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Rough Singular Integrals Along Submanifolds of Finite Type on Product Domains

Hussain Al-Qassem

Abstract

We establish the L^p boundedness of singular integrals on product domains with rough kernels in $L(\log L)^2$ and are supported by subvarieties.

Key words and phrases: Singular integrals, product domains, rough kernels, Block spaces.

1. Introduction and Results

Suppose that \mathbf{S}^{d-1} ($d = n$ or m) is the unit sphere of \mathbf{R}^d ($d \geq 2$) equipped with the normalized Lebesgue measure $d\sigma = d\sigma(x')$ which is normalized so that $\sigma(\mathbf{S}^{d-1}) = 1$. For a nonzero point $x \in \mathbf{R}^d$, we denote $x' = x/|x|$. Let $K(\cdot, \cdot)$ be the singular kernel on $\mathbf{R}^n \times \mathbf{R}^m$ given by

$$K(u, v) = \Omega(u', v') |u|^{-n} |v|^{-m}, \quad (1.1)$$

where $\Omega \in L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ and satisfies

$$\int_{\mathbf{S}^{n-1}} \Omega(u, \cdot) d\sigma(u) = 0 \text{ and } \int_{\mathbf{S}^{m-1}} \Omega(\cdot, v) d\sigma(v) = 0. \quad (1.2)$$

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Define the singular integral operator T_c and the corresponding maximal truncated singular integral operator T_c^* by

$$(T_c f)(x, y) = \text{p.v.} \int_{\mathbf{R}^n \times \mathbf{R}^m} f(x - u, y - v) K(u, v) \, dudv \quad (1.3)$$

and

$$(T_c^* f)(x, y) = \sup_{\varepsilon_1, \varepsilon_2 > 0} \left| \int_{S(\varepsilon_1, \varepsilon_2)} f(x - u, y - v) K(u, v) \, dudv \right| \quad (1.4)$$

where $S(\varepsilon_1, \varepsilon_2) = \{(u, v) \in \mathbf{R}^n \times \mathbf{R}^m : (|u|, |v|) \in [\varepsilon_1, 1) \times [\varepsilon_2, 1)\}$.

The L^p boundedness of the operators T_c and T_c^* , under various conditions on Ω , has been investigated by many authors ([1], [4], [6]–[9]). For example, R. Fefferman and E. Stein proved in [8] that T_c and T_c^* are bounded on $L^p(\mathbf{R}^{n+m})$ for $1 < p < \infty$ if Ω satisfies certain Lipschitz conditions. Subsequently in [4], Duoandikoetxea established the L^p ($1 < p < \infty$) boundedness of T_c under the weaker condition $\Omega \in L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ (with $q > 1$), and then in Fan-Guo-Pan [6] for the case when Ω belongs to certain block spaces which contains $L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ as a proper subspace (for $p = 2$, it was proved by Jiang and Lu in [9]). Recently, Al-Qassem and Pan [1] established the L^p ($1 < p < \infty$) boundedness of a more general class of operators than T_c and T_c^* and for when Ω belongs to certain block spaces.

Very recently, Al-Salman, Al-Qassem and Pan [2] were able to show that the L^p ($1 < p < \infty$) boundedness of T_c and T_c^* if $\Omega \in L(\log^+ L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$. Furthermore, the condition that $\Omega \in L(\log^+ L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ turns out to be the most desirable size condition for the L^p boundedness of T_c . This was made clear by the authors of [2], where it was shown that T_c may fail to be bounded on L^p for any p if the condition is replaced by the condition $\Omega \in L(\log^+ L)^{2-\varepsilon}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ for any $\varepsilon > 0$.

Let $\mathbf{B}_d(0, 1)$ ($d = n$ or m) denotes the unit ball centered at the origin in \mathbf{R}^d . For $N, M \in \mathbf{N}$, let $\Phi : \mathbf{B}_n(0, 1) \rightarrow \mathbf{R}^N$ and $\Psi : \mathbf{B}_m(0, 1) \rightarrow \mathbf{R}^M$ be sufficiently smooth mappings. Define the singular integral operator $T_{\Phi, \Psi}$ and its corresponding maximal truncated singular integral operator $T_{\Phi, \Psi}^*$ by

$$(T_{\Phi, \Psi} f)(x, y) = \text{p.v.} \int_{\mathbf{B}_n(0, 1) \times \mathbf{B}_m(0, 1)} f(x - \Phi(u), y - \Psi(v)) K(u, v) \, dudv \quad (1.5)$$

and

$$(T_{\Phi, \Psi}^* f)(x, y) = \sup_{\varepsilon_1, \varepsilon_2 > 0} \left| \int_{S(\varepsilon_1, \varepsilon_2)} f(x - \Phi(u), y - \Psi(v)) K(u, v) \, dudv \right|, \quad (1.6)$$

for $x \in \mathbf{R}^N$ and $y \in \mathbf{R}^M$.

For $\Phi(u) \equiv u$ and $\Psi(v) \equiv v$, one obtains essentially the singular integral operator T_c and its corresponding maximal operator T_c^* described in (1.3)–(1.4).

Our main result in this paper is the following:

Theorem 1.1. *Let $T_{\Phi, \Psi}$, and $T_{\Phi, \Psi}^*$ be given by (1.1)–(1.2) and (1.5)–(1.6). Suppose that $\Omega \in L(\log^+ L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$. If Φ and Ψ are of finite type at 0, then for $1 < p < \infty$ there exists a constant $C_p > 0$ such that*

$$\|T_{\Phi, \Psi}(f)\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \leq C_p \|f\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)}; \quad (1.7)$$

$$\|T_{\Phi, \Psi}^*(f)\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \leq C_p \|f\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \quad (1.8)$$

for any $f \in L^p(\mathbf{R}^N \times \mathbf{R}^M)$.

We point out that the one parameter case of Theorem 1.1 was studied by many authors (see for example [11], [5], [3]).

As in the one-parameter setting, we can show that the L^p boundedness of the operators $T_{\Phi, \Psi}$ and $T_{\Phi, \Psi}^*$ may fail for any p if either one of the mappings Φ and Ψ is not of finite type at 0.

