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Abstract

We establish the L^p boundedness of singular integrals on product domains with rough kernels in $L(\log L)^2$ and are supported by subvarieties.

Key words and phrases: Singular integrals, product domains, rough kernels, Block spaces.

1. Introduction and Results

Suppose that $\mathbf{S}^{d-1}$ ($d = n$ or m) is the unit sphere of $\mathbf{R}^d$ ($d \geq 2$) equipped with the normalized Lebesgue measure $d\sigma = d\sigma(x')$ which is normalized so that $\sigma(\mathbf{S}^{d-1}) = 1$. For a nonzero point $x \in \mathbf{R}^d$, we denote $x' = x/|x|$. Let $K(\cdot,\cdot)$ be the singular kernel on $\mathbf{R}^n \times \mathbf{R}^m$ given by

$$K(u,v) = \Omega(u',v')\,|u|^{-n}\,|v|^{-m}, \tag{1.1}$$

where $\Omega \in L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ and satisfies

$$\int_{\mathbf{S}^{n-1}} \Omega(u,\cdot)\,d\sigma(u) = 0 \text{ and } \int_{\mathbf{S}^{m-1}} \Omega(\cdot,v)\,d\sigma(v) = 0. \tag{1.2}$$

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Define the singular integral operator T_c and the corresponding maximal truncated singular integral operator T_c^* by

$$(T_c f)(x,y) = \text{p.v.} \int_{\mathbf{R}^n \times \mathbf{R}^m} f\,(x-u,\ y-v)\, K\,(u,v)\, dudv \tag{1.3}$$

and

$$(T_c^* f)(x,y) = \sup_{\varepsilon_1,\varepsilon_2>0} \left| \int_{S(\varepsilon_1,\varepsilon_2)} f\,(x-u,\ y-v)\, K\,(u,v)\, dudv \right| \tag{1.4}$$

where $S(\varepsilon_1,\varepsilon_2) = \{(u,v) \in \mathbf{R}^n \times \mathbf{R}^m : (|u|\,,|v|) \in [\varepsilon_1,1) \times [\varepsilon_2,1)\}\,.$

The L^p boundedness of the operators T_c and T_c^*, under various conditions on Ω, has been investigated by many authors ([1], [4], [6]–[9]). For example, R. Fefferman and E. Stein proved in [8] that T_c and T_c^* are bounded on $L^p(\mathbf{R}^{n+m})$ for $1 < p < \infty$ if Ω satisfies certain Lipschitz conditions. Subsequently in [4], Duoandikoetxea established the L^p $(1 < p < \infty)$ boundedness of T_c under the weaker condition $\Omega \in L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ (with $q > 1$), and then in Fan-Guo-Pan [6] for the case when Ω belongs to certain block spaces which contains $L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ as a proper subspace (for $p = 2$, it was proved by Jiang and Lu in [9]). Recently, Al-Qassem and Pan [1] established the L^p $(1 < p < \infty)$ boundedness of a more general class of operators than T_c and T_c^* and for when Ω belongs to certain block spaces.

Very recently, Al-Salman, Al-Qassem and Pan [2] were able to show that the L^p $(1 < p < \infty)$ boundedness of T_c and T_c^* if $\Omega \in L(\log^+ L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$. Furthermore, the condition that $\Omega \in L(\log^+ L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ turns out to be the most desirable size condition for the L^p boundedness of T_c. This was made clear by the authors of [2], where it was shown that T_c may fail to be bounded on L^p for any p if the condition is replaced by the condition $\Omega \in L(\log^+ L)^{2-\varepsilon}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ for any $\varepsilon > 0$.

Let $\mathbf{B}_d(0,1)$ $(d = n$ or $m)$ denotes the unit ball centered at the origin in $\mathbf{R}^d$. For $N, M \in \mathbf{N}$, let $\Phi : \mathbf{B}_n(0,1) \to \mathbf{R}^N$ and $\Psi : \mathbf{B}_m(0,1) \to \mathbf{R}^M$ be sufficiently smooth mappings. Define the singular integral operator $T_{\Phi,\Psi}$ and its corresponding maximal truncated singular integral operator $T_{\Phi,\Psi}^*$ by

$$(T_{\Phi,\Psi} f)(x,y) = \text{p.v.} \int_{\mathbf{B}_n(0,1) \times \mathbf{B}_m(0,1)} f\,(x-\Phi(u),\ y-\Psi(v))\, K\,(u,v)\, dudv \tag{1.5}$$

and

$$(T^*_{\Phi,\Psi}f)(x,y) = \sup_{\varepsilon_1,\varepsilon_2>0}\left|\int_{S(\varepsilon_1,\varepsilon_2)} f\left(x-\Phi(u),\ y-\Psi(v)\right) K\left(u,v\right) dudv\right|, \tag{1.6}$$

for $x \in \mathbf{R}^N$ and $y \in \mathbf{R}^M$.

For $\Phi(u) \equiv u$ and $\Psi(v) \equiv v$, one obtains essentially the singular integral operator T_c and its corresponding maximal operator T_c^* described in (1.3)–(1.4).

Our main result in this paper is the following:

Theorem 1.1. *Let $T_{\Phi,\Psi}$, and $T^*_{\Phi,\Psi}$ be given by (1.1)–(1.2) and (1.5)–(1.6). Suppose that $\Omega \in L(\log^+ L)^2(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$. If Φ and Ψ are of finite type at 0, then for $1<p<\infty$ there exists a constant $C_p>0$ such that*

$$\left\|T_{\Phi,\Psi}\left(f\right)\right\|_{L^p(\mathbf{R}^N\times\mathbf{R}^M)} \leq C_p\left\|f\right\|_{L^p(\mathbf{R}^N\times\mathbf{R}^M);} \tag{1.7}$$

$$\left\|T^*_{\Phi,\Psi}\ \left(f\right)\right\|_{L^p(\mathbf{R}^N\times\mathbf{R}^M)} \leq C_p\left\|f\right\|_{L^p(\mathbf{R}^N\times\mathbf{R}^M)} \tag{1.8}$$

for any $f \in L^p(\mathbf{R}^N\times\mathbf{R}^M)$.

We point out that the one parameter case of Theorem 1.1 was studied by many authors (see for example [11], [5], [3]).

As in the one-parameter setting, we can show that the L^p boundedness of the operators $T_{\Phi,\Psi}$ and $T^*_{\Phi,\Psi}$ may fail for any p if either one of the mappings Φ and Ψ is not of finite of type at 0.

