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On Intuitionistic Fuzzy Subhypernear-rings of Hypernear-Rings

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Abstract

In this paper, we introduce the concept of an intuitionistic fuzzy subhypernear-ring of a hypernear-ring and obtain some results in this connection.

Key Words: Fuzzy subhypernear-ring,intuitionistic fuzzy subhypernear-ring, upper (resp. lower) t-level cut, homomorphism.

1. Introduction

After the introduction of the concept of fuzzy sets by Zadeh [3], several researchers were conducted on the generalizations of the notion of fuzzy set. The idea of "intuitionistic fuzzy set" was first published by Atanassov [1], as a generalization of the notion of fuzzy set. In this paper, using Atanassov's idea, we establish the intuitionistic fuzzification of the concept of subhypernear-rings in hypernear-rings and investigate some of their properties. Also, for any intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ and a homomorphism f from hypernear-ring R to hypernear-ring R', we define IFS $A^f = (\mu_A^f, \gamma_A^f)$ in R by $\mu_A^f(x) := \mu_A(f(x))$, $\gamma_A^f(x) := \gamma_A(f(x))$ for all $x \in R$. Then we show that If an IFS $A = (\mu_A, \gamma_A)$ in R' is an intuitionistic fuzzy subhypernear-ring of R', then an IFS $A^f = (\mu_A^f, \gamma_A^f)$ in R is an intuitionistic fuzzy subhypernear-ring of R. We consider the notion of equivalence relations on the family of all intuitionistic fuzzy subhypernear-rings of a hypernear-ring and investigate some related properties.

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2. Preliminaries

First we shall present the fundamental definitions.

A hyperstructure is a set H together with a map $+ : H \times H \longrightarrow \mathcal{P}^*(H)$ called hyperoperation, where $\mathcal{P}^*(H)$ denotes the set of all the nonempty subsets of H. A *hypernear-ring* is an algebraic structure $(R, +, \cdot)$ which satisfies the following axioms:

(H1) $x + (y + z) = (x + y) + z$,

(H2) There is $0 \in R$ such that $x + 0 = 0 + x = x$.

(H3) For every $x \in R$ there exists one and only one $x' \in R$ such that $0 \in x + x'$ where we shall write $-x$ for x' and we call it the opposite of x,

(H4) $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$,

(H5) With respect to the multiplication, $(R, \cdot)$ is a semigroup having a bilaterally obsorbing element 0, that is, $x0 = 0x = 0$ for all $x \in R$.

(H6) The multiplication is distributive with respect to the hyperoperation $+$ on the left side, that is, $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in R$.

If $x \in R$ and A, B are subsets of R, then by $A + B, A + x$ and $x + B$ we mean

$$A + B = \bigcup_{a \in A, b \in B} a + b, A + x = A + \{x\}, x + B = \{x\} + B$$

A subhyper group $A \subseteq R$ is *normal* if we have $x + A - x \subseteq A$.

By a *fuzzy set* μ in a nonempty set X we mean a function $\mu : X \to [0, 1]$, and the complement of μ, denoted by $\overline{\mu}$, is the fuzzy set in X given by $\overline{\mu}(x) = 1 - \mu(x)$ for all $x \in X$.

A fuzzy set μ in R is called a *fuzzy subhypernear-ring* of R (see[2]) if it satisfies

(F1) $\min\{\mu(x), \mu(y)\} \leq \inf_{\alpha \in x+y}\{\mu(\alpha)\}$,

(F2) $\mu(x) \leq \mu(-x)$,

(F3) $\min\{\mu(x), \mu(y)\} \leq \mu(xy)$.

An intuitionistic fuzzy set (briefly, IFS) A in a nonempty set X is an object having the form

$$A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\}$$

where the functions $\mu_A : X \to [0, 1]$ and $\gamma_A : X \to [0, 1]$ denote the degree of membership and the degree of nonmembership, respectively, and

$$0 \leq \mu_A(x) + \gamma_A(x) \leq 1$$

for all $x \in X$.

For the sake of simplicity, we shall use the symbol $A = (\mu_A, \gamma_A)$ for the IFS $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\}$.

Definition 2.1 ([1]). Let X be a nonempty set and let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be IFSs in X. Then

(i) $A \subseteq B$ iff $\mu_A(x) \le \mu_B(x)$ and $\gamma_A(x) \ge \gamma_B(x)$ for all $x \in X$,

(ii) $A = B$ iff $A \subseteq B$ and $B \subseteq A$,

(iii) $\overline{A} = \{(x, \gamma_A(x), \mu_A(x)) : x \in X\}$,

(iv) $A \cap B = \{(x, \mu_A(x) \wedge \mu_B(x), \gamma_A(x) \vee \gamma_B(x)) : x \in X\}$,

(v) $A \cup B = \{(x, \mu_A(x) \vee \mu_B(x), \gamma_A(x) \wedge \gamma_B(x)) : x \in X\}$,

(vi) $\Box A = \{(x, \mu_A(x), 1 - \mu_A(x)) : x \in X\}$,

(vii) $\Diamond A = \{(x, 1 - \gamma_A(x), \gamma_A(x)) : x \in X\}$.