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2. Preliminaries

Definition 2.1. *Let U be an open set in \mathbf{R}^n , and let $\Psi : U \rightarrow \mathbf{R}^l$ be a smooth mapping. For $x_0 \in U$, we say that Ψ is of finite type at x_0 if, for each unit vector η in \mathbf{R}^l , there is a nonzero multi-index α such that*

$$D^\alpha [\Psi \cdot \eta](x_0) \neq 0.$$

Definition 2.2. *For $\mu \in \mathbf{N} \cup \{0\}$, let $a_\mu = 2^{(\mu+1)}$ and for $k, j \in \mathbf{Z}_-$, let $I_{k, j, \mu} = \{(u, v) \in \mathbf{R}^n \times \mathbf{R}^m : (|u|, |v|) \in [a_\mu^{k-1}, a_\mu^k] \times [a_\mu^{j-1}, a_\mu^j]\}$. For suitable mappings $\Theta :$*

$\mathbf{R}^n \rightarrow \mathbf{R}^N$, $\Upsilon : \mathbf{R}^m \rightarrow \mathbf{R}^M$, and $\Omega_\mu : \mathbf{S}^{n-1} \times \mathbf{S}^{m-1} \rightarrow \mathbf{R}$, we define the measures

$\{\lambda_{\Omega_\mu, \Theta, \Upsilon, k, j} : k, j \in \mathbf{Z}_-\}$ on $\mathbf{R}^N \times \mathbf{R}^M$ by

$$\int_{\mathbf{R}^N \times \mathbf{R}^M} f d\lambda_{\Omega_\mu, \Theta, \Upsilon, k, j} = \int_{I_{k, j, \mu}} f(\Theta(x), \Upsilon(y)) \Omega_\mu(x', y') |x|^{-n} |y|^{-m} dx dy. \quad (2.1)$$

We shall need the following result from [4]:

Lemma 2.3. *Let $\{\nu_{k, j} : k, j \in \mathbf{Z}\}$ be a sequence of Borel measures in $\mathbf{R}^n \times \mathbf{R}^m$ and let $\nu^*(f) = \sup_{k, j \in \mathbf{Z}} |\nu_{k, j} * f|$. Suppose that for some $q > 1$ and $B > 0$, we have*

$$\|\nu^*(f)\|_q \leq B \|f\|_q \quad (2.2)$$

for every f in $L^q(\mathbf{R}^n \times \mathbf{R}^m)$. Then the vector-valued inequality

$$\left\| \left(\sum_{k, j \in \mathbf{Z}} |\nu_{k, j} * g_{k, j}|^2 \right)^{\frac{1}{2}} \right\|_{p_0} \leq (B \sup_{k, j \in \mathbf{Z}} \|\nu_{k, j}\|)^{\frac{1}{2}} \left\| \left(\sum_{k, j \in \mathbf{Z}} |g_{k, j}|^2 \right)^{\frac{1}{2}} \right\|_{p_0} \quad (2.3)$$

holds for $|1/p_0 - 1/2| = 1/(2q)$ and for arbitrary functions $\{g_{k, j}\}$ on $\mathbf{R}^n \times \mathbf{R}^m$.

The following lemma can be found in [1], which is an extension of a result due to Duoandikoetxea in [4].

Lemma 2.4. *Let $M, N \in \mathbf{N}$ and $\{\sigma_{k, j}^{(l, s)} : k, j \in \mathbf{Z}, 0 \leq l \leq N, 0 \leq s \leq M\}$ be a family of Borel measures on $\mathbf{R}^n \times \mathbf{R}^m$ with $\sigma_{k, j}^{(l, 0)} = 0$ and $\sigma_{k, j}^{(0, s)} = 0$ for every $k, j \in \mathbf{Z}$. Let $\{a_l, b_s : 1 \leq l \leq N, 1 \leq s \leq M\} \subseteq \mathbf{R}^+ \setminus (0, 2)$, $\{B(l), D(s) : 1 \leq l \leq N, 1 \leq s \leq M\} \subseteq \mathbf{N}$, $\{\alpha_l, \beta_s : 1 \leq l \leq N, 1 \leq s \leq M\} \subseteq \mathbf{R}^+$, and let $L_l : \mathbf{R}^n \rightarrow \mathbf{R}^{B(l)}$ and $Q_s : \mathbf{R}^m \rightarrow \mathbf{R}^{D(s)}$ be linear transformations for $1 \leq l \leq N, 1 \leq s \leq M$. Suppose that for some $B > 1$ and $p_0 \in (2, \infty)$ the following hold for $k, j \in \mathbf{Z}, 1 \leq l \leq N, 1 \leq s \leq M$, and $(\xi, \eta) \in \mathbf{R}^n \times \mathbf{R}^m$:*

- (i) $\|\sigma_{k, j}^{(l, s)}\| \leq B^2$;
- (ii) $|\hat{\sigma}_{k, j}^{(l, s)}(\xi, \eta)| \leq B^2 |a_l^{k B} L_l(\xi)|^{-\frac{\alpha_l}{B}} |b_s^{j B} Q_s(\eta)|^{-\frac{\beta_s}{B}}$;
- (iii) $|\hat{\sigma}_{k, j}^{(l, s)}(\xi, \eta) - \hat{\sigma}_{k, j}^{(l-1, s)}(\xi, \eta)| \leq B^2 |a_l^{k B} L_l(\xi)|^{-\frac{\alpha_l}{B}} |b_s^{j B} Q_s(\eta)|^{-\frac{\beta_s}{B}}$;

- (iv) $\left| \hat{\sigma}_{k,j}^{(l,s)}(\xi, \eta) - \hat{\sigma}_{k,j}^{(l,s-1)}(\xi, \eta) \right| \leq B^2 |a_l^{kB} L_l(\xi)|^{-\frac{\alpha_l}{B}} |b_s^{jB} Q_s(\eta)|^{\frac{\beta_s}{B}};$
- (v) $\left| \hat{\sigma}_{k,j}^{(l,s)}(\xi, \eta) - \hat{\sigma}_{k,j}^{(l-1,s)}(\xi, \eta) - \hat{\sigma}_{k,j}^{(l,s-1)}(\xi, \eta) + \hat{\sigma}_{k,j}^{(l-1,s-1)}(\xi, \eta) \right|$
 $\leq B^2 |a_l^{kB} L_l(\xi)|^{\frac{\alpha_l}{B}} |b_s^{jB} Q_s(\eta)|^{\frac{\beta_s}{B}};$
- (vi) $\left| \hat{\sigma}_{k,j}^{(l,s-1)}(\xi, \eta) - \hat{\sigma}_{k,j}^{(l-1,s-1)}(\xi, \eta) \right| \leq B^2 |a_l^{kB} L_l(\xi)|^{\frac{\alpha_l}{B}};$
- (vii) $\left| \hat{\sigma}_{k,j}^{(l-1,s)}(\xi, \eta) - \hat{\sigma}_{k,j}^{(l-1,s-1)}(\xi, \eta) \right| \leq B^2 |b_s^{jB} Q_s(\eta)|^{\frac{\beta_s}{B}};$
- (viii) For arbitrary function $g_{k,j}$ on $\mathbf{R}^n \times \mathbf{R}^m$,

$$\left\| \left(\sum_{k,j \in \mathbf{Z}} \left| \sigma_{k,j}^{(l,s)} * g_{k,j} \right|^2 \right)^{\frac{1}{2}} \right\|_{p_0} \leq B^2 \left\| \left(\sum_{k,j \in \mathbf{Z}} |g_{k,j}|^2 \right)^{\frac{1}{2}} \right\|_{p_0}. \quad (2.4)$$

Then for $p'_0 < p < p_0$, there exists a positive constant C_p such that

$$\left\| \sum_{k,j \in \mathbf{Z}} \sigma_{k,j}^{(N,M)} * f \right\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)} \leq C_p B^2 \|f\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)} \quad (2.5)$$

$$\left\| \left(\sum_{k,j \in \mathbf{Z}} \left| \sigma_{k,j}^{(N,M)} * f \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)} \leq C_p B^2 \|f\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)} \quad (2.6)$$

hold for all f in $L^p(\mathbf{R}^n \times \mathbf{R}^m)$. The constant C_p is independent of the linear transformations $\{L_l\}_{l=1}^N$ and $\{Q_s\}_{s=1}^M$.

