^0\ S$ u, v
singular solution

u, v $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$
v *weak solution* a §

singular points u, v $x,$ S $v\ x,$
u, v C^2 S S°

a , u_a, v_a C^2 S u_a u v_a v
C^0 S a $+$

§ §
u, v
∂S

5. Examples

N §

Example 5.1. $a \geqslant$

$$N_a \quad \Big\{ z_1, z_2, z_3 \quad {}^3 \quad |z_1|^2 - \quad a \quad |z_2|^2 \quad |z_3|^2,$$
$$\big(z_1 z_2 z_3\big) \quad , \quad \big(z_1 z_2 z_3\big) \geqslant \quad \Big\}.$$

T^2 N_a $\mathcal{S}^1 \times \mathbb{R}^2$ $a >$ N_0 ${}_2$
, , N_a

$$^{i\theta_1}, \ ^{i\theta_2} \quad z_1, z_2, z_3 \quad - \quad ^{i\theta_1} z_1, \ ^{i\theta_2} z_2, \ ^{-i\theta_1 - i\theta_2} z_3 \ ,$$

${}_2$

§
u_a, v_a $\mathbb{R}^2$ $\mathbb{R}$

Example 5.2. α, β, γ $\mathbb{R}$ $u\ x, y$ αx β $v\ x, y$ αy γ
u, v a

Example 5.3. $u\ x, y$ y x $v\ x, y$ $\frac{1}{2}y^2$ ${}^2 x - \frac{1}{2}$ ${}^2 x$ u
v a N
^{3} §
m ^{m} $\mathbb{R}^3$

Example 5.4. $u\ x, y$ $|y| - \frac{1}{2}$ x $v\ x, y$ $-y$ x u, v
$\frac{\partial u}{\partial y}$ x a
N ^{3}
^{3}

a *singular solutions* $\mathbb{R}^2$

6. Generating u, v from a potential f

u, v $\frac{\partial u}{\partial x}$ $\frac{\partial v}{\partial y}$ *potential* f u, v $\frac{\partial f}{\partial y}$ u $\frac{\partial f}{\partial x}$ v

Proposition 6.1. *Let S be a domain in $\mathbb{R}^2$ and u, v C^1 S satisfy for a / . Then there exists f C^2 S with $\frac{\partial f}{\partial y}$ u, $\frac{\partial f}{\partial x}$ v and*

$$\Big(\Big(\frac{\partial f}{\partial x}\Big)^2 \quad y^2 \quad a^2\Big)^{-1/2}\frac{\partial^2 f}{\partial x^2} \qquad \frac{\partial^2 f}{\partial y^2} \qquad .$$

This f is unique up to addition of a constant, f f c. Conversely, all solutions of yield solutions of .

a

Dirichlet problem

Theorem 6.2. *Suppose S is a strictly convex domain in $\mathbb{R}^2$ invariant under x, y $x, -y$, and $k \geqslant$, α , . Let a $\mathbb{R}$ and ϕ $C^{k+3,\alpha}$ ∂S . Then if a / there exists a unique f $C^{k+3,\alpha}$ S with $f|_{\partial S}$ ϕ satisfying . If a there exists a unique f C^1 S with $f|_{\partial S}$ ϕ, which is twice weakly differentiable and satisfies with weak derivatives.*

Define u $\frac{\partial f}{\partial y}$ and v $\frac{\partial f}{\partial x}$. If a / then u, v $C^{k+2,\alpha}$ S satisfy , and if a then u, v C^0 S are a singular solution of , in the sense of Definition . . Furthermore, f depends continuously in C^1 S , and u, v depend continuously in C^0 S , on ϕ, a in $C^{k+3,\alpha}$ ∂S $\times \mathbb{R}$.

f, u, v

f, u, v ∂S

$|f| \leqslant$ $_{\partial S} |\phi|$ f' α βx γy

S u, v ∂S S

ϕ *a priori* f C^1 S u, v

C^0 S

a / v y *uniformly elliptic*

f $C^{k+3,\alpha}$ S

$f|_{\partial S}$ ϕ

a

§

f_a $C^{k+3,\alpha}$ S a , $f|_{\partial S}$ ϕ

f_a C^1 S a + f a

a priori f_a

a ,

u, v

3 *singular*

entirely unaffected

S°

7. Results motivated by complex analysis

§ § u_1, v_1 $-$ u_2, v_2 u_j, v_j

winding number

Definition 7.1. C γ C $\mathbb{R}^2 \setminus \{ \}$
winding number of γ *about* *along* C $\frac{1}{2\pi}\int_C \gamma^*$ θ
θ x^{-1} $y - y^{-1}$ x $\mathbb{R}^2 \setminus \{ \}$