Definition 2.2 ([1]). Let $\{A_i : i \in \Lambda\}$ be an arbitrary family of IFSs in X. Then

(i) $\cap A_i = \{(x, \wedge\mu_{A_i}(x), \vee\gamma_{A_i}(x)) : x \in X\}$,

(ii) $\cup A_i = \{(x, \vee\mu_{A_i}(x), \wedge\gamma_{A_i}(x)) : x \in X\}$.

3. Intuitionistic fuzzy subhypernear-rings of hypernear-rings

In what follows, let R denote a hypernear-ring unless otherwise specified. We first consider the intuitionistic fuzzification of the notion of subhypernear-rings in a hypernear-rings as follows.

Definition 3.1. An IFS $A = (\mu_A, \gamma_A)$ in R is called an *intuitionistic fuzzy subhypernear-ring* of R if it satisfies:

(IF1) $\min\{\mu_A(x), \mu_A(y)\} \le \inf\limits_{\alpha \in x+y}\{\mu_A(\alpha)\}$ and $\max\{\gamma_A(x), \gamma_A(y)\} \ge \sup\limits_{\alpha \in x+y}\{\gamma_A(\alpha)\}$,

(IF2) $\mu_A(x) \le \mu_A(-x)$ and $\gamma_A(x) \ge \gamma_A(-x)$

(IF3) $\min\{\mu_A(x), \mu_A(y)\} \le \mu_A(xy)$ and $\max\{\gamma_A(x), \gamma_A(y)\} \ge \gamma_A(xy)$

Lemma 3.2. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy subhypernear-ring of a hypernear-ring R. Then

$$\mu_A(x) \le \mu_A(0), \gamma_A(x) \ge \gamma_A(0)$$

for all $x \in R$.

Proof. We have

$$\mu_A(0) \geq \inf\{\mu_A(\alpha)\} \geq \min\{\mu_A(x), \mu_A(-x)\} = \mu_A(x)$$

$$\gamma_A(0) \leq \sup\{\gamma_A(\alpha)\} \leq \max\{\gamma_A(x), \gamma_A(-x)\} = \gamma_A(x).$$

□

Theorem 3.3. *If $\{A_i\}_{i\in\Lambda}$ is a family of intuitionistic fuzzy subhypernear-rings of R, then $\cap A_i$ is an intuitionistic fuzzy subhypernear-ring of R.*

Proof. Let $x, y, i \in R$. Then we have

$$\begin{aligned}
\inf_{\alpha\in x+y}\{\cap\mu_{A_i}(\alpha)\} &= \inf_{\alpha\in x+y}\{\inf\{\mu_{A_i}(\alpha)\}\} \\
&= \inf\{\inf_{\alpha\in x+y}\{\mu_{A_i}(\alpha)\}\} \\
&\geq \inf\{\min\{\mu_{A_i}(x), \mu_{A_i}(y)\}\} \\
&= \min\{\inf\{\mu_{A_i}(x)\}, \inf\{\mu_{A_i}(y)\}\} = \min\{\cap\mu_{A_i}(x), \cap\mu_{A_i}(y)\},
\end{aligned}$$

$$\begin{aligned}
\sup_{\alpha\in x+y}\{\cup\gamma_{A_i}(\alpha)\} &= \sup_{\alpha\in x+y}\{\sup\{\gamma_{A_i}(\alpha)\}\} \\
&= \sup\{\inf_{\alpha\in x+y}\{\gamma_{A_i}(\alpha)\}\} \\
&\leq \sup\{\max\{\gamma_{A_i}(x), \gamma_{A_i}(y)\}\} \\
&= \max\{\sup\{\gamma_{A_i}(x)\}, \sup\{\gamma_{A_i}(y)\}\} = \max\{\cup\gamma_{A_i}(x), \cup\gamma_{A_i}(y)\}.
\end{aligned}$$