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2. Preliminaries

Definition 2.1. *Let U be an open set in $\mathbf{R}^n$, and let $\Psi: U\to\mathbf{R}^l$* be *a smooth mapping. For $x_0\in U$, we say that Ψ is of finite type at x_0 if, for each unit vector η* in *$\mathbf{R}^l$, there is a nonzero multi-index α such that*

$$D^{\alpha}[\Psi\cdot\eta](x_0)\neq 0.$$

Definition 2.2. *For $\mu \in \mathbf{N}\cup\{0\}$, let $a_\mu = 2^{(\mu+1)}$ and for $k,j\in\mathbf{Z}_-$, let $I_{k,j,\mu} = \{(u,v)\in\mathbf{R}^n\times\mathbf{R}^m : (|u|,|v|)\in[a_\mu^{k-1},a_\mu^k)\times[\ a_\mu^{j-1},a_\mu^j)\}$. For suitable mappings Θ :*

$\mathbf{R}^n \to \mathbf{R}^N$, $\Upsilon : \mathbf{R}^m \to \mathbf{R}^M$, *and* $\Omega_\mu : \mathbf{S}^{n-1} \times \mathbf{S}^{m-1} \to \mathbf{R}$, *we define the measures* $\{\lambda_{\Omega_\mu,\Theta,\Upsilon,k,j} : k,j \in \mathbf{Z}_-\}$ *on* $\mathbf{R}^N \times \mathbf{R}^M$ *by*

$$\int_{\mathbf{R}^N \times \mathbf{R}^M} f d\lambda_{\Omega_\mu,\Theta,\Upsilon,k,j} = \int_{I_{k,j,\mu}} f\left(\Theta(x), \Upsilon(y)\right) \Omega_\mu\left(x', y'\right) |x|^{-n} |y|^{-m} \, dxdy. \tag{2.1}$$

We shall need the following result from [4]:

Lemma 2.3. *Let* $\{\nu_{k,j} : k,j \in \mathbf{Z}\}$ *be a sequence of Borel measures in* $\mathbf{R}^n \times \mathbf{R}^m$ *and let* $\nu^*(f) = \sup\limits_{k,j\in\mathbf{Z}} ||\nu_{k,j}| * f|$. *Suppose that for some* $q > 1$ *and* $B > 0$, *we have*

$$\|\nu^*(f)\|_q \le B \|f\|_q \tag{2.2}$$

for every f *in* $L^q(\mathbf{R}^n \times \mathbf{R}^m)$. *Then the vector-valued inequality*

$$\left\| \Big(\sum_{k,j\in\mathbf{Z}} |\nu_{k,j} * g_{k,j}|^2\Big)^{\frac{1}{2}} \right\|_{p_0} \le \Big(B \sup_{k,j\in\mathbf{Z}} \|\nu_{k,j}\|\Big)^{\frac{1}{2}} \left\| \Big(\sum_{k,j\in\mathbf{Z}} |g_{k,j}|^2\Big)^{\frac{1}{2}} \right\|_{p_0} \tag{2.3}$$

holds for $|1/p_0 - 1/2| = 1/(2q)$ *and for arbitrary functions* $\{g_{k,j}\}$ *on* $\mathbf{R}^n \times \mathbf{R}^m$.

The following lemma can be found in [1], which is an extension of a result due to Duoandikoetxea in [4].

Lemma 2.4. *Let* $M, N \in \mathbf{N}$ *and* $\left\{\sigma_{k,j}^{(l,s)} : k,j \in \mathbf{Z},\, 0 \le l \le N, 0 \le s \le M\right\}$ *be a family of Borel measures on* $\mathbf{R}^n \times \mathbf{R}^m$ *with* $\sigma_{k,j}^{(l,0)} = 0$ *and* $\sigma_{k,j}^{(0,s)} = 0$ *for every* $k,\, j \in \mathbf{Z}$. *Let* $\{a_l, b_s : 1 \le l \le N,\, 1 \le s \le M\} \subseteq \mathbf{R}^+ \backslash (0,2)$, $\{B(l),\, D(s) : 1 \le l \le N,\, 1 \le s \le M\} \subseteq \mathbf{N}$, $\{\alpha_l,\, \beta_s : 1 \le l \le N,\, 1 \le s \le M\} \subseteq \mathbf{R}^+$, *and let* $L_l : \mathbf{R}^n \to \mathbf{R}^{B(l)}$ and $Q_s : \mathbf{R}^m \to \mathbf{R}^{D(s)}$ *be linear transformations for* $1 \le l \le N$, $1 \le s \le M$. *Suppose that for some* $B > 1$ and $p_0 \in (2,\infty)$ the following hold for $k,\, j \in \mathbf{Z}$, $1 \le l \le N$, $1 \le s \le M$, *and* $(\xi,\eta) \in \mathbf{R}^n \times \mathbf{R}^m$:

(*i*) $\left\|\sigma_{k,j}^{(l,s)}\right\| \le B^2$;

(*ii*) $\left|\hat{\sigma}_{k,j}^{(l,s)}(\xi,\eta)\right| \le B^2 \left|a_l^{kB} L_l(\xi)\right|^{-\frac{\alpha_l}{B}} \left|b_s^{jB} Q_s(\eta)\right|^{-\frac{\beta_s}{B}}$;

(*iii*) $\left|\hat{\sigma}_{k,j}^{(l,s)}(\xi,\eta) - \hat{\sigma}_{k,j}^{(l-1,s)}(\xi,\eta)\right| \le B^2 \left|a_l^{kB} L_l(\xi)\right|^{\frac{\alpha_l}{B}} \left|b_s^{jB} Q_s(\eta)\right|^{-\frac{\beta_s}{B}}$;

(*iv*) $\left|\hat{\sigma}_{k,j}^{(l,s)}(\xi,\eta)-\hat{\sigma}_{k,j}^{(l,s-1)}(\xi,\eta)\right|\leq B^2\left|a_l^{kB}L_l(\xi)\right|^{-\frac{\alpha_l}{B}}\left|b_s^{jB}Q_s(\eta)\right|^{\frac{\beta_s}{B}}$;

(*v*) $\left|\hat{\sigma}_{k,j}^{(l,s)}(\xi,\eta)-\hat{\sigma}_{k,j}^{(l-1,s)}(\xi,\eta)-\hat{\sigma}_{k,j}^{(l,s-1)}(\xi,\eta)+\hat{\sigma}_{k,j}^{(l-1,s-1)}(\xi,\eta)\right|$

$$\leq B^2\left|a_l^{kB}L_l\left(\xi\right)\right|^{\frac{\alpha_l}{B}}\left|b_s^{jB}Q_s\left(\eta\right)\right|^{\frac{\beta_s}{B}};$$

(*vi*) $\left|\hat{\sigma}_{k,j}^{(l,s-1)}\left(\xi,\eta\right)-\hat{\sigma}_{k,j}^{(l-1,s-1)}\left(\xi,\eta\right)\right|\leq B^2\left|a_l^{kB}L_l\left(\xi\right)\right|^{\frac{\alpha_l}{B}}$;