We shall need the following oscillatory estimates from [5].

Lemma 2.5. *Let $\Phi : \mathbf{B}_n(0, 1) \rightarrow \mathbf{R}^d$ be a smooth mapping and Ω be a homogeneous function on \mathbf{R}^n of degree 0. Suppose that Φ is of finite type at 0 and $\Omega \in L^2(\mathbf{S}^{n-1})$. Then there are $N_0 \in \mathbf{N}$, $\delta \in (0, 1]$, $C > 0$ and $j_0 \in \mathbf{Z}_-$ such that*

$$\left| \int_{2^{j-1} \leq |y| < 2^j} e^{-i\xi \cdot \Phi(y)} \frac{\Omega(y)}{|y|^n} dy \right| \leq C \|\Omega\|_{L^2(\mathbf{S}^{n-1})} (2^{jN_0} |\xi|)^{-\delta}$$

for all $j \leq j_0$ and $\xi \in \mathbf{R}^d$.

Lemma 2.6. *Let $l \in \mathbf{N}$ and $R(\cdot)$ be a real-valued polynomial on \mathbf{R}^n with $\deg(R) \leq l-1$. Suppose that $P(y) = \sum_{|\alpha|=l} c_\alpha y^\alpha + R(y)$, Ω is a homogeneous function of degree zero, and*

$\Omega \in L^2(\mathbf{S}^{n-1})$. *Then there exists a constant $C > 0$ such that*

$$\left| \int_{2^{j-1} \leq |y| < 2^j} e^{-iP(y)} \frac{\Omega(y)}{|y|^n} dy \right| \leq C \|\Omega\|_{L^2(\mathbf{S}^{n-1})} (2^{jl} \sum_{|\alpha|=l} |c_\alpha|)^{-\frac{1}{4}}$$

holds for all $j \in \mathbf{Z}$ and $\{c_\alpha\} \subset \mathbf{R}$.

Lemma 2.7. *Let $\Phi : \mathbf{B}_n(0, 1) \rightarrow \mathbf{R}^N$ and $\Psi : \mathbf{B}_m(0, 1) \rightarrow \mathbf{R}^M$ be C^∞ mappings. Let $\mu \in \mathbf{N} \cup \{0\}$ and $\Omega_\mu(\cdot, \cdot)$ be a function on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ satisfying the conditions: (i) $\|\Omega_\mu\|_{L^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})} \leq (a_\mu)^2$ and (ii) $\|\Omega_\mu\|_{L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})} \leq 1$. Suppose that Φ and Ψ are of finite type at 0. Then there are $N_0, M_0 \in \mathbf{N}$, $\delta \in (0, 1]$, $C > 0$ and $k_0, j_0 \in \mathbf{Z}_-$ such*

that

$$\left| \hat{\lambda}_{\Omega_\mu, \Phi, \Psi, k, j}(\xi, \eta) \right| \leq C (\mu + 1)^2 (a_\mu^{N_0 k} |\xi|)^{-\frac{\delta}{\mu+1}} (a_\mu^{M_0 j} |\eta|)^{-\frac{\delta}{\mu+1}} \tag{2.7}$$

for all $k \leq k_0$, $j \leq j_0$, and $(\xi, \eta) \in \mathbf{R}^N \times \mathbf{R}^M$.

Proof. By the definition of $\lambda_{\Omega_\mu, \Phi, \Psi, k, j}$, we get

$$\left| \hat{\lambda}_{\Omega_\mu, \Phi, \Psi, k, j}(\xi, \eta) \right| \leq C (\mu + 1) \int_{\mathbf{S}^{m-1}} S_k(y, \xi) d\sigma(y) \tag{2.8}$$

where

$$S_k(y, \xi) = \left| \int_{a_\mu^{k-1} \leq |x| < a_\mu^k} e^{-i\xi \cdot \Phi(x)} \frac{\Omega_\mu(x, y)}{|x|^n} dx \right|.$$

Now, by Lemma 2.5 we have

$$\begin{aligned} |S_k(y, \xi)| &\leq \sum_{s=1}^{\mu+1} \left| \int_{a_\mu^{(k-1)2^{s-1}} \leq |x| < a_\mu^{(k-1)2^s}} e^{-i\xi \cdot \Phi(x)} \frac{\Omega_\mu(x, y)}{|x|^n} dx \right| \\ &\leq C \sum_{s=1}^{\mu+1} \|\Omega_\mu(\cdot, y)\|_{L^2(\mathbf{S}^{n-1})} (a_\mu^{N_0(k-1)2^{N_0 s}} |\xi|)^{-\delta}. \end{aligned}$$

Therefore, by (i), (2.8) and Hölder's inequality we have

$$\left| \hat{\lambda}_{\Omega_\mu, \Phi, \Psi, k, j}(\xi, \eta) \right| \leq C (\mu + 1)^2 a_\mu^{(\delta N_0 + 2)} (a_\mu^{N_0 k} |\xi|)^{-\delta}$$

which when combined with the trivial bound $\left| \hat{\lambda}_{\Omega_\mu, \Phi, \Psi, k, j}(\xi, \eta) \right| \leq C (\mu + 1)^2$ implies

$$\left| \hat{\lambda}_{\Omega_\mu, \Phi, \Psi, k, j}(\xi, \eta) \right| \leq C (\mu + 1)^2 (a_\mu^{N_0 k} |\xi|)^{-\frac{\delta}{\mu+1}}. \tag{2.9}$$

Similarly, we have

$$\left| \hat{\lambda}_{\Omega_\mu, \Phi, \Psi, k, j}(\xi, \eta) \right| \leq C (\mu + 1)^2 (a_\mu^{M_0 j} |\eta|)^{-\frac{\delta}{\mu+1}}. \tag{2.10}$$

Hence, by (2.9), (2.10) we obtain (2.7) to complete the proof. □

By Lemma 2.6 and the same argument as in the proof of Lemma 2.7 we get the following:

lemma 2.8. *Let $N_0, M_0 \in \mathbf{N}$, and $\Omega_\mu(\cdot, \cdot)$ be as in Lemma 2.7. Let $R_1(\cdot)$ and $R_2(\cdot)$ be real-valued polynomials on \mathbf{R}^n and \mathbf{R}^m , respectively with $\deg(R_1) \leq N_0 - 1$ and $\deg(R_2) \leq M_0 - 1$. Let $P(x) = \sum_{|\alpha|=N_0} c_\alpha x^\alpha + R_1(x)$, and $Q(y) = \sum_{|\beta|=M_0} d_\beta y^\beta + R_2(y)$.*