Theorem 7.1. *Let* S *be a domain in* , *and suppose* f S *is a holomorphic function, with* f / *on* ∂S. *Then the number of zeroes of* f *in* S°, *counted with multiplicity, is equal to the winding number of* $f|_{\partial S}$ *about* *along* ∂S.

multiplicity S°

Definition 7.2. S $\mathbb{R}^2$ a $\mathbb{R}$ u_j, v_j S $\mathbb{R}$ j ,
C^1 S a / C^0 S a

b, c S *zero* u_1, v_1 $-$ u_2, v_2 S u_1, v_1 u_2, v_2 b, c
b, c *singular* a c v_1 $b,$ v_2 $b,$ b, c
singular point u_1, v_1 u_2, v_2 b, c *nonsingular zero*
b, c *isolated* $\epsilon >$ x, y u_1, v_1 $-$ u_2, v_2
S $<$ $x - b$ ^{2} $y - c$ $^2 \leqslant \epsilon^2$
b, c S° u_1, v_1 $-$ u_2, v_2 *multiplicity* b, c
u_1, v_1 $-$ u_2, v_2
γ_ϵ b, c ϵ b, c $\epsilon >$ γ_ϵ b, c S°
b, c u_1, v_1 $-$ u_2, v_2 γ_ϵ b, c

§

Theorem 7.2. *In the situation above, the multiplicity of any isolated zero* b, c *of* u_1, v_1 $-$ u_2, v_2 *in* S° *is a positive integer.*

b, c

b, c

Proposition 7.3. *In the situation above, suppose* $u_1, v_1 \; - \; u_2, v_2$ *has an isolated, non-singular zero at* $b, c \quad S^\circ$*. Then there exists* $k \geqslant$ *and* $C \quad \setminus \{ \; \}$ *such that*

$$\lambda u_1 \; x, y \quad iv_1 \; x, y \quad \lambda u_2 \; x, y \quad iv_2 \; x, y \quad C\big(\lambda \; x - b \quad i \; y - c \;\big)^k$$
$$O\big(|x-b|^{k+1} \quad |y-c|^{k+1}\big),$$

where $\lambda \quad \sqrt{}\big(v_1 \; b, c \; ^2 \quad c^2 \quad a^2\big)^{1/4}$.

$\lambda \; u_1 - u_2 \quad i \; v_1 - v_2$ $\lambda x \quad iy$ $\lambda b \quad ic$ $\lambda \; x - b \quad i \; y - c \; ^k$ $k \geqslant$ k *multiplicity* b, c

$b, c \quad u_1, v_1 \; - \; u_2, v_2$ $k \geqslant \quad \partial^j u_1 \; b, c \quad \partial^j u_2 \; b, c \quad \partial^j v_1 \; b, c \quad \partial^j v_2 \; b, c$ $j \quad , \ldots, k -$ $\partial^k u_1 \; b, c \; / \; \partial^k u_2 \; b, c \quad \partial^k v_1 \; b, c \; / \; \partial^k v_2 \; b, c$

$u_1, v_1 \; \not\equiv \; u_2, v_2$ $u_1, v_1 \; - \; u_2, v_2 \quad S^\circ$

Theorem 7.4. *Let* S *be a domain in* $\mathbb{R}^2$*, and* $a \quad \mathbb{R}$*. If* $a \; /$ *let* $u_j, v_j \quad C^1 \; S$ *satisfy for* $j \quad , \; ,$ *and if* a *let* $u_j, v_j \quad C^0 \; S$ *be singular solutions of for* $j \quad , \; .$ *Then either* $u_1, v_1 \; \equiv \; u_2, v_2$ *, or there are at most countably many zeroes of* $u_1, v_1 \; - \; u_2, v_2$ *in* S°*, all isolated.*