(*vii*) $\left|\hat{\sigma}_{k,j}^{(l-1,s)}\left(\xi,\eta\right)-\hat{\sigma}_{k,j}^{(l-1,s-1)}\left(\xi,\eta\right)\right|\leq B^2\left|b_s^{jB}Q_s\left(\eta\right)\right|^{\frac{\beta_s}{B}}$;

(*viii*) *For arbitrary function* $g_{k,j}$ *on* $\mathbf{R}^n\times\mathbf{R}^m$,

$$\left\|\left(\sum_{k,j\in\mathbf{Z}}\left|\sigma_{k,j}^{(l,s)}*g_{k,j}\right|^2\right)^{\frac{1}{2}}\right\|_{p_0}\leq B^2\left\|\left(\sum_{k,j\in\mathbf{Z}}\left|g_{k,j}\right|^2\right)^{\frac{1}{2}}\right\|_{p_0}. \tag{2.4}$$

Then for $p_0'<p<p_0$, *there exists a positive constant* C_p *such that*

$$\left\|\sum_{k,j\in\mathbf{Z}}\sigma_{k,j}^{(N,M)}*f\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}\leq C_pB^2\left\|f\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \tag{2.5}$$

$$\left\|\left(\sum_{k,j\in\mathbf{Z}}\left|\sigma_{k,j}^{(N,M)}*f\right|^2\right)^{\frac{1}{2}}\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}\leq C_pB^2\left\|f\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \tag{2.6}$$

hold for all f *in* $L^p(\mathbf{R}^n\times\mathbf{R}^m)$. *The constant* C_p *is independent of the linear transformations* $\{L_l\}_{l=1}^N$ *and* $\{Q_s\}_{s=1}^M$.

We shall need the following oscillatory estimates from [5].

Lemma 2.5. *Let* $\Phi:\mathbf{B}_n(0,1)\to\mathbf{R}^d$ *be a smooth mapping and* Ω *be a homogeneous function on* $\mathbf{R}^n$ *of degree* 0. *Suppose that* Φ *is of finite type at* 0 *and* $\Omega\in L^2(\mathbf{S}^{n-1})$. *Then there are* $N_0\in\mathbf{N}$, $\delta\in(0,1]$, $C>0$ *and* $j_0\in\mathbf{Z}_-$ *such that*

$$\left|\int_{2^{j-1}\leq|y|<2^j}e^{-i\xi\cdot\Phi(y)}\frac{\Omega(y)}{|y|^n}dy\right|\leq C\left\|\Omega\right\|_{L^2(\mathbf{S}^{n-1})}(2^{jN_0}\left|\xi\right|)^{-\delta}$$

for all $j\leq j_0$ *and* $\xi\in\mathbf{R}^d$.

Lemma 2.6. *Let $l \in \mathbf{N}$ and $R(\cdot)$ be a real-valued polynomial on $\mathbf{R}^n$ with $\deg(R) \leq l-1$. Suppose that $P(y) = \sum_{|\alpha|=l} c_\alpha y^\alpha + R(y)$, Ω is a homogeneous function of degree zero, and $\Omega \in L^2(\mathbf{S}^{n-1})$. Then there exists a constant $C > 0$ such that*

$$\left| \int_{2^{j-1} \leq |y| < 2^j} e^{-iP(y)} \frac{\Omega(y)}{|y|^n} dy \right| \leq C \|\Omega\|_{L^2(\mathbf{S}^{n-1})} (2^{jl} \sum_{|\alpha|=l} |c_\alpha|)^{-\frac{1}{4l}}$$

holds for all $j \in \mathbf{Z}$ and $\{c_\alpha\} \subset \mathbf{R}$.

Lemma 2.7. *Let $\Phi : \mathbf{B}_n(0,1) \to \mathbf{R}^N$ and $\Psi : \mathbf{B}_m(0,1) \to \mathbf{R}^M$ be C^∞ mappings. Let $\mu \in \mathbf{N} \cup \{0\}$ and $\Omega_\mu(\cdot,\cdot)$ be a function on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ satisfying the conditions: (i) $\|\Omega_\mu\|_{L^2(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})} \leq (a_\mu)^2$ and (ii) $\|\Omega_\mu\|_{L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})} \leq 1$. Suppose that Φ and Ψ are of finite type at 0. Then there are $N_0, M_0 \in \mathbf{N}$, $\delta \in (0,1]$, $C > 0$ and $k_0, j_0 \in \mathbf{Z}_-$ such*

that

$$\left| \hat{\lambda}_{\Omega_\mu,\Phi,\Psi,k,j}(\xi,\eta) \right| \leq C(\mu+1)^2 (a_\mu^{N_0 k} |\xi|)^{-\frac{\delta}{\mu+1}} (a_\mu^{M_0 j} |\eta|)^{-\frac{\delta}{\mu+1}} \tag{2.7}$$

for all $k \leq k_0$, $j \leq j_0$, and $(\xi,\eta) \in \mathbf{R}^N \times \mathbf{R}^M$.

Proof. By the definition of $\lambda_{\Omega_\mu,\Phi,\Psi,k,j}$, we get

$$\left| \hat{\lambda}_{\Omega_\mu,\Phi,\Psi,k,j}(\xi,\eta) \right| \leq C(\mu+1) \int_{\mathbf{S}^{m-1}} S_k(y,\xi)\, d\sigma(y) \tag{2.8}$$

where

$$S_k(y,\xi) = \left| \int_{a_\mu^{k-1} \leq |x| < a_\mu^k} e^{-i\xi\cdot\Phi(x)} \frac{\Omega_\mu(x,y)}{|x|^n} dx \right|.$$

Now, by Lemma 2.5 we have

$$\begin{aligned} |S_k(y,\xi)| &\leq \sum_{s=1}^{\mu+1} \left| \int_{a_\mu^{(k-1)} 2^{s-1} \leq |x| < a_\mu^{(k-1)} 2^s} e^{-i\xi\cdot\Phi(x)} \frac{\Omega_\mu(x,y)}{|x|^n} dx \right| \\ &\leq C \sum_{s=1}^{\mu+1} \|\Omega_\mu(\cdot,y)\|_{L^2(\mathbf{S}^{n-1})} (a_\mu^{N_0(k-1)} 2^{N_0 s} |\xi|)^{-\delta}. \end{aligned}$$

Therefore, by (i), (2.8) and Hölder's inequality we have

$$\left|\hat{\lambda}_{\Omega_\mu,\Phi,\Psi,k,j}(\xi,\eta)\right| \leq C\,(\mu+1)^2\, a_\mu^{(\delta N_0+2)}(a_\mu^{N_0 k}\,|\xi|)^{-\delta}$$

which when combined with the trivial bound $\left|\hat{\lambda}_{\Omega_\mu,\Phi,\Psi,k,j}(\xi,\eta)\right| \leq C\,(\mu+1)^2$ implies

$$\left|\hat{\lambda}_{\Omega_\mu,\Phi,\Psi,k,j}(\xi,\eta)\right| \leq C\,(\mu+1)^2\,(a_\mu^{N_0 k}\,|\xi|)^{-\frac{\delta}{\mu+1}}. \tag{2.9}$$

