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## On General Fibonacci Sequences in Groups

*Engin Özkan*

### Abstract

In this paper, we have constituted 3-step general Fibonacci sequences in a nilpotent group with exponent  $p$  ( $p$  is a prime number) and nilpotency class 4 and given formulas to find the  $\alpha$  term of the sequence.

**Key Words:** General Fibonacci sequences; nilpotent group; nilpotency class; fundamental period.

### 1. Introduction

Let  $s_i$  denote the 3-step general recurrence defined by  $s_i = ls_{i-1} + ms_{i-2} + ns_{i-3}$  for some  $l, m, n \in \mathbb{N}$ . We assume that  $p$  does not divide  $n$ ; then we get the definition of a 3-step general standard Fibonacci sequence as  $(0, 0, 1, l, l^2 + m, l(l^2 + m) + lm + n, \dots)$  in  $Z/pZ$ . If  $p$  were permitted to divide  $n$ , then the sequence would ultimately be periodic, but would never return to  $0, 0, 1$ . This sequence or loop must be periodic and we use the letter  $k$  to denote the fundamental period of  $s_i$  that is the shortest period of that sequence. The fundamental period of a sequence satisfying a linear recurrence is sometimes called the Wall number of that sequence. Obviously  $k$  depends on  $p$ .

In the recent years, there has been much interest in applications of Fibonacci numbers and sequences. Takahashi gives a fast algorithm which is based on the product of Lucas numbers to compute large Fibonacci numbers [8]. Fibonacci sequences have been an interesting subject in applied mathematics. West has shown by using transfer matrices

that the number  $|S_n(123, 3214)|$  of permutations avoiding the patterns 123 and 3214 is the Fibonacci number  $F_{2n}$  [11].

The study of Fibonacci sequences in groups began with the earlier work of Wall [10], where the ordinary Fibonacci sequences in cyclic groups were investigated. Vinson was particularly interested in ranks of apparition in ordinary Fibonacci sequences [9]. In the mid 1980's, Wilcox extended the problem to abelian groups [12]. Prolific co-operation among Campbell, Doostie and Robertson expanded the theory to some finite simple groups [3]. Ryba constructed and analyzed a pair of sequences of representations of the symmetric groups [7]. Aydın and Smith proved in [2] that the lengths of ordinary 2-step Fibonacci sequences are equal to the lengths of the 2-step Fibonacci recurrences in finite nilpotent groups of nilpotency class 4 and a prime number  $p$  exponent. Furthermore, Aydın and Dikici proved in [1] that the length of the 2-step general Fibonacci sequences are equal to the length of the 2-step general Fibonacci recurrences constructed by two generating elements of finite nilpotent groups of nilpotency class 2 and exponent a prime number  $p$ . Dikici and Smith proved in [5] that, for the 3-step Fibonacci recurrence and any finite  $p$ -group of exponent  $p$  and nilpotency class 2, the length of a fundamental period of any loop satisfying the recurrence must divide the period of the ordinary 3-step Fibonacci sequence in the field  $GF(p)$ . Dikici and Özkan proved in [4] for the 3-step general Fibonacci recurrence and any finite  $p$ -group of exponent  $p$  and nilpotency class 2, the length of a fundamental period of any loop satisfying the recurrence must divide the period of the ordinary 3-step general Fibonacci sequence in the field  $GF(p)$ .

One of the latest works in this area is [6] in which if  $G$  is a non-trivial finite  $p$ -group of exponent  $p$  and nilpotency class 4, then the followings hold:

i.  $k(G) = k$ , except for finite primes;

and

ii.  $k(G) = kp$  for all primes. ( $p$  is a prime number and  $p > 3$ ).

**Definition 1** *Let  $H$  and  $K$  be normal in  $G$  and  $K \leq H$ . If  $H/K$  is contained in the centre of  $G/K$ , then  $H/K$  is called a central factor of  $G$ . A group  $G$  is called nilpotent if it has a finite series of normal subgroups*

$$G = G_0 \geq G_1 \geq \dots \geq G_r = 1$$

such that  $G_{i-1}/G_i$  is a central factor of  $G$  for each  $i = 1, 2, \dots, r$ . The smallest possible  $r$  is called the nilpotency class of  $G$ .

**The Main Result**

Let  $G$  be a nilpotent group with nilpotency class 4 and prime number exponent  $p > 3$ .  $G$  has two generators  $x$  and  $y$ . A presentation of  $G$  is

$$G = \langle x, y, z, t, u : (y, x) = z, (z, x) = t, (t, x) = u \rangle,$$

where pairs of generators with unspecified commutator are implicitly deemed to commute. The subgroup  $\langle y, z, t, u \rangle$  is abelian and  $u$  is in the centre of  $G$ . Every element of  $G$  has a unique representation as

$$x^a y^b z^c t^d u^e,$$

where the exponents are elements of  $GF(p)$ . Having established this way of writing elements, we can even think of groups of elements as vectors of dimension 5 over  $GF(p)$ , i.e. as  $(a, b, c, d, e)$ .