Then there exists a constant $C > 0$ such that for all $k, j \in \mathbf{Z}$ and $c_\alpha, d_\beta \in \mathbf{R}$,

$$\begin{aligned} & \left| \int_{I_{k,j,\mu}} e^{-i(P(x)+Q(y))} \frac{\Omega_\mu(x,y)}{|x|^n |y|^m} dx dy \right| \\ & \leq C (\mu + 1)^2 (a_\mu^{N_0 k}) \sum_{|\alpha|=N_0} |c_\alpha|^{-\frac{1}{4N_0(\mu+1)}} (a_\mu^{M_0 j}) \sum_{|\beta|=M_0} |d_\beta|^{-\frac{1}{4M_0(\mu+1)}}. \end{aligned}$$

3. Certain maximal functions

Definition 3.1. *For suitable mappings $\Theta : \mathbf{R}^n \rightarrow \mathbf{R}^N$, $\Upsilon : \mathbf{R}^m \rightarrow \mathbf{R}^M$, and $\Omega_\mu : \mathbf{S}^{n-1} \times \mathbf{S}^{m-1} \rightarrow \mathbf{R}$, we define the maximal function $\lambda_{\Omega_\mu, \Theta, \Upsilon}^*$ on $\mathbf{R}^n \times \mathbf{R}^m$ by*

$$\lambda_{\Omega_\mu, \Theta, \Upsilon}^* f(x, y) = \sup_{k \leq k_0, j \leq j_0} \left| \lambda_{\Omega_\mu, \Theta, \Upsilon, k, j} * f(x, y) \right|, \tag{3.1}$$

where k_0 and j_0 are given as in Lemma 2.7.

For $l \in \mathbf{N}$, let \mathcal{A}_l denote the class of polynomials of l variables with real coefficients. For $d \in \mathbf{N}$ and $\mathcal{R} = (\mathcal{R}_1, \dots, \mathcal{R}_d) \in (\mathcal{A}_1)^d$ define the maximal function $\mathcal{M}_{\mathcal{R}}f$ on \mathbf{R}^d by

$$\mathcal{M}_{\mathcal{R}}f(x) = \sup_{r>0} \frac{1}{r} \int_{-r}^r |f(x - \mathcal{R}(t))| dt.$$

The following result can be found in [11], pp. 476–478.

Lemma 3.2. *For $1 < p \leq \infty$ there exists a positive constant C_p such that*

$$\|\mathcal{M}_{\mathcal{R}}f\|_p \leq C_p \|f\|_p$$

for $f \in L^p(\mathbf{R}^d)$. The constant C_p may depend on the degrees of the polynomials $\mathcal{R}_1, \dots, \mathcal{R}_d$, but it is independent of their coefficients.

By Lemma 3.2 we get immediately the following theorem.

Lemma 3.3. *Let $\mathcal{P} = (P_1, \dots, P_N) : \mathbf{R}^n \rightarrow \mathbf{R}^N$ and $\mathcal{Q} = (Q_1, \dots, Q_M) : \mathbf{R}^m \rightarrow \mathbf{R}^M$ be polynomial mappings. Let $\Omega_\mu(\cdot, \cdot)$ be as in Lemma 2.7. Then for $1 < p \leq \infty$ there exists a constant C_p such that*

$$\left\| \lambda_{\Omega_\mu, \mathcal{P}, \mathcal{Q}}^*(f) \right\|_p \leq C_p (\mu + 1)^2 \|f\|_p \tag{3.2}$$

for $f \in L^p(\mathbf{R}^N \times \mathbf{R}^M)$.

Lemma 3.4. *Let $\Phi : \mathbf{B}_n(0, 1) \rightarrow \mathbf{R}^N$ and $\Psi : \mathbf{B}_m(0, 1) \rightarrow \mathbf{R}^M$ be C^∞ mappings and $\mathcal{P} = (P_1, \dots, P_N) : \mathbf{R}^n \rightarrow \mathbf{R}^N$ and $\mathcal{Q} = (Q_1, \dots, Q_M) : \mathbf{R}^m \rightarrow \mathbf{R}^M$ be polynomial mappings. Let $\Omega_\mu(\cdot, \cdot)$ be as in Lemma 2.7. Suppose that Φ and Ψ are of finite type at 0. Then for $1 < p \leq \infty$ and $f \in L^p(\mathbf{R}^N \times \mathbf{R}^M)$ there exists a positive constant C_p which is independent of μ such that*

$$\left\| \lambda_{\Omega_\mu, \mathcal{P}, \Psi}^*(f) \right\|_p \leq C_p (\mu + 1)^2 \|f\|_p \tag{3.3}$$

and

$$\left\| \lambda_{\Omega_\mu, \Phi, \mathcal{Q}}^*(f) \right\|_p \leq C_p (\mu + 1)^2 \|f\|_p \tag{3.4}$$

for $f \in L^p(\mathbf{R}^N \times \mathbf{R}^M)$.

Proof. We shall only present the proof of (3.3). The proof of (3.4) will be similar. It is easy to see that $\lambda_{\Omega_\mu, \mathcal{P}, \Psi}^* f(x, y)$ is dominated by

$$\sup_{j \leq j_0} \int_{a_\mu^{j-1} \leq |v| < a_\mu^j} \frac{1}{|v|^m} \int_{\mathbf{S}^{n-1}} |\Omega_\mu(u, v)| |(\mathcal{M}_{\mathcal{P}, \mu, u} f(\cdot, y - \Psi(v)))(x)| d\sigma(u) dv$$

where $\mathcal{M}_{\mathcal{P}, \mu, u} h(x) = \sup_{k \leq k_0} \int_{a_\mu^{k-1}}^{a_\mu^k} |h(x - \mathcal{P}(tu))| \frac{dt}{t}$. By Lemma 3.2 we immediately get

$$\left\| \lambda_{\Omega_\mu, \mathcal{P}, \Psi}^*(f) \right\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \leq C_p(\mu + 1) \left(\int_{\mathbf{R}^M} \left\| \mathcal{H}_{\Psi, \Omega_\mu^0} f(\cdot, y) \right\|_{L^p(\mathbf{R}^N)}^p dy \right)^{\frac{1}{p}}, \quad (3.5)$$

where $\mathcal{H}_{\Psi, \Omega_\mu^0} g(y) = \sup_{j \leq j_0} \int_{a_\mu^{j-1} \leq |v| < a_\mu^j} |g(y - \Psi(v))| \frac{\Omega_\mu^0(v)}{|v|^m} dv$ and Ω_μ^0 is a function on \mathbf{S}^{m-1}

defined by $\Omega_\mu^0(v) = \int_{\mathbf{S}^{n-1}} |\Omega_\mu(u, v)| d\sigma(u)$. It is easy to verify that Ω_μ^0 satisfies (i) $\|\Omega_\mu^0\|_{L^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})} \leq (a_\mu)^2$ and (ii) $\|\Omega_\mu^0\|_{L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})} \leq 1$. By the arguments in the proof of the L^p boundedness of the corresponding maximal function in the one-parameter setting in ([3], Lemma 3.6) we obtain

$$\left\| \mathcal{H}_{\Psi, \Omega_\mu^0} f(\cdot, y) \right\|_{L^p(\mathbf{R}^N)} \leq C_p(\mu + 1) \|f(\cdot, y)\|_{L^p(\mathbf{R}^N)} \quad (3.6)$$

for every $f \in L^p(\mathbf{R}^N)$. By (3.5) and (3.6) we get (3.3). This finishes the proof of our lemma. \square