Similarly, we have

$$\left|\hat{\lambda}_{\Omega_\mu,\Phi,\Psi,k,j}(\xi,\eta)\right| \leq C\,(\mu+1)^2\,(a_\mu^{M_0 j}\,|\eta|)^{-\frac{\delta}{\mu+1}}. \tag{2.10}$$

Hence. by (2.9), (2.10) we obtain (2.7) to complete the proof. □

By Lemma 2.6 and the same argument as in the proof of Lemma 2.7 we get the following:

lemma 2.8. *Let $N_0, M_0 \in \mathbf{N}$, and $\Omega_\mu(\cdot,\cdot)$ be as in Lemma 2.7. Let $R_1(\cdot)$ and $R_2(\cdot)$ be real-valued polynomials on $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively with* $\deg(R_1) \leqslant N_0 - 1$ *and* $\deg(R_2) \leqslant M_0-1$. *Let $P(x) = \sum_{|\alpha|=N_0} c_\alpha x^\alpha + R_1(x)$, and $Q(y) = \sum_{|\beta|=M_0} d_\beta y^\beta + R_2(y)$. Then there exists a constant $C>0$ such that for all $k,\, j \in \mathbf{Z}$ and $c_\alpha, d_\beta \in \mathbf{R}$,*

$$\begin{aligned} &\left|\int_{I_{k,j,\mu}} e^{-i(P(x)+Q(y))}\frac{\Omega_\mu(x,y)}{|x|^n\,|y|^m}dxdy\right| \\ \leq\;& C\,(\mu+1)^2\,(a_\mu^{N_0 k}\sum_{|\alpha|=N_0}|c_\alpha|)^{-\frac{1}{4N_0(\mu+1)}}(a_\mu^{M_0 j}\sum_{|\beta|=M_0}\left|d_\beta\right|)^{-\frac{1}{4M_0(\mu+1)}}. \end{aligned}$$

3. Certain maximal functions

Definition 3.1. *For suitable mappings $\Theta : \mathbf{R}^n \to \mathbf{R}^N$, $\Upsilon : \mathbf{R}^m \to \mathbf{R}^M$, and $\Omega_\mu : \mathbf{S}^{n-1}\times\mathbf{S}^{m-1} \to \mathbf{R}$, we define the maximal function $\lambda^*_{\Omega_\mu,\Theta,\Upsilon}$ on $\mathbf{R}^N \times \mathbf{R}^M$ by*

$$\lambda^*_{\Omega_\mu,\Theta,\Upsilon} f(x,y) = \sup_{k\leq k_0, j\leq j_0} \left|\left|\lambda_{\Omega,\Theta,\Upsilon,k,j}\right| * f(x,y)\right|, \tag{3.1}$$

where k_0 and j_0 are given as in Lemma 2.7.

For $l \in \mathbf{N}$, let $\mathcal{A}_l$ denote the class of polynomials of l variables with real coefficients. For $d \in \mathbf{N}$ and $\mathcal{R} = (\mathcal{R}_1, ..., \mathcal{R}_d) \in (\mathcal{A}_1)^d$ define the maximal function $\mathcal{M}_{\mathcal{R}} f$ on $\mathbf{R}^d$ by

$$\mathcal{M}_{\mathcal{R}} f(x) = \sup_{r>0} \frac{1}{r} \int_{-r}^{r} |f(x - \mathcal{R}(t))| \, dt.$$

The following result can be found in [11], pp. 476–478.

Lemma 3.2. *For $1 < p \leq \infty$ there exists a positive constant C_p such that*

$$\|\mathcal{M}_{\mathcal{R}} f\|_p \leq C_p \|f\|_p$$

for $f \in L^p(\mathbf{R}^d)$. The constant C_p may depend on the degrees of the polynomials $\mathcal{R}_1, ..., \mathcal{R}_d$, but it is independent of their coefficients.

By Lemma 3.2 we get immediately the following theorem.

Lemma 3.3. *Let $\mathcal{P} = (P_1, \ldots, P_N) : \mathbf{R}^n \to \mathbf{R}^N$ and $\mathcal{Q} = (Q_1, \ldots, Q_M) : \mathbf{R}^m \to \mathbf{R}^M$ be polynomial mappings. Let $\Omega_\mu(\cdot, \cdot)$ be as in Lemma 2.7. Then for $1 < p \leq \infty$ there exists a constant C_p such that*

$$\left\| \lambda^*_{\Omega_\mu, \mathcal{P}, \mathcal{Q}}(f) \right\|_p \leq C_p(\mu + 1)^2 \|f\|_p \tag{3.2}$$

for $f \in L^p(\mathbf{R}^N \times \mathbf{R}^M)$.

Lemma 3.4. *Let $\Phi : \mathbf{B}_n(0,1) \to \mathbf{R}^N$ and $\Psi : \mathbf{B}_m(0,1) \to \mathbf{R}^M$ be C^∞ mappings and $\mathcal{P} = (P_1, \ldots, P_N) : \mathbf{R}^n \to \mathbf{R}^N$ and $\mathcal{Q} = (Q_1, \ldots, Q_M) : \mathbf{R}^m \to \mathbf{R}^M$ be polynomial mappings. Let $\Omega_\mu(\cdot, \cdot)$ be as in Lemma 2.7. Suppose that Φ and Ψ are of finite type at 0. Then for $1 < p \leq \infty$ and $f \in L^p(\mathbf{R}^N \times \mathbf{R}^M)$ there exists a positive constant C_p which is independent of μ such that*

$$\left\| \lambda^*_{\Omega_\mu, \mathcal{P}, \Psi}(f) \right\|_p \leq C_p(\mu + 1)^2 \|f\|_p \tag{3.3}$$

and

$$\left\| \lambda^*_{\Omega_\mu, \Phi, \mathcal{Q}}(f) \right\|_p \leq C_p(\mu + 1)^2 \|f\|_p \tag{3.4}$$

for $f \in L^p(\mathbf{R}^N \times \mathbf{R}^M)$.