Let  $(x^a y^b z^c t^d u^e)$  be an element of  $G$ . We claim that

$$\begin{aligned} (x^a y^b z^c t^d u^e)^n &= \left( x^{na} y^{nb} z^{nc + \binom{n}{2} ba_t nd + \binom{n}{2} ca + \binom{n}{2} b \binom{a}{2} + \binom{n}{3} ba^2} u^{ne + \binom{n}{2} da + \binom{n}{2} c \binom{a}{2} + \binom{n}{2} b \binom{a}{3}} \right. \\ &\quad \left. u^{[\binom{n-1}{2} + \binom{n-2}{2} + \dots + \binom{2}{2}] ca^2 + [\binom{n-1}{3} + \binom{n-2}{3} + \dots + \binom{3}{3}] ba^3 + 2[\binom{n-1}{2} + \binom{n-2}{2} + \dots + \binom{2}{2}] ba \binom{a}{2}} \right) \end{aligned} \tag{1}$$

We show this by induction on  $n$ . Since  $\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} = 0$  our argument is obvious for  $n = 1$ . So assume that

$$\begin{aligned} (x^a y^b z^c t^d u^e)^{n-1} &= \left( x^{(n-1)a} y^{(n-1)b} z^{(n-1)c + \binom{n-1}{2} ba_t (n-1)d + \binom{n-1}{2} ca + \binom{n-1}{2} b \binom{a}{2} + \binom{n-1}{3} ba^2} \right. \\ &\quad \left. u^{(n-1)e + \binom{n-1}{2} da + \binom{n-1}{2} c \binom{a}{2} + \binom{n-1}{2} b \binom{a}{3}} \right. \\ &\quad \left. u^{[\binom{n-2}{2} + \binom{n-3}{2} + \dots + \binom{2}{2}] ca^2 + [\binom{n-2}{3} + \binom{n-3}{3} + \dots + \binom{3}{3}] ba^3 + 2[2\binom{n-2}{2} + \binom{n-3}{2} + \dots + \binom{2}{2}] ba \binom{a}{2}} \right) \end{aligned}$$

Multiplication of both sides by  $x^a y^b z^c t^d u^e$  gives the left hand side of (1). Thus we have

$$(x^a y^b z^c t^d u^e)^n = \left( x^{(n-1)a} y^{(n-1)b} z^{(n-1)c + \binom{n-1}{2} ba} t^{(n-1)d + \binom{n-1}{2} ca + \binom{n-1}{2} b \binom{a}{2} + \binom{n-1}{3} ba^2} \right. \\ \left. u^{(n-1)e + \binom{n-1}{2} da + \binom{n-1}{2} c \binom{a}{2} + \binom{n-1}{2} b \binom{a}{3}} \right) \\ u^{[\binom{n-2}{2} + \binom{n-3}{2} + \dots + \binom{2}{2}]ca^2 + [\binom{n-2}{3} + \binom{n-3}{3} + \dots + \binom{3}{3}]ba^3 + 2[\binom{n-2}{2} + \binom{n-3}{2} + \dots + \binom{2}{2}]ba \binom{a}{2}} (x^a y^b z^c t^d u^e)$$

Notice that  $u$  is in the centre of  $G$  and  $(y, x) = z$ ,  $(z, x) = t$  and  $(t, x) = u$ , so that

$$(x^a y^b z^c t^d u^e)^n = x^{na} y^{nb} z^{nc + \binom{n}{2} ba} t^{nd + \binom{n}{2} ca + \binom{n}{2} b \binom{a}{2} + \binom{n}{3} ba^2} \\ u^{ne + \binom{n}{2} da + \binom{n}{2} c \binom{a}{2} + \binom{n}{2} b \binom{a}{3} + [\binom{n-1}{2} + \binom{n-2}{2} + \dots + \binom{2}{2}]ca^2} \\ u^{[\binom{n-1}{3} + \binom{n-2}{3} + \dots + \binom{3}{3}]ba^3 + 2[\binom{n-1}{2} + \binom{n-2}{2} + \dots + \binom{2}{2}]ba \binom{a}{2}}$$