Lemma 3.5. *Let $\Phi : \mathbf{B}_n(0, 1) \rightarrow \mathbf{R}^N$ and $\Psi : \mathbf{B}_m(0, 1) \rightarrow \mathbf{R}^M$ be C^∞ mappings and let $\Omega_\mu(\cdot, \cdot)$ be as in Lemma 2.7. Suppose that Φ and Ψ are of finite type at 0. Then for $1 < p \leq \infty$ and $f \in L^p(\mathbf{R}^N \times \mathbf{R}^M)$ there exists a positive constant C_p which is independent of μ such that*

$$\left\| \lambda_{\Omega_\mu, \Phi, \Psi}^*(f) \right\|_p \leq C_p(\mu + 1)^2 \|f\|_p. \quad (3.7)$$

Proof. Without loss of generality, we may assume that $\Omega_\mu \geq 0$. Let $N_0, M_0 \in \mathbf{N}$, $\delta \in (0, 1]$, $C > 0$ and $k_0, j_0 \in \mathbf{Z}_-$ be as in Lemma 2.7. For $\Phi = (\Phi_1, \dots, \Phi_N)$ and $\Psi = (\Psi_1, \dots, \Psi_M)$ we let $\mathcal{P} = (P_1, \dots, P_N)$ and $\mathcal{Q} = (Q_1, \dots, Q_M)$ be defined by

$$P_l(x) = \sum_{|\alpha| \leq N_0 - 1} \frac{1}{\alpha!} \frac{\partial^\alpha \Phi_l}{\partial x^\alpha}(0) x^\alpha \quad \text{and} \quad Q_s(y) = \sum_{|\beta| \leq M_0 - 1} \frac{1}{\beta!} \frac{\partial^\beta \Psi_s}{\partial y^\beta}(0) y^\beta,$$

for $1 \leq s \leq M$ and $1 \leq l \leq N$. Then, for $k \leq k_0$ and $j \leq j_0$ we have

$$\left| \hat{\lambda}_{\Omega_\mu, \Phi, \Psi, k, j}(\xi, \eta) - \hat{\lambda}_{\Omega_\mu, \mathcal{P}, \Psi, k, j}(\xi, \eta) \right| \leq C(\mu + 1) (a_\mu^{N_0 k} |\xi|) \int_{\mathbf{S}^{n-1}} H_j(x, \eta) d\sigma(x),$$

where

$$H_{j, \mu}(x, \eta) = \left| \int_{a_\mu^{j-1} \leq |y| < a_\mu^j} e^{-i\eta \cdot \Psi(y)} \frac{\Omega_\mu(x, y)}{|y|^m} dy \right|.$$

Thus by Lemma 2.5 and the argument in the proof of (2.8) we get

$$\begin{aligned} & \left| \hat{\lambda}_{\Omega_\mu, \Phi, \Psi, k, j}(\xi, \eta) - \hat{\lambda}_{\Omega_\mu, \mathcal{P}, \Psi, k, j}(\xi, \eta) \right| \\ & \leq C(\mu + 1)^2 (a_\mu^{N_0 k} |\xi|)^{\frac{\delta}{\mu+1}} (a_\mu^{M_0 j} |\eta|)^{-\frac{\delta}{\mu+1}} \text{ for } k \leq k_0 \text{ and } j \leq j_0. \end{aligned} \tag{3.8}$$

Similarly, it is easy to verify that, for $k \leq k_0$ and $j \leq j_0$, the following estimates hold:

$$\begin{aligned} & \left| \hat{\lambda}_{\Omega_\mu, \Phi, \Psi, k, j}(\xi, \eta) - \hat{\lambda}_{\Omega_\mu, \Phi, \mathcal{Q}, k, j}(\xi, \eta) \right| \\ & \leq C(\mu + 1)^2 (a_\mu^{N_0 k} |\xi|)^{-\frac{\delta}{\mu+1}} (a_\mu^{M_0 j} |\eta|)^{\frac{\delta}{\mu+1}}; \end{aligned} \tag{3.9}$$

$$\begin{aligned} & \left| \hat{\lambda}_{\Omega_\mu, \Phi, \Psi, k, j}(\xi, \eta) - \hat{\lambda}_{\Omega_\mu, \mathcal{P}, \Psi, k, j}(\xi, \eta) - \hat{\lambda}_{\Omega_\mu, \Phi, \mathcal{Q}, k, j}(\xi, \eta) + \hat{\lambda}_{\Omega_\mu, \mathcal{P}, \mathcal{Q}, k, j}(\xi, \eta) \right| \\ & \leq C(\mu + 1)^2 (a_\mu^{N_0 k} |\xi|)^{\frac{\delta}{\mu+1}} (a_\mu^{M_0 j} |\eta|)^{\frac{\delta}{\mu+1}}; \end{aligned} \tag{3.10}$$

$$\left| \hat{\lambda}_{\Omega_\mu, \Phi, \mathcal{Q}, k, j, \rho}(\xi, \eta) - \hat{\lambda}_{\Omega_\mu, \mathcal{P}, \mathcal{Q}, k, j}(\xi, \eta) \right| \leq C(\mu + 1)^2 (a_\mu^{N_0 k} |\xi|)^{\frac{\delta}{\mu+1}}; \tag{3.11}$$

$$\left| \hat{\lambda}_{\Omega_\mu, \mathcal{P}, \Psi, k, j}(\xi, \eta) - \hat{\lambda}_{\Omega_\mu, \mathcal{P}, \mathcal{Q}, k, j}(\xi, \eta) \right| \leq C(\mu + 1)^2 (a_\mu^{M_0 j} |\eta|)^{\frac{\delta}{\mu+1}}. \tag{3.12}$$