Proof. We shall only present the proof of (3.3). The proof of (3.4) will be similar. It is easy to see that $\lambda^*_{\Omega_\mu,\mathcal{P},\Psi}f(x,y)$ is dominated by

$$\sup_{j\le j_0}\int_{a_\mu^{j-1}\le|v|<a_\mu^j}\frac{1}{|v|^m}\int_{\mathbf{S}^{n-1}}|\Omega_\mu(u,v)|\,|(\mathcal{M}_{\mathcal{P},\mu,u}f(\cdot,y-\Psi(v)))(x)|\,d\sigma(u)dv$$

where $\mathcal{M}_{\mathcal{P},\mu,u}h(x)=\sup\limits_{k\le k_0}\int_{a_\mu^{k-1}}^{a_\mu^k}|h(x-\mathcal{P}(tu))|\,\frac{dt}{t}$. By Lemma 3.2 we immediately get

$$\left\|\lambda^*_{\Omega_\mu,\mathcal{P},\Psi}(f)\right\|_{L^p(\mathbf{R}^N\times\mathbf{R}^M)}\le C_p(\mu+1)\Big(\int_{\mathbf{R}^M}\left\|\mathcal{H}_{\Psi,\Omega_\mu^0}f(\cdot,y)\right\|^p_{L^p(\mathbf{R}^N)}\,dy\Big)^{\frac{1}{p}}\ , \tag{3.5}$$

where $\mathcal{H}_{\Psi,\Omega^0_\mu}g(y)=\sup\limits_{j\le j_0}\int_{a_\mu^{j-1}\le|v|<a_\mu^j}|g(y-\Psi(v))|\,\frac{\Omega^0_\mu(v)}{|v|^m}dv$ and Ω^0_μ is a function on $\mathbf{S}^{m-1}$ defined by $\Omega^0_\mu(v)=\int_{\mathbf{S}^{n-1}}|\Omega_\mu(u,v)|\,d\sigma(u)$. It is easy to verify that Ω^0_μ satisfies (i) $\left\|\Omega^0_\mu\right\|_{L^2(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\le(a_\mu)^2$ and (ii) $\left\|\Omega^0_\mu\right\|_{L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\le 1$. By the arguments in the proof of the L^p boundedness of the corresponding maximal function in the one-parameter setting in ([3], Lemma 3.6) we obtain

$$\left\|\mathcal{H}_{\Psi,\Omega^0_\mu}f(\cdot,y)\right\|_{L^p(\mathbf{R}^N)}\le C_p(\mu+1)\,\|f(\cdot,y)\|_{L^p(\mathbf{R}^N)} \tag{3.6}$$

for every $f\in L^p(\mathbf{R}^N)$. By (3.5) and (3.6) we get (3.3). This finishes the proof of our lemma. □

Lemma 3.5. *Let $\Phi:\mathbf{B}_n(0,1)\to\mathbf{R}^N$ and $\Psi:\mathbf{B}_m(0,1)\to\mathbf{R}^M$ be C^∞ mappings and let $\Omega_\mu(\cdot,\cdot)$ be as in Lemma 2.7. Suppose that Φ and Ψ are of finite type at 0. Then for $1<p\le\infty$ and $f\in L^p\left(\mathbf{R}^N\times\mathbf{R}^M\right)$ there exists a positive constant C_p which is independent of μ such that*

$$\left\|\lambda^*_{\Omega_\mu,\Phi,\Psi}(f)\right\|_p\le C_p(\mu+1)^2\,\|f\|_p\,. \tag{3.7}$$

Proof. Without loss of generality, we may assume that $\Omega_\mu\ge0$. Let $N_0,M_0\in\mathbf{N}$, $\delta\in(0,1]$, $C>0$ and $k_0,j_0\in\mathbf{Z}_-$ be as in Lemma 2.7. For $\Phi=(\Phi_1,\dots,\Phi_N)$ and $\Psi=(\Psi_1,\dots,\Psi_M)$ we let $\mathcal{P}=(P_1,\dots,P_N)$ and $\mathcal{Q}=(Q_1,\dots,Q_M)$ be defined by

$$P_l(x)=\sum_{|\alpha|\le N_0-1}\frac{1}{\alpha!}\frac{\partial^\alpha\Phi_l}{\partial x^\alpha}(0)x^\alpha \text{ and } Q_s(y)=\sum_{|\beta|\leqslant M_0-1}\frac{1}{\beta!}\frac{\partial^\beta\Psi_s}{\partial y^\beta}(0)y^\beta,$$

for $1 \leqslant s \leqslant M$ and $1 \leqslant l \leqslant N$. Then, for $k \leq k_0$ and $j \leq j_0$ we have

$$\left|\hat{\lambda}_{\Omega_\mu,\Phi,\Psi,k,j}(\xi,\eta) - \hat{\lambda}_{\Omega_\mu,\mathcal{P},\Psi,k,j}(\xi,\eta)\right| \leq C(\mu+1)\left(a_\mu^{N_0 k}|\xi|\right)\int_{\mathbf{S}^{n-1}} H_j(x,\eta)\,d\sigma(x),$$

where

$$H_{j,\mu}(x,\eta) = \left|\int_{a_\mu^{j-1}\leq|y|<a_\mu^j} e^{-i\eta\cdot\Psi(y)}\frac{\Omega_\mu(x,y)}{|y|^m}dy\right|.$$

Thus by Lemma 2.5 and the argument in the proof of (2.8) we get

$$\begin{aligned}&\left|\hat{\lambda}_{\Omega_\mu,\Phi,\Psi,k,j}(\xi,\eta) - \hat{\lambda}_{\Omega_\mu,\mathcal{P},\Psi,k,j}(\xi,\eta)\right|\\ \leq\ & C(\mu+1)^2(a_\mu^{N_0 k}|\xi|)^{\frac{\delta}{\mu+1}}(a_\mu^{M_0 j}|\eta|)^{-\frac{\delta}{\mu+1}} \text{ for } k\leq k_0 \text{ and } j\leq j_0. \qquad (3.8)\end{aligned}$$

Similarly, it is easy to verify that, for $k \leq k_0$ and $j \leq j_0$, the following estimates hold:

$$\begin{aligned}&\left|\hat{\lambda}_{\Omega_\mu,\Phi,\Psi,k,j}(\xi,\eta) - \hat{\lambda}_{\Omega_\mu,\Phi,\mathcal{Q},k,j}(\xi,\eta)\right|\\ \leq\ & C(\mu+1)^2(a_\mu^{N_0 k}|\xi|)^{-\frac{\delta}{\mu+1}}(a_\mu^{M_0 j}|\eta|)^{\frac{\delta}{\mu+1}}; \qquad (3.9)\end{aligned}$$

$$\begin{aligned}&\left|\hat{\lambda}_{\Omega_\mu,\Phi,\Psi,k,j}(\xi,\eta) - \hat{\lambda}_{\Omega_\mu,\mathcal{P},\Psi,k,j}(\xi,\eta) - \hat{\lambda}_{\Omega_\mu,\Phi,\mathcal{Q},k,j}(\xi,\eta) + \hat{\lambda}_{\Omega_\mu,\mathcal{P},\mathcal{Q},k,j}(\xi,\eta)\right|\\ \leq\ & C(\mu+1)^2(a_\mu^{N_0 k}|\xi|)^{\frac{\delta}{\mu+1}}(a_\mu^{M_0 j}|\eta|)^{\frac{\delta}{\mu+1}}; \qquad (3.10)\end{aligned}$$

$$\left|\hat{\lambda}_{\Omega_\mu,\Phi,\mathcal{Q},k,j,\rho}(\xi,\eta) - \hat{\lambda}_{\Omega_\mu,\mathcal{P},\mathcal{Q},k,j}(\xi,\eta)\right| \leq C(\mu+1)^2(a_\mu^{N_0 k}|\xi|)^{\frac{\delta}{\mu+1}}; \qquad (3.11)$$

$$\left|\hat{\lambda}_{\Omega_\mu,\mathcal{P},\Psi,k,j}(\xi,\eta) - \hat{\lambda}_{\Omega_\mu,\mathcal{P},\mathcal{Q},k,j}(\xi,\eta)\right| \leq C(\mu+1)^2(a_\mu^{M_0 j}|\eta|)^{\frac{\delta}{\mu+1}}. \qquad (3.12)$$