$u_1, v_1 \; \not\equiv \; u_2, v_2$ b, c $u_1, v_1 \; - \; u_2, v_2 \quad S^\circ$ u_j, v_j $b, c \quad \partial^j u_1 \; b, c \quad \partial^j u_2 \; b, c$ $\partial^j v_1 \; b, c \quad \partial^j v_2 \; b, c$ $j \geqslant$ $u_1, v_1 \; \equiv \; u_2, v_2$ $k \geqslant$ $\partial^k u_1 \; b, c \; / \; \partial^k u_2 \; b, c \quad \partial^k v_1 \; b, c \; / \; \partial^k v_2 \; b, c$ $C \quad \setminus \{ \; \}$ b, c $u_1, v_1 \; \not\equiv \; u_2, v_2$ $u_1, v_1 \; - \; u_2, v_2 \quad S^\circ$ u, v v x §

Theorem 7.5. *Let* S *be a domain in* $\mathbb{R}^2$*, and* $a \quad \mathbb{R}$*. If* $a \; /$ *let* $u_j, v_j \quad C^1 \; S$ *satisfy for* $j \quad , \; ,$ *and if* a *let* $u_j, v_j \quad C^0 \; S$ *be singular solutions of for* $j \quad , \; .$ *Suppose* $u_1, v_1 \; / \; u_2, v_2$ *at every point of* ∂S*. Then* $u_1, v_1 \; - \; u_2, v_2$ *has finitely many zeroes in* S*, all isolated. Let there be* n *zeroes, with multiplicities* $k_1, \ldots, k_n$*. Then the winding number of* $u_1, v_1 \; - \; u_2, v_2$ *about along* ∂S *is* $\sum_{i=1}^n k_i$*.*

u_j, v_j f_j § $f_j|_{\partial S} \quad \phi_j$ $u_1, v_1 \; - \; u_2, v_2$ $\partial S \quad k$

$\phi_1 - \phi_2$ l l ∂S $|k| \leqslant l -$

Theorem 7.6. *Suppose S is a strictly convex domain in $\mathbb{R}^2$ invariant under x, y $x, -y$, and a $\mathbb{R}$, $k \geqslant$, α , , and ϕ_1, ϕ_2 $C^{k+3,\alpha}$ ∂S. Let u_j, v_j C^0 S be the (singular) solution of or constructed in Theorem . from ϕ_j, for j , .*

Suppose $\phi_1 - \phi_2$ has l local maxima and l local minima on ∂S. Then u_1, v_1 $-$ u_2, v_2 has finitely many zeroes in S°, all isolated. Let there be n zeroes in S° with multiplicities $k_1, \ldots, k_n$. Then $\sum_{i=1}^n k_i \leqslant l -$.

l u_1, v_1 / u_2, v_2 S° §

8. A rough classification of singular points

§ u, v §

Definition 8.1. S $\mathbb{R}^2$ u, v C^0 S S x, y $x, -y$ u', v' C^0 S u' x, y u $x, -y$ v' x, y $-v$ $x, -y$ u', v'

singular point *singularity* u, v $b,$ S v $b,$ u, v u, v $-$ u', v' u, v $-$ u', v' x u, v *isolated* u, v $-$ u', v' *multiplicity* $b,$ u, v S° u, v $-$ u', v' $b,$

Theorem 8.1. *Let S be a domain in $\mathbb{R}^2$ invariant under x, y $x, -y$, and u, v C^0 S a singular solution of . If u $x, -y$ $\equiv u$ x, y and v $x, -y$ $\equiv -v$ x, y then u, v is singular along the x-axis in S, and the singularities are nonisolated. Otherwise there are at most countably many singularities of u, v in S°, all isolated.*

$b,$
v $x,$ $b,$

Definition 8.2. S $\mathbb{R}^2$ u, v C^0 S $b,$ u, v S° $\epsilon >$ $\overline{B}_\epsilon$ $b,$ $\subset S^\circ$ $b,$ u, v $\overline{B}_\epsilon$ $b,$ $< |x - b| \leqslant \epsilon$ $x,$ S° v $x,$ / v $b - \epsilon, b$ $\times \{ \}$ b, b ϵ $\times \{ \}$

v x $<$ x $b - \epsilon, b$ v x $>$ x b, b ϵ $b,$ *increasing type*

v x $>$ x $b - \epsilon, b$ v x $<$ x b, b ϵ $b,$ *decreasing type*

$v\ x\ <\ \ x\ \ b-\epsilon, b\ \ v\ x\ <\ \ x\ \ b, b\ \ \epsilon$
$b,$ *maximum type*
$v\ x\ >\ \ x\ \ b-\epsilon, b\ \ v\ x\ >\ \ x\ \ b, b\ \ \epsilon$
$b,$ *minimum type*