Let $\Lambda^1 \in \mathcal{S}(\mathbf{R}^N)$, and $\Lambda^2 \in \mathcal{S}(\mathbf{R}^M)$ be two Schwartz functions such that $(\Lambda^i)^\hat{(\zeta_i)} = 1$ for $|\zeta_i| \leq \frac{1}{2}$ and $(\Lambda^i)^\hat{(\zeta_i)} = 0$ for $|\zeta_i| \geq 1$, $i = 1, 2$ and define

$$(\Lambda_k^1)^\hat{(x)} = (\Lambda^1)^\hat{(a_\mu^{N_0 k} x)} \text{ and } (\Lambda_j^2)^\hat{(y)} = (\Lambda^2)^\hat{(a_\mu^{M_0 j} y)}.$$

Define the sequence of measures $\{\nu_{k, j, \mu}\}$ by

$$\begin{aligned} \nu_{k, j, \mu}(\xi, \eta) &= \hat{\lambda}_{\Omega_\mu, \Phi, \Psi, k, j}(\xi, \eta) - (\Lambda_k^1)^\hat{(\xi)} \hat{\lambda}_{\Omega_\mu, \mathcal{P}, \Psi, k, j}(\xi, \eta) - (\Lambda_j^2)^\hat{(\eta)} \times \\ & \quad \hat{\lambda}_{\Omega_\mu, \Phi, \mathcal{Q}, k, j}(\xi, \eta) + (\Lambda_k^1)^\hat{(\xi)} (\Lambda_j^2)^\hat{(\eta)} \hat{\lambda}_{\Omega_\mu, \mathcal{P}, \mathcal{Q}, k, j}(\xi, \eta). \end{aligned} \tag{3.13}$$

Then by (2.7), (3.8)–(3.12), (3.13) we have

$$|\hat{\nu}_{k,j,\mu}(\xi, \eta)| \leq C(\mu + 1)^2; \quad (3.14)$$

and

$$|\hat{\nu}_{k,j,\mu}(\xi, \eta)| \leq C(\mu + 1)^2 (a_\mu^{N_0 k} |\xi|)^{\pm \frac{\delta}{(\mu+1)}} (a_\mu^{M_0 j} |\eta|)^{\pm \frac{\delta}{2(\mu+1)}}. \quad (3.15)$$

Now let

$$\mathbf{g}_\mu f(x, y) = \left(\sum_{k \leq k_0, j \leq j_0} |\nu_{k,j,\mu} * f(x, y)|^2 \right)^{\frac{1}{2}} \quad (3.16)$$

and

$$\nu_\mu^*(f)(x, y) = \sup_{k \leq k_0, j \leq j_0} |\nu_{k,j,\mu} * f(x, y)|. \quad (3.17)$$

Thus,

$$\begin{aligned} \lambda_{\Omega_\mu, \Phi, \Psi}^* f(x, y) &\leq \mathbf{g}_\mu f(x, y) + C(\mathcal{M}_{\mathbf{R}^N} \otimes id_{\mathbf{R}^M}) \circ (\lambda_{\Omega_\mu, \mathcal{P}, \Psi}^* f(x, y)) + \\ &\quad 2C(id_{\mathbf{R}^N} \otimes \mathcal{M}_{\mathbf{R}^M}) \circ (\lambda_{\Omega_\mu, \Phi, \mathcal{Q}}^* f(x, y)) + 2C(\mathcal{M}_{\mathbf{R}^N} \otimes id_{\mathbf{R}^M}) \\ &\quad \circ (id_{\mathbf{R}^N} \otimes \mathcal{M}_{\mathbf{R}^M}) \circ (\lambda_{\Omega_\mu, \mathcal{P}, \mathcal{Q}}^* f(x, y)) \end{aligned} \quad (3.18)$$

and

$$\begin{aligned} \nu_\mu^* f(x, y) &\leq \mathbf{g}_\mu f(x, y) + 2C(\mathcal{M}_{\mathbf{R}^N} \otimes id_{\mathbf{R}^M}) \circ (\lambda_{\Omega_\mu, \mathcal{P}, \Psi}^* f(x, y)) + \\ &\quad 2C(id_{\mathbf{R}^N} \otimes \mathcal{M}_{\mathbf{R}^M}) \circ (\lambda_{\Omega_\mu, \Phi, \mathcal{Q}}^* f(x, y)) + 2C(\mathcal{M}_{\mathbf{R}^N} \otimes id_{\mathbf{R}^M}) \\ &\quad \circ (id_{\mathbf{R}^N} \otimes \mathcal{M}_{\mathbf{R}^M}) \circ (\lambda_{\Omega_\mu, \mathcal{P}, \mathcal{Q}}^* f(x, y)), \end{aligned} \quad (3.19)$$

where $\mathcal{M}_{\mathbf{R}^d}$ denotes the classical Hardy-Littlewood maximal function on \mathbf{R}^d .

Now by Lemmas 3.3, 3.4 and the boundedness of $\mathcal{M}_{\mathbf{R}^d}$ on L^p spaces, for $1 < p < \infty$ there exists a positive constant C_p independent of μ such that

$$\left\| (\mathcal{M}_{\mathbf{R}^N} \otimes id_{\mathbf{R}^M}) \circ (\lambda_{\Omega_\mu, \mathcal{P}, \Psi}^* f) \right\|_p \leq C_p (\mu + 1)^2 \|f\|_p, \quad (3.20)$$

$$\left\| (id_{\mathbf{R}^N} \otimes \mathcal{M}_{\mathbf{R}^M}) \circ (\lambda_{\Omega_\mu, \Phi, \mathcal{Q}}^* f) \right\|_p \leq C_p (\mu + 1)^2 \|f\|_p, \quad (3.21)$$

and

$$\left\| (\mathcal{M}_{\mathbf{R}^N} \otimes id_{\mathbf{R}^M}) \circ (id_{\mathbf{R}^N} \otimes \mathcal{M}_{\mathbf{R}^M}) \circ (\lambda_{\Omega_\mu, \mathcal{P}, \mathcal{Q}}^* f) \right\|_p \leq C_p (\mu + 1)^2 \|f\|_p \quad (3.22)$$

for every $f \in L^p(\mathbf{R}^N \times \mathbf{R}^M)$.

By (3.14), (3.15) and Plancherel's theorem, there exists a positive constant $C > 0$ independent of μ such that

$$\|\mathbf{g}_\mu f\|_2 \leq C(\mu + 1)^2 \|f\|_2. \tag{3.23}$$

Therefore, by (3.19)–(3.22), we get

$$\|\nu_\mu^*(f)\|_2 \leq C(\mu + 1)^2 \|f\|_2. \tag{3.24}$$

Thus, by (3.14), (3.24) and using Lemma 2.3 with $p_0 = 4$ and $q = 2$, we get

$$\left\| \left(\sum_{k \leq k_0, j \leq j_0} |\nu_{k,j,\mu} * g_{k,j}|^2 \right)^{1/2} \right\|_4 \leq C(\mu + 1)^2 \left\| \left(\sum_{k \leq k_0, j \leq j_0} |g_{k,j}|^2 \right)^{1/2} \right\|_4 \tag{3.25}$$

for arbitrary functions $\{g_{k,j}\}_{k,j \in \mathbf{Z}}$ on $\mathbf{R}^N \times \mathbf{R}^M$.