Also, we have

$$\cap\mu_{A_i}(x) = \inf\{\mu_{A_i}(x)\} \leq \inf\{\mu_{A_i}(-x)\} = \cap\mu_{A_i}(-x),$$

$$\cup\gamma_{A_i}(x) = \sup\{\gamma_{A_i}(x)\} \geq \sup\{\gamma_{A_i}(-x)\} = \cup\gamma_{A_i}(-x),$$

$$\begin{aligned}
\cap\mu_{A_i}(xy) &= \inf\{\mu_{A_i}(xy)\} \\
&\leq \inf\{\min\{\mu_{A_i}(x), \mu_{A_i}(y)\}\} \\
&= \min\{\inf\{\mu_{A_i}(x)\}, \inf\{\mu_{A_i}(y)\}\} \\
&= \min\{\cap\mu_{A_i}(x), \cap\mu_{A_i}(y)\},
\end{aligned}$$

and

$$\begin{aligned}\cup\gamma_{A_i}(xy) &= \sup\{\gamma_{A_i}(xy)\}\\ &\geq \sup\{\max\{\gamma_{A_i}(x),\gamma_{A_i}(y)\}\}\\ &= \max\{\sup\{\gamma_{A_i}(x)\},\sup\{\gamma_{A_i}(y)\}\}\\ &= \max\{\cup\gamma_{A_i}(x),\cup\gamma_{A_i}(y)\},\end{aligned}$$

□

Lemma 3.4. *An IFS $A=(\mu_A,\gamma_A)$ is an intuitionistic fuzzy subhypernear-ring of R if and only if the fuzzy sets μ_A and $\overline{\gamma}_A$ are fuzzy subhypernear-rings of R.*

Proof. Let $A=(\mu_A,\gamma_A)$ be an intuitionistic fuzzy subhypernear-ring of R. Clearly μ_A is a fuzzy subhypernear-ring of R. For every $x,y\in R$, we have

$$\begin{aligned}\sup_{\alpha\in x+y}\{\overline{\gamma}_A(\alpha)\} &= \sup_{\alpha\in x+y}\{1-\gamma_A(\alpha)\}\\ &= 1-\max\{\gamma_A(x),\gamma_A(y)\}\\ &= \min\{1-\gamma_A(x),1-\gamma_A(y)\}\\ &= \min\{\overline{\gamma}_A(x),\overline{\gamma}_A(y)\}.\end{aligned}$$

Next,

$$\overline{\gamma}_A(x)=1-\gamma_A(x)\leq 1-\gamma_A(-x)=\overline{\gamma}_A(-x)$$

and $\overline{\gamma}_A(xy)=1-\gamma_A(xy)\geq 1-\max\{\gamma_A(x),\gamma_A(y)\}=\min\{\overline{\gamma}_A(x),\overline{\gamma}_A(y)\}$. Hence $\overline{\gamma}_A$ is a fuzzy subhypernear-ring of R. Conversely, μ_A and γ_A are fuzzy subhypernear-rings of R. For every $x,y\in R$, we get $\inf\limits_{\alpha\in x+y}\{\mu_A(\alpha)\}\geq\min\{\mu_A(x),\mu_A(y)\}$ and

$$\begin{aligned}1-\sup_{\alpha\in x+y}\{\gamma_A(\alpha)\} &= \inf_{\alpha\in x+y}\{\overline{\gamma}_A(\alpha)\}\\ &\geq \min\{\overline{\gamma}_A(x),\overline{\gamma}_A(y)\}\\ &= \min\{1-\gamma_A(x),1-\gamma_A(y)\}\\ &= 1-\max\{\gamma_A(x),\gamma_A(y)\},\end{aligned}$$

that is, $\sup\limits_{\alpha\in x+y}\{\gamma_A(\alpha\}\leq\max\{\gamma_A(x),\gamma_A(y)\}$. Also, we have $\mu_A(x)\leq\mu_A(-x)$ and

$$1-\gamma_A(x)=\overline{\gamma}_A(x)\leq\gamma_A(-x)=1-\gamma_A(-x),$$

that is, $\gamma_A(x) \geq \gamma_A(-x)$. Finally, we have

$$\min\{\mu_A(x), \mu_A(y)\} \leq \mu_A(xy)$$

and

$$\begin{aligned} 1 - \gamma_A(xy) &= \overline{\gamma}_A(xy) \\ &\geq \min\{\overline{\gamma}_A(x), \overline{\gamma}_A(y)\} \\ &= \min\{1 - \gamma_A(x), 1 - \gamma_A(y)\} \\ &= \max\{\gamma_A(x), \gamma_A(y)\}, \end{aligned}$$

that is, $\gamma_A(xy) \leq \max\{\gamma_A(x), \gamma_A(y)\}$. Hence $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy subhypernear-ring of R. $\square$

Theorem 3.5. *Let $A = (\mu_A, \gamma_A)$ be an IFS in R. Then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy subhypernear-ring of R if and only if $\Box A = (\mu_A, \overline{\mu}_A)$ and $\Diamond A = (\overline{\gamma}_A, \gamma_A)$ are intuitionistic fuzzy subhypernear-rings R.*