Proposition 8.2. *Let $u, v\ \ C^0\ S$ be a singular solution of on a domain S in $\mathbb{R}^2$, and $b,$ be an isolated singularity of u, v in S° with multiplicity k. If $b,$ is of increasing or decreasing type then k is odd, and if $b,$ is of maximum or minimum type then k is even.*

Theorem 8.3. *Suppose S is a strictly convex domain in $\mathbb{R}^2$ invariant under x, y $x, -y$, and $\phi\ \ C^{k+3,\alpha}\ \partial S$ for $k \geqslant$ and α , . Let $u, v\ \ C^0\ S$ be the singular solution of constructed in Theorem . from ϕ with a .*

Define $\phi'\ \ C^{k+3,\alpha}\ \partial S$ by $\phi'\ x, y\ \ -\phi\ x, -y$. Suppose $\phi - \phi'$ has l local maxima and l local minima on ∂S. Then u, v has finitely many singularities in S°. Let there be n singularities in S° with multiplicities $k_1, \ldots, k_n$. Then $\sum_{i=1}^n k_i \leqslant l -$.

S $\mathbb{R}^2$ ϕ
$j\theta$, $j\theta$ ∂S

Theorem 8.4. *There exist examples of singular solutions u, v of with isolated singularities of every possible multiplicity $n \geqslant$, and with both possible types allowed by Proposition . .*

3

infinite number
§ $n \geqslant$ *real codimension n*

9. Special Lagrangian fibrations

special Lagrangian fibrations
3
SYZ Conjecture §

Definition 9.1. S $\mathbb{R}^2$ x, y $x, -y$
U $\mathbb{R}^3$ α , U $C^{3,\alpha}\ \partial S$
$a, b, c\ /\ a, b', c'$ U $a, b, c\ -\ a, b', c'$
∂S
$\boldsymbol{\alpha}$ a, b, c U $f_{\boldsymbol{\alpha}}$ $C^{3,\alpha}\ S$
$f_{\boldsymbol{\alpha}}|_{\partial S}$ $\boldsymbol{\alpha}$ $u_{\boldsymbol{\alpha}}$ $\frac{\partial f_{\boldsymbol{\alpha}}}{\partial y}$ $v_{\boldsymbol{\alpha}}$ $\frac{\partial f_{\boldsymbol{\alpha}}}{\partial x}$

$u_{\boldsymbol\alpha}, v_{\boldsymbol\alpha}$ a / a $u_{\boldsymbol\alpha}, v_{\boldsymbol\alpha}$
$\boldsymbol\alpha$ U C^0 S
$\boldsymbol\alpha$ a, b, c U $N_{\boldsymbol\alpha}$ ^{3}

$$N_{\boldsymbol\alpha} \quad \{ z_1, z_2, z_3 \quad {}^3 \quad z_1 z_2 \quad v_{\boldsymbol\alpha}\ x, y \quad iy, \quad z_3 \quad x \quad iu_{\boldsymbol\alpha}\ x, y\ , \quad |z_1|^2 - |z_2|^2 \quad a, \quad x, y \quad S^\circ \}.$$

$N_{\boldsymbol\alpha}$ ^{3}
a /

$N_{\boldsymbol\alpha}$ *SL fibration*

Theorem 9.1. *In the situation of Definition . , if* $\boldsymbol\alpha$ / $\boldsymbol\alpha'$ *in* U *then* $N_{\boldsymbol\alpha} \cap N_{\boldsymbol\alpha'}$ $\emptyset$. *There exists an open set* $V \subset$ ^{3} *and a continuous, surjective map* F V U *such that* F^{-1} $\boldsymbol\alpha$ $N_{\boldsymbol\alpha}$ *for all* $\boldsymbol\alpha$ U. *Thus,* F *is a special Lagrangian fibration of* $V \subset$ ^{3}*, which may include singular fibres.*