By (3.15), (3.25) and invoking Lemma 2.4, we obtain that

$$\|\mathbf{g}_\mu f\|_p \leq C_p(\mu + 1)^2 \|f\|_p \tag{3.26}$$

holds for $4/3 < p < 4$ and $f \in L^p(\mathbf{R}^N \times \mathbf{R}^M)$ with a positive constant C_p independent of μ .

By replacing $p = 2$ with $p = (4/3) + \varepsilon$ ($\varepsilon > 0$) in (3.23) and repeating the preceding arguments we get

$$\|\mathbf{g}_\mu f\|_p \leq C_p(\mu + 1)^2 \|f\|_p \tag{3.27}$$

for $8/7 < p < 8$ and $f \in L^p(\mathbf{R}^N \times \mathbf{R}^M)$. By continuing this process, we get

$$\|\mathbf{g}_\mu f\|_p \leq C_p(\mu + 1)^2 \|f\|_p \tag{3.28}$$

for $1 < p < \infty$ and $f \in L^p(\mathbf{R}^N \times \mathbf{R}^M)$, where C_p is a constant independent of μ . Hence by (3.18), (3.20)–(3.22) and (3.27) we obtain (3.7) to complete the proof.

4. Proof of the main theorem

Assume that $\Omega \in L(\log^+ L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$. As in [2] we decompose Ω as follows: For $\mu \in \mathbf{N}$ let $E_\mu = \{(x, y) \in \mathbf{S}^{n-1} \times \mathbf{S}^{m-1}: 2^{\mu-1} \leq |\Omega(x, y)| < 2^\mu\}$, $b_\mu = \Omega \chi_{E_\mu}$ and

$C_\mu = \|b_\mu\|_1$. Let $\mathbf{D} = \{\mu \in \mathbf{N} : C_\mu \geq 2^{-4\mu}\}$,

$$\begin{aligned} \Omega_\mu(x, y) &= (C_\mu)^{-1} \left(b_\mu(x, y) - \int_{\mathbf{S}^{n-1}} b_\mu(u, y) d\sigma(u) - \int_{\mathbf{S}^{m-1}} b_\mu(x, v) d\sigma(v) \right. \\ &\quad \left. + \int_{\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}} b_\mu(u, v) d\sigma(u) d\sigma(v) \right) \end{aligned}$$

for $\mu \in \mathbf{D}$ and

$$\Omega_0 = \Omega - \sum_{\mu \in \mathbf{D}} \Omega_\mu.$$

Then it is easy to verify that

$$\int_{\mathbf{S}^{n-1}} \Omega_\mu(u, \cdot) d\sigma(u) = \int_{\mathbf{S}^{m-1}} \Omega_\mu(\cdot, v) d\sigma(v) = 0, \tag{4.1}$$

$$\|\Omega_\mu\|_1 \leq 4, \quad \|\Omega_\mu\|_2 \leq 4(a_\mu)^2, \tag{4.2}$$

$$\Omega(x, y) = \sum_{\mu \in \mathbf{D} \cup \{0\}} C_\mu \Omega_\mu(x, y), \tag{4.3}$$

$$\sum_{\mu \in \mathbf{D} \cup \{0\}} (\mu + 1)^2 C_\mu \leq C \|\Omega\|_{L(\log L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})}, \tag{4.4}$$

for $\mu \in \mathbf{D} \cup \{0\}$ where we used $C_0 = 1$.

By (4.4)

$$\|T_{\Phi, \Psi} f\|_p \leq \sum_{\mu \in \mathbf{D} \cup \{0\}} |C_\mu| \|T_{\Omega_\mu} f\|_p \tag{4.5}$$

where

$$T_{\Omega_\mu} f(x, y) = \text{p.v.} \int_{\mathbf{B}_n(0,1) \times \mathbf{B}_m(0,1)} f(x - \Phi(u), y - \Psi(v)) \frac{\Omega_\mu(u, v)}{|u|^n |v|^m} dudv. \tag{4.6}$$

Let N_0, M_0, \mathcal{P} and \mathcal{Q} be given as in the proof of Lemma 3.5. For $1 \leq l \leq N$, $1 \leq s \leq M$ let $c_{l,\alpha} = \frac{1}{\alpha!} \frac{\partial^\alpha \Phi_l}{\partial x^\alpha}(0)$ and $d_{s,\beta} = \frac{1}{\beta!} \frac{\partial^\beta \Psi_s}{\partial y^\beta}(0)$. For $0 \leq \tau \leq N_0, 0 \leq \kappa \leq M_0$ we

define $P_\tau = (P_{l,\tau}, \dots, P_{N,\tau})$ and $Q_\kappa = (Q_{1,\kappa}, \dots, Q_{M,\kappa})$ by

$$P_{l,\tau}(x) = \sum_{|\alpha| \leq \tau} c_{l,\alpha} x^\alpha, \quad \text{for } l = 1, \dots, N, \quad 0 \leq \tau \leq N_0 - 1; \quad (4.7)$$

$$Q_{s,\kappa}(y) = \sum_{|\beta| \leq \kappa} d_{s,\beta} y^\beta, \quad \text{for } s = 1, \dots, M, \quad 0 \leq \kappa \leq M_0 - 1; \quad (4.8)$$

$P_{N_0} = \Phi$ and $Q_{M_0} = \Psi$. For each $0 \leq \tau \leq N_0; 0 \leq \kappa \leq M_0$, let $\lambda_{\Omega_\mu, k, j}^{(\tau, \kappa)} = \lambda_{\Omega_\mu, P_\tau, Q_\kappa, k, j}$. Let $\omega(\tau)$ and $\gamma(\kappa)$ denote the number of multi-indices $\alpha \in (\mathbf{N} \cup \{0\})^n$ and $\beta \in (\mathbf{N} \cup \{0\})^m$ satisfying $|\alpha| = \tau$ and $|\beta| = \kappa$, respectively. Label the coordinates of $\mathbf{R}^{\omega(\tau)}$ and $\mathbf{R}^{\gamma(\kappa)}$ by the of multi-indices α and β with $|\alpha| = \tau$ and $|\beta| = \kappa$, respectively. That is, $\mathbf{R}^{\omega(\tau)} = \{(x_\alpha)\}_{|\alpha|=\tau}$ and $\mathbf{R}^{\gamma(\kappa)} = \{(y_\beta)\}_{|\beta|=\kappa}$. For $0 \leq \tau \leq N_0$ and $0 \leq \kappa \leq M_0$, we define the linear transformations $L_\tau : \mathbf{R}^N \rightarrow \mathbf{R}^{\omega(\tau)}$ and $Q_\kappa : \mathbf{R}^M \rightarrow \mathbf{R}^{\gamma(\kappa)}$ by

$$(L_\tau(\xi))_\alpha = \sum_{l=1}^{\tau} c_{l,\alpha} \xi_l \quad \text{and} \quad (Q_\kappa(\eta))_\beta = \sum_{s=1}^{\kappa} d_{s,\beta} \eta_s$$