Proof. If $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy subhypernear-ring of R, then $\mu_A = \overline{\overline{\mu}}_A$ and γ_A are fuzzy subhypernear-ring of R from Lemma 3.4, hence $\Box A = (\mu_A, \overline{\mu}_A)$ and $\Diamond A = (\overline{\gamma}_A, \gamma_A)$ are intuitionistic fuzzy subhypernear-ring of R. Conversely if $\Box A = (\mu_A, \overline{\mu}_A)$ and $\Diamond A = (\overline{\gamma}_A, \gamma_A)$ are intuitionistic fuzzy subhypernear-ring of R, then the fuzzy sets μ_A and $\overline{\gamma}_A$ are fuzzy subhypernear-ring of R, hence $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy subhypernear-ring of R. $\square$

For any $t \in [0, 1]$ and a fuzzy set μ in a nonempty set R, the set

$$U(\mu; t) = \{x \in R \mid \mu(x) \geq t\} \text{ (resp. } L(\mu; t) = \{x \in R \mid \mu(x) \leq t\})$$

is called an *upper* (resp. *lower*) *t-level cut* of μ.

Theorem 3.6. *An IFS $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy subhypernear-ring of R if and only if for all $s, t \in [0, 1]$, the sets $U(\mu_A; t)$ and $L(\gamma_A; s)$ are either empty or subhypernear-ring of R.*

Proof. Let the set $U(\mu_A; t)$ and $L(\gamma_A; s)$ be either empty or subhypernear-ring of R for each $s, t \in [0, 1]$. For any $x \in S$, let $\mu_A(x) = t$ and $\gamma_A(x) = s$. Then $x \in U(\mu_A; t) \cap L(\gamma_A; s)$, and so $U(\mu_A; t) \neq \emptyset \neq L(\gamma_A; s)$. If there are $x, y \in R$ such that

$\inf\limits_{\alpha \in x+y}\{\mu_A(\alpha)\} \leq \min\{\mu_A(x), \mu_A(y)\}$, then $\inf\limits_{\alpha \in x+y}\{\mu_A(\alpha)\} < t_0 < \min\{\mu_A(x), \mu_A(y)\}$ by taking $t_0 := \frac{1}{2}\left\{\inf\limits_{\alpha \in x+y}\{\mu_A(\alpha)\} + \min\{\mu_A(x), \mu_A(y)\}\right\}$. Hence $t_0 < \mu_A(x)$ and $t_0 < \mu_A(y)$, and so $x \in U(\mu_A; t_0)$ and $y \in U(\mu_A; t_0)$. Since $U(\mu_A; t_0)$ is a subhypernear-ring of R, we have $x + y \in U(\mu_A; t_0)$. So, $\mu_A(x + y) \geq t_0$. This leads to a contradiction. Now let $x \in R$ be such that $\mu_A(x) \geq \mu_A(-x)$. Puttting $s_0 := \frac{1}{2}\left\{\mu_A(x) + \mu_A(-x)\right\}$, then $\mu_A(-x) < s_0 < \mu_A(x)$, and so $x \in U(\mu_A; s_0)$ but $-x \notin U(\mu_A; s_0)$. This leads to a contradiction. If there are $x, y \in R$ such that $\min\{\mu_A(x), \mu_A(y)\} \geq \mu_A(xy)$, then $\mu_A(xy) < r_0 < \min\{\mu_A(x), \mu_A(y)\}$ by taking

$$r_0 := \frac{1}{2}\left\{\mu_A(xy) + \min\{\mu_A(x), \mu_A(y)\}\right\}.$$

Hence $x \in U(\mu_A; r_0), y \in (\mu_A; r_0)$ and $xy \notin U(\mu_A; r_0)$. This leads to a contradiction. If there are $a, b \in R$ such that $\sup\limits_{\alpha \in a+b}\{\gamma_A(\alpha)\} \leq \max\{\gamma_A(a), \gamma_A(b)\}$, then $\sup\limits_{\alpha \in a+b}\{\gamma_A(\alpha)\} > t_0 > \max\{\gamma_A(a), \gamma_A(b)\}$ by taking $u_0 := \frac{1}{2}\left\{\sup\limits_{\alpha \in a+b}\{\gamma_A(\alpha)\} + \max\{\gamma_A(a), \gamma_A(b)\right\}$. Hence $u_0 > \gamma_A(a)$ and $u_0 > \gamma_A(b)$, and so $a \in L(\gamma_A; u_0)$ and $b \in L(\gamma_A; u_0)$. Since $L(\gamma_A; u_0)$ is a subhypernear-ring of R, we have $a + b \in L(\gamma_A; u_0)$. So, $\gamma_A(a + b) \leq u_0$. This leads to a contradiction. Now let $a \in R$ be such that $\gamma_A(a) \geq \gamma_A(-a)$. Puttting $v_0 := \frac{1}{2}\left\{\gamma_A(a) + \gamma_A(-a)\right\}$, then $\gamma_A(-a) > v_0 > \gamma_A(a)$, and so $a \in L(\gamma_A; v_0)$ but $-a \notin L(\gamma_A; v_0)$. This leads a contradiction. If there are $a, b \in R$ such that $\max\{\gamma_A(a), \gamma_A(b)\} \leq \gamma_A(ab)$, then $\gamma_A(ab) > r_0 > \max\{\gamma_A(a), \gamma_A(b)\}$ by taking

$$w_0 := \frac{1}{2}\left\{\gamma_A(ab) + \max\{\gamma_A(a), \gamma_A(b)\}\right\}.$$