Let $\Lambda^1 \in \mathcal{S}\left(\mathbf{R}^N\right)$, and $\Lambda^2 \in \mathcal{S}\left(\mathbf{R}^M\right)$ be two Schwartz functions such that $(\Lambda^i\hat{)}(\zeta_i) = 1$ for $|\zeta_i| \leq \frac{1}{2}$ and $(\Lambda^i\hat{)}(\zeta_i) = 0$ for $|\zeta_i| \geq 1$, $i = 1, 2$ and define

$$(\Lambda_k^1\hat{)}(x) = (\Lambda^1\hat{)}\left(a_\mu^{N_0 k}x\right) \text{ and } (\Lambda_j^2\hat{)}(y) = (\Lambda^2\hat{)}\left(a_\mu^{M_0 j}y\right).$$

Define the sequence of measures $\{\nu_{k,j,\mu}\}$ by

$$\begin{aligned}\nu_{k,j,\mu}(\xi,\eta) &= \hat{\lambda}_{\Omega_\mu,\Phi,\Psi,k,j}(\xi,\eta) - (\Lambda_k^1\hat{)}(\xi)\,\hat{\lambda}_{\Omega_\mu,\mathcal{P},\Psi,k,j}(\xi,\eta) - (\Lambda_j^2\hat{)}(\eta)\times\\ &\quad \hat{\lambda}_{\Omega_\mu,\Phi,\mathcal{Q},k,j}(\xi,\eta) + (\Lambda_k^1\hat{)}(\xi)(\Lambda_j^2\hat{)}(\eta)\hat{\lambda}_{\Omega_\mu,\mathcal{P},\mathcal{Q},k,j}(\xi,\eta). \qquad (3.13)\end{aligned}$$

Then by (2.7), (3.8)–(3.12), (3.13) we have

$$|\hat{\nu}_{k,j,\mu}(\xi,\eta)| \leq C(\mu+1)^2; \tag{3.14}$$

and

$$|\hat{\nu}_{k,j,\mu}(\xi,\eta)| \leq C(\mu+1)^2 (a_\mu^{N_0 k}\,|\xi|)^{\pm\frac{\delta}{(\mu+1)}}\;(a_\mu^{M_0 j}\,|\eta|)^{\pm\frac{\delta}{2(\mu+1)}}. \tag{3.15}$$

Now let

$$\mathbf{g}_\mu f(x,y) = \left(\sum_{k\leq k_0, j\leq j_0} |\nu_{k,j,\mu} * f(x,y)|^2\right)^{\frac{1}{2}} \tag{3.16}$$

and

$$\nu_\mu^*(f)(x,y) = \sup_{k\leq k_0, j\leq j_0} \left||\nu_{k,j,\mu}| * f(x,y)\right|. \tag{3.17}$$

Thus,

$$\begin{aligned}\lambda^*_{\Omega_\mu,\Phi,\Psi} f(x,y) \quad &\leq \quad \mathbf{g}_\mu f(x,y) + C(\mathcal{M}_{\mathbf{R}^N}\otimes id_{\mathbf{R}^M})\circ(\lambda^*_{\Omega_\mu,\mathcal{P},\Psi} f(x,y)) + \\ &\quad\; 2C(id_{\mathbf{R}^N}\otimes\mathcal{M}_{\mathbf{R}^M})\circ(\lambda^*_{\Omega_\mu,\Phi,\mathcal{Q}})f(x,y)) + 2C(\mathcal{M}_{\mathbf{R}^N}\otimes id_{\mathbf{R}^M}) \\ &\quad\; \circ(id_{\mathbf{R}^N}\otimes\mathcal{M}_{\mathbf{R}^M})\circ(\lambda^*_{\Omega_\mu,\mathcal{P},\mathcal{Q}} f(x,y))\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}\nu_\mu^* f(x,y) \quad &\leq \quad \mathbf{g}_\mu f(x,y) + 2C(\mathcal{M}_{\mathbf{R}^N}\otimes id_{\mathbf{R}^M})\circ(\lambda^*_{\Omega_\mu,\mathcal{P},\Psi} f(x,y)) + \\ &\quad\; 2C(id_{\mathbf{R}^N}\otimes\mathcal{M}_{\mathbf{R}^M})\circ(\lambda^*_{\Omega_\mu,\Phi,\mathcal{Q}})f(x,y)) + 2C(\mathcal{M}_{\mathbf{R}^N}\otimes id_{\mathbf{R}^M}) \\ &\quad\; \circ(id_{\mathbf{R}^N}\otimes\mathcal{M}_{\mathbf{R}^M})\circ(\lambda^*_{\Omega_\mu,\mathcal{P},\mathcal{Q}} f(x,y)),\end{aligned} \tag{3.19}$$

where $\mathcal{M}_{\mathbf{R}^d}$ denotes the classical Hardy-Littlewood maximal function on $\mathbf{R}^d$.

Now by Lemmas 3.3, 3.4 and the boundedness of $\mathcal{M}_{\mathbf{R}^d}$ on L^p spaces, for $1 < p < \infty$ there exists a positive constant C_p independent of μ such that

$$\left\|(\mathcal{M}_{\mathbf{R}^N}\otimes id_{\mathbf{R}^M})\circ(\lambda^*_{\Omega_\mu,\mathcal{P},\Psi} f)\right\|_p \quad\leq\quad C_p(\mu+1)^2\,\|f\|_p\,, \tag{3.20}$$

$$\left\|(id_{\mathbf{R}^N}\otimes\mathcal{M}_{\mathbf{R}^M})\circ(\lambda^*_{\Omega_\mu,\Phi,\mathcal{Q}} f)\right\|_p \quad\leq\quad C_p(\mu+1)^2\,\|f\|_p\,, \tag{3.21}$$

and

$$\left\|(\mathcal{M}_{\mathbf{R}^N}\otimes id_{\mathbf{R}^M})\circ(id_{\mathbf{R}^N}\otimes\mathcal{M}_{\mathbf{R}^M})\circ(\lambda^*_{\Omega_\mu,\mathcal{P},\mathcal{Q}} f)\right\|_p \leq C_p(\mu+1)^2\,\|f\|_p \tag{3.22}$$

for every $f \in L^p\left(\mathbf{R}^N \times \mathbf{R}^M\right)$.