for $|\alpha| = \tau, |\beta| = \kappa, 0 \leq \tau \leq N_0 - 1$ and $0 \leq \kappa \leq M_0 - 1$, where $\omega(N_0) = N_0$ and $\gamma(M_0) = M_0$. Then by Lemmas 2.7, 2.8, (2.7), (3.8)–(3.12) and the same argument as in proofs of (2.7), we get

$$\left\| \lambda_{\Omega_\mu, k, j}^{(\tau, \kappa)} \right\| \leq C(\mu + 1)^2; \quad (4.9)$$

$$\left| \hat{\lambda}_{\Omega_\mu, k, j}^{(\tau, \kappa)}(\xi, \eta) \right| \leq C(\mu + 1)^2 \left| a_\mu^{\tau k} L_\tau(\xi) \right|^{-\frac{\alpha_\tau}{\mu}} \left| a_\mu^{\kappa j} Q_\kappa(\eta) \right|^{-\frac{\alpha_\kappa}{\mu+1}}; \quad (4.10)$$

$$\left| \hat{\lambda}_{\bar{b}_\mu, k, j, \rho_\mu}^{(\tau, \kappa)}(\xi, \eta) - \hat{\lambda}_{\bar{b}_\mu, k, j, \rho_\mu}^{(\tau-1, \kappa)}(\xi, \eta) \right| \leq C(\mu + 1)^2 \left| a_\mu^{\tau k} L_\tau(\xi) \right|^{\frac{\alpha_\tau}{\mu+1}} \left| a_\mu^{\kappa j} Q_\kappa(\eta) \right|^{-\frac{\alpha_\kappa}{\mu+1}}; \quad (4.11)$$

$$\left| \hat{\lambda}_{\Omega_\mu, k, j}^{(\tau, \kappa)}(\xi, \eta) - \hat{\lambda}_{\Omega_\mu, k, j}^{(\tau, \kappa-1)}(\xi, \eta) \right| \leq C(\mu + 1)^2 \left| a_\mu^{\tau k} L_\tau(\xi) \right|^{-\frac{\alpha_\tau}{\mu+1}} \left| a_\mu^{\kappa j} Q_\kappa(\eta) \right|^{\frac{\alpha_\kappa}{\mu+1}}; \quad (4.12)$$

$$\begin{aligned} & \left| \hat{\lambda}_{\Omega_\mu, k, j}^{(\tau, \kappa)}(\xi, \eta) - \hat{\lambda}_{\Omega_\mu, k, j}^{(\tau-1, \kappa)}(\xi, \eta) - \hat{\lambda}_{\Omega_\mu, k, j}^{(\tau, \kappa-1)}(\xi, \eta) + \hat{\lambda}_{\Omega_\mu, k, j}^{(\tau-1, \kappa-1)}(\xi, \eta) \right| \\ & \leq C(\mu + 1)^2 \left| a_\mu^{\tau k} L_\tau(\xi) \right|^{\frac{\alpha_\tau}{\mu+1}} \left| a_\mu^{\kappa j} Q_\kappa(\eta) \right|^{\frac{\alpha_\kappa}{\mu+1}}; \end{aligned} \quad (4.13)$$

$$\left| \hat{\lambda}_{\Omega_\mu, k, j}^{(\tau, \kappa-1)}(\xi, \eta) - \hat{\lambda}_{\Omega_\mu, k, j}^{(\tau-1, \kappa-1)}(\xi, \eta) \right| \leq C(\mu+1)^2 \left| a_\mu^{\tau k} L_\tau(\xi) \right|^{\frac{\alpha_\tau}{\mu+1}}; \quad (4.14)$$

$$\left| \hat{\lambda}_{\Omega_\mu, k, j}^{(\tau-1, \kappa)}(\xi, \eta) - \hat{\lambda}_{\Omega_\mu, k, j}^{(\tau-1, \kappa-1)}(\xi, \eta) \right| \leq C(\mu+1)^2 \left| a_\mu^{\kappa j} Q_\kappa(\eta) \right|^{\frac{\alpha_\kappa}{\mu+1}} \quad (4.15)$$

for $\mu \in \mathbf{D} \cup \{0\}$, $1 \leq \tau \leq N_0$ and $1 \leq \kappa \leq M_0$.

By invoking Lemmas 3.3–3.5, (4.9)–(4.15), and Lemmas 2.3, 2.4 we get

$$\|T_{\Omega_\mu} f\|_p = \left\| \sum_{k \leq k_0, j \leq j_0} \lambda_{\Omega_\mu, k, j}^{(N_0, M_0)} * f \right\|_p \leq C_p (\mu+1)^2 \|f\|_p, \quad (4.16)$$

for every $f \in L^p(\mathbf{R}^N \times \mathbf{R}^M)$, $\mu \in \mathbf{D} \cup \{0\}$, and for all p , $1 < p < \infty$. Hence, (1.7) follows by (4.4), (4.5) and (4.16).

One may construct a proof for (1.8) by using the above estimates and employing the techniques in [1]. We omit the details.

References

- [1] Al-Qassem, H., Pan, Y.: L^p boundedness for singular integrals with rough kernels on product domains, Hokkaido Mathematical Journal, 31 (2002), 555-613.
- [2] Al-Salman, A., Al-Qassem, H., Pan, Y.: Singular Integrals on Product Domains, Preprint.
- [3] Al-Salman, A., Pan, Y.: Singular integrals with rough kernels in $L \log^+ L(\mathbf{S}^{n-1})$, J. London Math. Soc. (2) 66 (2002) 153-174.
- [4] Duoandikoetxea, J.: Multiple singular integrals and maximal functions along hypersurfaces, Ann. Ins. Fourier (Grenoble) 36 (1986) 18 5-206.
- [5] Fan, D., Guo, K., Pan, Y.: Singular integrals along submanifolds of finite type, Mich. Math. J. 45 (1998), 135-142.
- [6] Fan, D., Guo, K., Pan, Y.: Singular integrals with rough kernels on product spaces, Hokkaido Mathematical Journal, Vol. 28 (1999), 435-460.
- [7] Fefferman, R.: Singular integrals on product domains, Bull. Amer. Math. Soc. 4 (1981), 195-201.
- [8] Fefferman, R., Stein, E. M.: Singular integrals on product spaces, Adv. in Math., 45 (1982), 117-143.

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- [9] Jiang, Y., Lu, S.: A class of singular integral operators with rough kernels on product domains. Hokkaido Mathematical Journal, Vol. 24 (1995), 1-7.
- [10] Stein, E. M.: Problems in harmonic analysis related to curvature and oscillatory integrals, Proc. Int l. Cong. Math. (1986), 196-221.
- [11] Stein, E. M.: Harmonic analysis real-variable methods, orthogonality and oscillatory integrals, Princeton University Press, Princeton, NJ, 1993.

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