Hence $a \in L(\gamma_A; w_0), b \in (\gamma_A; w_0)$ and $ab \notin L(\gamma_A; w_0)$. This leads to a contradiction and this completes the proof. □

Theorem 3.7. *Let $\{I_t \mid t \in \Lambda\}$ be a collection of subhypernear-rings of R such that*

(i) $R = \cup_{t \in \Lambda} I_t$,

(ii) *$s > t$ if and only if $I_s \subset I_t$ for all $s, t \in \Lambda$.*

Then an IFS $A = (\mu_A, \gamma_A)$ in R defined by

$$\mu_A(x) := \sup\{t \in \Lambda \mid x \in I_t\},\ \gamma_A(x) := \inf\{t \in \Lambda \mid x \in I_t\}$$

for all $x \in R$ is an intuitionistic fuzzy subhypernear-ring of R.

Proof. According to Theorem 3.6, it is sufficient to show that nonempty level sets $U(\mu_A; t)$ and $L(\gamma_A; s)$ are subhypernear-rings of R of R for every $s, t \in [0, 1]$. In order to prove that $U(\mu_A; t)$ $(\neq \emptyset)$ is a subhypernear-ring of R, we consider the following two cases:

$$(1^\circ)\ \ t = \sup\{q \in \Lambda \mid q < t\},\ (2^\circ)\ \ t \neq \sup\{q \in \Lambda \mid q < t\}.$$

Case (1°) implies that

$$x \in U(\mu_A; t) \Leftrightarrow x \in I_q \quad \text{for all}\ \ q < t \Leftrightarrow x \in \cap_{q<t} I_q,$$

so that $U(\mu_A; t) = \cap_{q<t} I_q$, which is a subhypernear-ring of R. For the case (2°), we claim that $U(\mu_A; t) = \cup_{q\geq t} I_q$. If $x \in \cup_{q\geq t} I_q$, then $x \in I_q$ for some $q \geq t$. It follows that $\mu_A(x) \geq q \geq t$, so that $x \in U(\mu_A; t)$. This shows that $\cup_{q\geq t} I_q \subseteq U(\mu_A; t)$. Now assume that $x \notin \cup_{q\geq t} I_q$. Then $x \notin I_q$ for all $q \geq t$. Since $t \neq \sup\{q \in \Lambda \mid q < t\}$, there exists $\varepsilon > 0$ such that $(t - \varepsilon, t) \cap \Lambda = \emptyset$. Hence $x \notin I_q$ for all $q > t - \varepsilon$, which means that if $x \in I_q$, then $q \leq t - \varepsilon$. Thus $\mu_A(x) \leq t - \varepsilon < t$, and so $x \notin U(\mu_A; t)$. Therefore $U(\mu_A; t) \subseteq \cup_{q\geq t} I_q$, and thus $U(\mu_A; t) = \cup_{q\geq t} I_q$ which is a subhypernear-ring of R. Next we prove that $L(\gamma_A; s)$ $(\neq \emptyset)$ is a subhypernear-ring of R. We consider the following two cases:

$$(3^\circ)\ \ s = \inf\{r \in \Lambda \mid s < r\},\ (4^\circ)\ \ s \neq \inf\{r \in \Lambda \mid s < r\}.$$

For the case (3°) we have

$$x \in L(\gamma_A; s) \Leftrightarrow x \in I_r \ \text{ for all }\ s < r \Leftrightarrow x \in \cap_{q\geq t} I_r,$$

and hence $L(\gamma_A; s) = \cap_{s<r} I_r$ which is a subhypernear-rings of R. For the case (4°), there exists $\varepsilon > 0$ such that $(s, s + \varepsilon) \cap \Lambda = \emptyset$. We will show that $L(\gamma_A; s) = \cup_{s\geq r} I_r$. If $x \in \cup_{s\geq r} I_r$, then $x \in I_r$ for some $r \leq s$. It follows that $\gamma_A(x) \leq r \leq s$ so that $x \in L(\gamma_A; s)$. Hence $\cup_{s\geq r} I_r \subseteq L(\gamma_A; s)$. Conversely if $x \notin \cup_{s\geq r} I_r$, then $x \notin I_r$ for all $r \leq s$, which implies that $x \notin I_r$ for all $r < s + \varepsilon$, that is, if $x \in I_r$, then $r \geq s + \varepsilon$. Thus $\gamma_A(x) \geq s + \varepsilon > s$, that is, $x \notin L(\gamma_A; s)$. Therefore $L(\gamma_A; s) \subseteq \cup_{s\geq r} I_r$ and consequently $L(\gamma_A; s) = \cup_{s\geq r} I_r$ which is a subhypernear-ring of R. This completes the proof. □

A mapping f from a hypernear-ring R to a hypernear-ring R' is called a *homomorphism* if $f(x+y) = f(x)+f(y)$, $f(x \cdot y) = f(x) \cdot f(y)$ and $f(0) = 0$ for all $x, y \in R$. From the above definition, we get $f(-x) = -f(x)$.