By (3.14), (3.15) and Plancherel's theorem, there exists a positive constant $C > 0$ independent of μ such that

$$\|\mathbf{g}_\mu f\|_2 \le C(\mu+1)^2 \|f\|_2 . \tag{3.23}$$

Therefore, by (3.19)–(3.22), we get

$$\left\|\nu_\mu^*(f)\right\|_2 \le C(\mu+1)^2 \|f\|_2 . \tag{3.24}$$

Thus, by (3.14), (3.24) and using Lemma 2.3 with $p_0 = 4$ and $q = 2$, we get

$$\left\|\left(\sum_{k\le k_0, j\le j_0} |\nu_{k,j,\mu} * g_{k,j}|^2\right)^{1/2}\right\|_4 \le C(\mu+1)^2 \left\|\left(\sum_{k\le k_0, j\le j_0} |g_{k,j}|^2\right)^{1/2}\right\|_4 \tag{3.25}$$

for arbitrary functions $\{g_{k,j}\}_{k,j\in\mathbf{Z}}$ on $\mathbf{R}^N \times \mathbf{R}^M$.

By (3.15), (3.25) and invoking Lemma 2.4, we obtain that

$$\|\mathbf{g}_\mu f\|_p \le C_p(\mu+1)^2 \|f\|_p \tag{3.26}$$

holds for $4/3 < p < 4$ and $f \in L^p(\mathbf{R}^N \times \mathbf{R}^M)$ with a positive constant C_p independent of μ.

By replacing $p = 2$ with $p = (4/3) + \varepsilon$ ($\varepsilon > 0$) in (3.23) and repeating the preceding arguments we get

$$\|\mathbf{g}_\mu f\|_p \le C_p(\mu+1)^2 \|f\|_p \tag{3.27}$$

for $8/7 < p < 8$ and $f \in L^p(\mathbf{R}^N \times \mathbf{R}^M)$. By continuing this process, we get

$$\|\mathbf{g}_\mu f\|_p \le C_p(\mu+1)^2 \|f\|_p \tag{3.28}$$

for $1 < p < \infty$ and $f \in L^p(\mathbf{R}^N \times \mathbf{R}^M)$, where C_p is a constant independent of μ. Hence by (3.18), (3.20)–(3.22) and (3.27) we obtain (3.7) to complete the proof.

4. Proof of the main theorem

Assume that $\Omega \in L(\log^+ L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$. As in [2] we decompose Ω as follows: For $\mu \in \mathbf{N}$ let $E_\mu = \{(x,y) \in \mathbf{S}^{n-1} \times \mathbf{S}^{m-1}\colon\ 2^{\mu-1} \le |\Omega\ (x,y)| < 2^\mu\}$, $b_\mu = \Omega\chi_{\mathbf{E}_\mu}$ and

$C_\mu = \|b_\mu\|_1$. Let $\mathbf{D} = \left\{\mu \in \mathbf{N} : C_{_\mu} \geq 2^{-4\mu}\right\}$,

$$\begin{aligned}\Omega_\mu(x,y) &= (C_\mu)^{-1}\left(b_\mu(x,y) - \int_{\mathbf{S}^{n-1}} b_\mu(u,y)d\sigma(u) - \int_{\mathbf{S}^{m-1}} b_\mu(x,v)d\sigma(v)\right.\\ &\quad \left. + \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} b_\mu(u,v)d\sigma(u)d\sigma(v)\right)\end{aligned}$$

for $\mu \in \mathbf{D}$ and

$$\Omega_0 = \Omega - \sum_{\mu\in\mathbf{D}} \Omega_\mu.$$

Then it is easy to verify that

$$\int_{\mathbf{S}^{n-1}} \Omega_\mu\left(u,\cdot\right)d\sigma\left(u\right) = \int_{\mathbf{S}^{m-1}} \Omega_\mu\left(\cdot,v\right)d\sigma\left(v\right) = 0, \tag{4.1}$$

$$\|\Omega_\mu\|_1 \leq 4,\ \left\|\Omega_{_\mu}\right\|_2 \leq 4(a_{_\mu})^2, \tag{4.2}$$

$$\Omega(x,y) = \sum_{\mu\in\mathbf{D}\cup\{0\}} C_\mu\Omega_\mu(x,y), \tag{4.3}$$

$$\sum_{\mu\in\mathbf{D}\cup\{0\}} (\mu+1)^2 C_\mu \leq C\left\|\Omega\right\|_{L(\log L)^2(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}, \tag{4.4}$$

for $\mu \in \mathbf{D}\cup\{0\}$ where we used $C_0 = 1$.

By (4.4)

$$\left\|T_{\Phi,\Psi}f\right\|_p \leqslant \sum_{\mu\in\mathbf{D}\cup\{0\}} \left|C_{_\mu}\right| \left\|T_{\Omega_\mu}f\right\|_p \tag{4.5}$$

where

$$T_{\Omega_\mu}f(x,y) = \text{p.v.}\int_{\mathbf{B}_n(0,1)\times\mathbf{B}_m(0,1)} f\left(x-\Phi(u), y-\Psi(v)\right)\frac{\Omega_\mu\left(u,v\right)}{|u|^n\,|v|^m}dudv. \tag{4.6}$$

Let N_0, M_0, $\mathcal{P}$ and $\mathcal{Q}$ be given as in the proof of Lemma 3.5. For $1 \leqslant l \leqslant N$, $1 \leqslant s \leqslant M$ let $c_{l,\alpha} = \frac{1}{\alpha!}\frac{\partial^\alpha \Phi_l}{\partial x^\alpha}(0)$ and $d_{s,\beta} = \frac{1}{\beta!}\frac{\partial^\beta \Psi_s}{\partial y^\beta}(0)$. For $0 \leqslant \tau \leqslant N_0, 0 \leqslant \kappa \leqslant M_0$ we define $P_\tau = (P_{l,\tau}, \ldots, P_{N,\tau})$ and $Q_\kappa = (Q_{1,\kappa}, \ldots, Q_{M,\kappa})$ by

$$P_{l,\tau}(x) = \sum_{|\alpha|\leqslant\tau} c_{l,\alpha}\, x^{\alpha}, \qquad \text{for } l = 1, \ldots, N,\ 0 \leqslant \tau \leqslant N_0 - 1; \tag{4.7}$$

$$Q_{s,\kappa}(y) = \sum_{|\beta|\leqslant\kappa} d_{s,\beta}\, y^{\beta}, \qquad \text{for } s = 1, \ldots, M,\ 0 \leqslant \kappa \leqslant M_0 - 1; \tag{4.8}$$