$N_{\boldsymbol\alpha}$
V $\bigcup_{\boldsymbol\alpha \in U} N_{\boldsymbol\alpha}$ α a, b, c α' a', b', c' U
a / a' $N_{\boldsymbol\alpha} \cap N_{\boldsymbol\alpha'}$ $\emptyset$ $|z_1|^2 - |z_2|^2$ a $N_{\boldsymbol\alpha}$ a' $N_{\boldsymbol\alpha'}$
a a' $\boldsymbol\alpha$ $-$ $\boldsymbol\alpha'$ ∂S
l $u_{\boldsymbol\alpha}, v_{\boldsymbol\alpha}$ $-$ $u_{\boldsymbol\alpha'}, v_{\boldsymbol\alpha'}$
S° $N_{\boldsymbol\alpha} \cap N_{\boldsymbol\alpha'}$ $\emptyset$ $N_{\boldsymbol\alpha}$

^{3}

Example 9.1. S $\mathbb{R}^2$ x, y $x, -y$
α , ϕ $C^{3,\alpha}$ ∂S U $\mathbb{R}^3$ $\mathbb{R}^3$ $C^{3,\alpha}$ ∂S a, b, c
ϕ bx cy a, b, c / a, b', c' a, b, c $-$ a, b', c' $b - b'$ x $c - c'$ y
C^∞ ∂S $b - b', c - c'$ S
$b - b'$ x $c - c'$ y ∂S
S, U
$V \subset$ ^{3} F V ^{3}
c U $\mathbb{R}^3$ $N_{\boldsymbol\alpha}$ ^{3}

$$V \quad \{ z_1, z_2, z_3 \quad {}^3 \quad z_3, \quad z_1 z_2 \quad S^\circ \}.$$

Example 9.2. F ^{3} $\mathbb{R} \times$

$$F\ z_1, z_2, z_3 \quad a, b\ , \qquad a \quad |z_1|^2 - |z_2|^2$$

$$b \quad \begin{cases} z_3, & a \quad z_1 \quad z_2 \quad , \\ z_3 \quad z_1 z_2 / |z_1|, & a \geqslant \ , \ z_1 \ / \ , \\ z_3 \quad z_1 z_2 / |z_2|, & a < \ . \end{cases}$$

3 $|z_1|$ $|z_2|$

F^{-1} a, b T^2 , , b a

$\mathcal{S}^1 \times \mathbb{R}^2$ a / $N_{|a|}$

3

a

$\mathcal{S}^1$

10. The SYZ Conjecture

Mirror Symmetry

X, X *String Theory*

X

Super Conformal Field Theory X

$H^{p,q}$ X X

X, X

X, X

not

X X *mirror*

$H^{1,1}$ X $\sim H^{2,1}$ X $H^{2,1}$ X $\sim H^{1,1}$ X

X

X

SYZ Conjecture

The SYZ Conjecture. *Suppose* X *and* X *are mirror Calabi–Yau -folds. Then (under some additional conditions) there should exist a compact topological -manifold* B *and surjective, continuous maps* f X B *and* f X B*, such that*

There exists a dense open set $B_0 \subset B$*, such that for each* b B_0*, the fibres* f^{-1} b *and* f^{-1} b *are nonsingular special Lagrangian -tori* T^3 *in* X *and* X*. Furthermore,* f^{-1} b *and* f^{-1} b *are in some sense dual to one another.*

For each b $B \setminus B_0$*, the fibres* f^{-1} b *and* f^{-1} b *are expected to be singular special Lagrangian -folds in* X *and* X*.*

f, f *special Lagrangian fibrations*

discriminant

X, X

f, f

f, f *Lagrangian fibrations*

f, f *smooth*

generic
f X B
X X
almost

f
§ [3]

- f X B B
  F
- f X B
  B
  §
-
- X f X B
  piecewise smooth
- f X B B

  f X B
  B
  $\mathcal{S}^1$
- X, X f X B
  f X B f
  f *not* B

§

Acknowledgements:

References

I: T

I *A* *G*

II: G *D ff* *G*

Th

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Ex

(, 2000)

$\mathbb{C}$ *v v q*

–

– *M* *R* *G*

v . *I. N*

v . *II. Ex*

v . *III. P*

h

– *I*

– *h*

v

M *T*

B479

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