Let f be a map from a set X to a set Y. If $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \mu_B)$ are IFSs in X and Y respectively, then the *preimage* of B under f, denoted by $f^{-1}(B)$, is an IFS in X defined by

$$f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\gamma_B)).$$

Theorem 3.8. *Let $f : S \to S'$ be a homomorphism of hypernear-rings. If $B = (\mu_B, \gamma_B)$ is an intuitionistic fuzzy subhypernear-ring of R', then the preimage $f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\gamma_B))$ of B under f is an intuitionistic fuzzy subhypernear-ring of R.*

Proof. Assume that $B = (\mu_B, \gamma_B)$ is an intuitionistic fuzzy subhypernear-ring of R and let $x, y \in R$. Then we have

$$\begin{aligned}
\inf_{\alpha \in x+y}\{f^{-1}(\mu_B)(\alpha)\} &= \inf_{f(\alpha)\in f(x)+f(y)}\{\mu_B(f(\alpha))\} \geq \min\{\mu_B(f(x)), \mu_B(f(y))\} \\
&= \min\{f^{-1}(\mu_B)(x), f^{-1}(\mu_B)(y)\},
\end{aligned}$$

$$\begin{aligned}
\sup_{\alpha \in x+y}\{f^{-1}(\gamma_B)(\alpha)\} &= \sup_{f(\alpha)\in f(x)+f(y)}\{\gamma_B(f(\alpha))\} \leq \sup\{\gamma_B(f(x)), \gamma_B(f(y))\} \\
&= \sup\{f^{-1}(\gamma_B)(x), f^{-1}(\gamma_B)(y)\}.
\end{aligned}$$

Also, we have

$$\begin{aligned}
f^{-1}(\mu_B)(x) &= \mu_B(f(x)) \leq \mu_B(-f(x)) = \mu_B(f(-x)) \\
&= f^{-1}(\mu_B)(-x)
\end{aligned}$$

$$\begin{aligned}
f^{-1}(\gamma_B)(x) &= \gamma_B(f(x)) \geq \gamma_B(-f(x)) = \gamma_B(f(-x)) \\
&= f^{-1}(\gamma_B)(-x)
\end{aligned}$$

$$\begin{aligned}
f^{-1}(\mu_B)(x \cdot y) &= \mu_B(f(x \cdot y)) = \mu_B(f(x) \cdot f(y)) \\
&\geq \min\{\mu_B(f(x)), \mu_B(f(y))\} \\
&= \min\{f^{-1}(\mu_B)(x), f^{-1}(\mu_B)(y)\},
\end{aligned}$$

$$\begin{aligned}
f^{-1}(\gamma_B)(x\cdot y) &= \gamma_B(f(x\cdot y)) = \gamma_B(f(x)\cdot f(y)) \\
&\leq \sup\{\gamma_B(f(x)), \gamma_B(f(y))\} \\
&= \sup\{f^{-1}(\gamma_B)(x), f^{-1}(\gamma_B)(y)\}.
\end{aligned}$$

Therefore $f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\gamma_B))$ is an intuitionistic fuzzy subhypernear-ring of R. □

Let $f : S \to S'$ be a homomorphism of hypernear-rings. For any IFS $A = (\mu_A, \gamma_A)$ in R', we define a new IFS $A^f = (\mu_A^f, \gamma_A^f)$ in R by

$$\mu_A^f(x) := \mu_A(f(x)),\ \gamma_A^f(x) := \gamma_A(f(x))$$

for all $x \in R$.

Theorem 3.9. *Let $f : R \to R'$ be a homomorphism of hypernear-rings. If an IFS $A = (\mu_A, \gamma_A)$ in R' is an intuitionistic fuzzy subhypernear-ring of R', then an IFS $A^f = (\mu_A^f, \gamma_A^f)$ in R is an intuitionistic fuzzy subhypernear-ring of R.*

Proof. Let $x, y \in R$.