$P_{N_0} = \Phi$ and $Q_{M_0} = \Psi$. For each $0 \leqslant \tau \leqslant N_0$; $0 \leqslant \kappa \leqslant M_0$, let $\lambda^{(\tau,\kappa)}_{\Omega_\mu,k,j} = \lambda_{\Omega_\mu,P_\tau,Q_\kappa,k,j}$. Let $\omega(\tau)$ and $\gamma(\kappa)$ denote the number of multi-indices $\alpha \in (\mathbf{N}\cup\{0\})^n$ and $\beta \in (\mathbf{N}\cup\{0\})^m$ satisfying $|\alpha| = \tau$ and $|\beta| = \kappa$, respectively. Label the coordinates of $\mathbf{R}^{\omega(\tau)}$ and $\mathbf{R}^{\gamma(\kappa)}$ by the of multi-indices α and β with $|\alpha| = \tau$ and $|\beta| = \kappa$, respectively. That is, $\mathbf{R}^{\omega(\tau)} = \{(x_\alpha)\}_{|\alpha|=\tau}$ and $\mathbf{R}^{\gamma(\kappa)} = \{(y_\beta)\}_{|\beta|=\kappa}$. For $0 \leqslant \tau \leqslant N_0$ and $0 \leqslant \kappa \leqslant M_0$, we define the linear transformations $L_\tau : \mathbf{R}^N \to \mathbf{R}^{\omega(\tau)}$ and $Q_\kappa : \mathbf{R}^M \to \mathbf{R}^{\gamma(\kappa)}$ by

$$\left(L_\tau(\xi)\right)_\alpha = \sum_{l=1}^{\tau} c_{l,\alpha}\ \xi_l \text{ and } \left(Q_\kappa(\eta)\right)_\beta = \sum_{s=1}^{\kappa} d_{s,\beta}\ \eta_s$$

for $|\alpha| = \tau, |\beta| = \kappa$, $0 \leqslant \tau \leqslant N_0 - 1$ and $0 \leqslant \kappa \leqslant M_0 - 1$, where $\omega(N_0) = N_0$ and $\gamma(M_0) = M_0$. Then by Lemmas 2.7, 2.8, (2.7), (3.8)–(3.12) and the same argument as in proofs of (2.7), we get

$$\left\|\lambda^{(\tau,\kappa)}_{\Omega_\mu,k,j}\right\| \leqslant C(\mu+1)^2; \tag{4.9}$$

$$\left|\hat{\lambda}^{(\tau,\kappa)}_{\Omega_\mu,k,j}(\xi,\eta)\right| \leqslant C(\mu+1)^2 \left|a_\mu^{\tau k} L_\tau(\xi)\right|^{-\frac{\alpha_\tau}{\mu}} \left|a_\mu^{\kappa j} Q_\kappa(\eta)\right|^{-\frac{\alpha_\kappa}{\mu+1}}; \tag{4.10}$$

$$\left|\hat{\lambda}^{(\tau,\kappa)}_{\tilde{b}_\mu,k,j,\rho_\mu}(\xi,\eta) - \hat{\lambda}^{(\tau-1,\kappa)}_{\tilde{b}_\mu,k,j,\rho_\mu}(\xi,\eta)\right| \leqslant C(\mu+1)^2 \left|a_\mu^{\tau k} L_\tau(\xi)\right|^{\frac{\alpha_\tau}{\mu+1}} \left|a_\mu^{\kappa j} Q_\kappa(\eta)\right|^{-\frac{\alpha_\kappa}{\mu+1}}; \tag{4.11}$$

$$\left|\hat{\lambda}^{(\tau,\kappa)}_{\Omega_\mu,k,j}(\xi,\eta) - \hat{\lambda}^{(\tau,\kappa-1)}_{\Omega_\mu,k,j}(\xi,\eta)\right| \leqslant C(\mu+1)^2 \left|a_\mu^{\tau k} L_\tau(\xi)\right|^{-\frac{\alpha_\tau}{\mu+1}} \left|a_\mu^{\kappa j} Q_\kappa(\eta)\right|^{\frac{\alpha_\kappa}{\mu+1}}; \tag{4.12}$$

$$\begin{aligned} &\left|\hat{\lambda}^{(\tau,\kappa)}_{\Omega_\mu,k,j}(\xi,\eta) - \hat{\lambda}^{(\tau-1,\kappa)}_{\Omega_\mu,k,j}(\xi,\eta) - \hat{\lambda}^{(\tau,\kappa-1)}_{\Omega_\mu,k,j}(\xi,\eta) + \hat{\lambda}^{(\tau-1,\kappa-1)}_{\Omega_\mu,k,j}(\xi,\eta)\right| \\ \leqslant\ & C(\mu+1)^2 \left|a_\mu^{\tau k} L_\tau(\xi)\right|^{\frac{\alpha_\tau}{\mu+1}} \left|a_\mu^{\kappa j} Q_\kappa(\eta)\right|^{\frac{\alpha_\kappa}{\mu+1}}; \end{aligned} \tag{4.13}$$

$$\left|\hat{\lambda}^{(\tau,\kappa-1)}_{\Omega_\mu,k,j}(\xi,\eta)-\hat{\lambda}^{(\tau-1,\kappa-1)}_{\Omega_\mu,k,j}(\xi,\eta)\right| \leqslant C(\mu+1)^2\left|a_{\mu}^{\tau k}L_\tau(\xi)\right|^{\frac{\alpha_\tau}{\mu+1}}; \tag{4.14}$$

$$\left|\hat{\lambda}^{(\tau-1,\kappa)}_{\Omega_\mu,k,j}(\xi,\eta)-\hat{\lambda}^{(\tau-1,\kappa-1)}_{\Omega_\mu,k,j}(\xi,\eta)\right| \leq C(\mu+1)^2\left|a_{\mu}^{\kappa j}Q_\kappa(\eta)\right|^{\frac{\alpha_\kappa}{\mu+1}} \tag{4.15}$$

for $\mu\in\mathbf{D}\cup\{0\}, 1\leqslant\tau\leqslant N_0$ and $1\leqslant\kappa\leqslant M_0$.

By invoking Lemmas 3.3–3.5, (4.9)–(4.15), and Lemmas 2.3, 2.4 we get

$$\left\|T_{\Omega_\mu}f\right\|_p=\left\|\sum_{k\leq k_0,j\leq j_0}\lambda^{(N_0,M_0)}_{\Omega_\mu,k,j}*f\right\|_p\leq C_p\,(\mu+1)^2\left\|f\right\|_p, \tag{4.16}$$

for every $f\in L^p(\mathbf{R}^N\times\mathbf{R}^M)$, $\mu\in\mathbf{D}\cup\{0\}$, and for all p, $1<p<\infty$. Hence, (1.7) follows by (4.4), (4.5) and (4.16).

One may construct a proof for (1.8) by using the above estimates and employing the techniques in [1]. We omit the details.

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