$$\begin{aligned}
\inf_{\alpha\in x+y}\{\mu_A^f(\alpha)\} &= \inf_{f(\alpha)\in f(x)+f(y)}\{\mu_A(f(\alpha))\} \geq \min\{\mu_A(f(x)), \mu_A(f(y))\} \\
&= \min\{\mu_A^f(x), \mu_A^f(y)\},
\end{aligned}$$

$$\begin{aligned}
\sup_{\alpha\in x+y}\{\gamma_A^f(\alpha)\} &= \sup_{f(\alpha)\in f(x)+f(y)}\{\gamma_A(f(\alpha))\} \leq \max\{\gamma_A(f(x)), \gamma_A(f(y))\} \\
&= \max\{\gamma_A^f(x), \gamma_A^f(y)\}.
\end{aligned}$$

Also, we have

$$\begin{aligned}
\mu_A^f(x) &= \mu_A(f(x)) \leq \mu_A(-f(x)) = \mu_A(f(-x)) \\
&= \mu_A^f(-x)
\end{aligned}$$

$$\begin{aligned}
\gamma_A^f(x) &= \gamma_A(f(x)) \geq \gamma_A(-f(x)) = \gamma_A(f(-x)) \\
&= \gamma_A^f(-x)
\end{aligned}$$

$$
\begin{aligned}
\mu_A(x \cdot y) &= \mu_A(f(x \cdot y)) = \mu_A(f(x) \cdot f(y)) \\
&\geq \min\{\mu_A(f(x)), \mu_A(f(y))\} \\
&= \min\{\mu_A)(x), \mu_A)(y)\},
\end{aligned}
$$

$$
\begin{aligned}
\gamma_A(x \cdot y) &= \gamma_A(f(x \cdot y)) = \gamma_A(f(x) \cdot f(y)) \\
&\leq \max\{\gamma_A(f(x)), \gamma_A(f(y))\} \\
&= \sup\{\gamma_A(x), \gamma_A(y)\}.
\end{aligned}
$$

Hence $A^f = (\mu_A^f, \gamma_A^f)$ is an intuitionistic fuzzy subhypernear-ring of R. $\square$

Let $IF(R)$ be the family of all intuitionistic fuzzy subhypernear-rings of R and let $t \in [0,1]$. Define binary relations U^t and L^t on $IF(R)$ as follows:

$$(A,B) \in U^t \Leftrightarrow U(\mu_A;t) = U(\mu_B;t),\ (A,B) \in L^t \Leftrightarrow L(\gamma_A;t) = L(\gamma_B;t),$$

respectively, for $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ in $IF(R)$. Then clearly U^t and L^t are equivalence relations on $IF(R)$. For any $A = (\mu_A, \gamma_A) \in IF(R)$, let $[A]_{U^t}$ (resp. $[A]_{L^t}$) denote the equivalence class of A modulo U^t (resp. L^t), and denote by $IF(R)/U^t$ (resp. $IF(R)/L^t$) the system of all equivalence classes modulo U^t (resp. L^t); so

$$IF(R)/U^t := \{[A]_{U^t} \mid A = (\mu_A, \gamma_A) \in IF(R)\}$$

$$(\text{resp. } IF(R)/L^t := \{[A]_{L^t} \mid A = (\mu_A, \gamma_A) \in IF(R)\}).$$

Now let $I(R)$ denote the family of all subhypernear-rings of R and let $t \in [0,1]$. Define maps f_t and g_t from $IF(R)$ to $I(R) \cup \{\emptyset\}$ by $f_t(A) = U(\mu_A;t)$ and $g_t(A) = L(\gamma_A;t)$, respectively, for all $A = (\mu_A, \gamma_A) \in IF(R)$. Then f_t and g_t are clearly well-defined.

Theorem 3.10. *For any $t \in (0,1)$ the maps f_t and g_t are surjective from $IF(S)$ to $I(R) \cup \{\emptyset\}$.*

Proof. Let $t \in (0,1)$. Note that $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1})$ is in $IF(R)$, where $\mathbf{0}$ and $\mathbf{1}$ are fuzzy sets in R defined by $\mathbf{0}(x) = 0$ and $\mathbf{1}(x) = 1$ for all $x \in R$. Obviously $f_t(\mathbf{0}_\sim) = U(\mathbf{0};t) = \emptyset = L(\mathbf{1};t) = g_t(\mathbf{0}_\sim)$. Let $G(\neq \emptyset) \in I(R)$. For $G_\sim = (\chi_G, \overline{\chi}_G) \in IF(S)$, we have $f_t(G_\sim) = U(\chi_G;t) = G$ and $g_t(G_\sim) = L(\overline{\chi}_G;t) = G$. Hence f_t and g_t are surjective. $\square$

Theorem 3.11. *The quotient sets $IF(R)/U^t$ and $IF(R)/L^t$ are equipotent to $I(R) \cup \{\emptyset\}$ for every $t \in (0,1)$.*

Proof. For $t \in (0,1)$ let f_t^* (resp. g_t^*) be a map from $IF(R)/U^t$ (resp. $IF(R)/L^t$) to $I(R) \cup \{\emptyset\}$ defined by $f_t^*([A]_{U^t}) = f_t(A)$ (resp. $g_t^*([A]_{L^t}) = g_t(A)$) for all $A = (\mu_A, \gamma_A) \in IF(R)$. If $U(\mu_A;t) = U(\mu_B;t)$ and $L(\gamma_A;t) = L(\gamma_B;t)$ for $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ in $IF(R)$, then $(A,B) \in U^t$ and $(A,B) \in L^t$; hence $[A]_{U^t} = [B]_{U^t}$ and $[A]_{L^t} = [B]_{L^t}$. Therefore the maps f_t^* and g_t^* are injective. Now let $G(\neq \emptyset) \in I(R)$. For $G_\sim = (\chi_G, \overline{\chi}_G) \in IF(R)$, we have

$$f_t^*([G_\sim]_{U^t}) = f_t(G_\sim) = U(\chi_G;t) = G,$$

$$g_t^*([G_\sim]_{L^t}) = g_t(G_\sim) = L(\overline{\chi}_G;t) = G.$$

Finally, for $\mathbf{0}_\sim$=$(\mathbf{0},\mathbf{1}) \in IF(R)$ we get

$$f_t^*([\mathbf{0}_\sim]_{U^t}) = f_t(\mathbf{0}_\sim) = U(\mathbf{0};t) = \emptyset,$$

$$g_t^*([\mathbf{0}_\sim]_{L^t}) = g_t(\mathbf{0}_\sim) = L(\mathbf{0};t) = \emptyset.$$

This shows that f_t^* and g_t^* are surjective, and we are done. □

For any $t \in [0,1]$, we define another relation R^t on $IF(R)$ as follows:

$$(A,B) \in R^t \Leftrightarrow U(\mu_A;t) \cap L(\gamma_A;t) = U(\mu_B;t) \cap L(\gamma_B;t)$$

for any $A = (\mu_A, \gamma_A)$, $B = (\mu_B, \gamma_B) \in IF(R)$. Then the relation R^t is also an equivalence relation on $IF(R)$.

Theorem 3.12. *For any $t \in (0,1)$, the map $\phi_t : IF(R) \to I(R) \cup \{\emptyset\}$ defined by $\phi_t(A) = f_t(A) \cap g_t(A)$ for each $A = (\mu_A, \gamma_A) \in IF(R)$ is surjective.*

Proof. Let $t \in (0,1)$. For $\mathbf{0}_\sim$=$(\mathbf{0},\mathbf{1}) \in IF(R)$,

$$\phi_t(\mathbf{0}_\sim) = f_t(\mathbf{0}_\sim) \cap g_t(\mathbf{0}_\sim) = U(\mathbf{0};t) \cap L(\mathbf{1};t) = \emptyset.$$

For any $H \in IF(R)$, there exists $H_\sim = (\chi_H, \overline{\chi}_H) \in IF(R)$ such that

$$\phi_t(H_\sim) = f_t(H_\sim) \cap g_t(H_\sim) = U(\chi_H;t) \cap L(\overline{\chi}_H;t) = H.$$

This completes the proof. □

Theorem 3.13. *For any $t \in (0,1)$, the quotient set $IF(R)/R^t$ is equipotent to $I(R) \cup \{\emptyset\}$.*

Proof. Let $t \in (0,1)$ and let $\phi_t^* : IF(R)/R^t \to I(R) \cup \{\emptyset\}$ be a map defined by $\phi_t^*([A]_{R^t}) = \phi_t(A)$ for all $[A]_{R^t} \in IF(R)/R^t$. If $\phi_t^*([A]_{R^t}) = \phi_t^*([B]_{R^t})$ for any $[A]_{R^t}$, $[B]_{R^t} \in IF(R)/R^t$ then $f_t(A) \cap g_t(A) = f_t(B) \cap g_t(B)$, that is, $U(\mu_A;t) \cap L(\gamma_A;t) = U(\mu_B;t) \cap L(\gamma_B;t)$, hence $(A,B) \in R^t$. It follows that $[A]_{R^t} = [B]_{R^t}$ so that ϕ_t^* is injective. For $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1}) \in IF(R)$,

$$\phi_t^*([\mathbf{0}_\sim]_{R^t}) = \phi_t(\mathbf{0}_\sim) = f_t(\mathbf{0}_\sim) \cap g_t(\mathbf{0}_\sim) = U(\mathbf{0};t) \cap L(\mathbf{1};t) = \emptyset.$$

If $H \in IF(R)$, then for $H_\sim = (\chi_H, \overline{\chi}_H) \in IF(R)$, we have

$$\phi_t^*([H_\sim]_{R^t}) = \phi_t(H_\sim) = f_t(H_\sim) \cap g_t(H_\sim) = U(\chi_H;t) \cap L(\overline{\chi}_H;t) = H.$$

Hence ϕ_t^* is surjective, completing the proof. □

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