Let upper side of  $x, y, z, t$  and  $u$  be  $A, B, C, D$  and  $E$ , respectively. That is,

$$A = na \\ B = nb$$

$$C = nc + \binom{n}{2} ba$$

$$D = nd + \binom{n}{2} ca + \binom{n}{2} b \binom{a}{2} + \binom{n}{3} ba^2$$

and

$$E = ne + \binom{n}{2} da + nc \binom{n}{2} + \binom{n}{2} b \binom{n}{3} + [\binom{2}{2} + \binom{3}{2} + \dots + \binom{n-2}{2}] \\ + \binom{n-1}{2} ca^2 + [\binom{3}{3} + \binom{4}{3} + \dots + \binom{n-2}{3} + \binom{n-1}{3}] ba^3 + 2[\binom{2}{2}] \\ + \binom{3}{2} + \dots + \binom{n-2}{2} + \binom{n-1}{2}] ba \binom{a}{2}.$$

Since  $\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \dots + \binom{n-1}{r} = \binom{n}{r+1}$ ,

$$E = ne + \binom{n}{2}da + \binom{n}{2}c\binom{a}{2} + \binom{n}{2}b\binom{a}{3} + \binom{n}{3}ca^2 + \binom{n}{4}ba^3 + 2\binom{n}{3}ba\binom{a}{2}.$$

Let  $(x^{a_0}y^{b_0}z^{c_0}t^{d_0}u^{e_0})$ ,  $(x^{a_1}y^{b_1}z^{c_1}t^{d_1}u^{e_1})$  and  $(x^{a_2}y^{b_2}z^{c_2}t^{d_2}u^{e_2})$  be three elements of  $G$ . We get the formula for multiplication of powers of these three elements. That is, using (1),

$$(x^{a_0}y^{b_0}z^{c_0}t^{d_0}u^{e_0})^l(x^{a_1}y^{b_1}z^{c_1}t^{d_1}u^{e_1})^m(x^{a_2}y^{b_2}z^{c_2}t^{d_2}u^{e_2})^n = (x^{a_3}y^{b_3}z^{c_3}t^{d_3}u^{e_3})$$

where  $(x^{a_0}y^{b_0}z^{c_0}t^{d_0}u^{e_0})^l = x^{A_0}y^{B_0}z^{C_0}t^{D_0}u^{E_0}$ ,  $(x^{a_1}y^{b_1}z^{c_1}t^{d_1}u^{e_1})^m = x^{A_1}y^{B_1}z^{C_1}t^{D_1}u^{E_1}$  and  $(x^{a_2}y^{b_2}z^{c_2}t^{d_2}u^{e_2})^n = x^{A_2}y^{B_2}z^{C_2}t^{D_2}u^{E_2}$  for simplicity. Furthermore,

$$A_0 = la_0$$

$$B_0 = lb_0$$

$$C_0 = lc_0 + \binom{l}{2}b_0a_0$$

$$D_0 = ld_0 + \binom{l}{2}c_0a_0 + \binom{l}{2}b_0\binom{a_0}{2} + \binom{l}{2}b_0a_0^2$$

and

$$E_0 = le_0 + \binom{l}{2}d_0a_0 + \binom{l}{2}c_0\binom{a_0}{2} + \binom{l}{2}b_0\binom{a_0}{3} + \binom{l}{3}c_0a_0^2 + \binom{l}{4}4b_0a_0^3 + 2\binom{l}{3}b_0a_0\binom{a_0}{2}.$$

Similarly,

$$A_1 = ma_1$$

$$B_1 = mb_1$$

$$C_1 = mc_1 + \binom{m}{2}b_1a_1$$

$$D_1 = md_1 + \binom{m}{2}c_1a_1 + \binom{m}{2}b_1\binom{a_1}{2} + \binom{m}{3}b_1a_1^2$$

and

$$E_1 = le_1 + \binom{m}{2}d_1a_1 + \binom{m}{2}c_1\binom{a_1}{2} + \binom{m}{2}b_1\binom{a_1}{3} + \binom{m}{3}c_1a_1^2 + \binom{m}{4}4b_1a_1^3 + 2\binom{m}{3}b_1a_1\binom{a_1}{2}.$$

Finally,

$$A_2 = na_2$$

$$B_2 = nb_2$$

$$C_2 = nc_2 + \binom{n}{2}b_2a_2$$

$$D_2 = nd_2 + \binom{n}{2}c_2a_2 + \binom{n}{2}b_2\binom{a_2}{2} + \binom{n}{3}b_2a_2^2$$

and

$$E_2 = ne_2 + \binom{n}{2}d_2a_2 + \binom{n}{2}c_2\binom{a_2}{2} + \binom{n}{2}b_2\binom{a_2}{3} + \binom{n}{3}c_2a_2^2 + \binom{n}{4}b_2a_2^3 + 2\binom{n}{3}b_2a_2\binom{a_2}{2}.$$

Thus,

$$\begin{aligned} & (x^{A_0}y^{B_0}z^{C_0}t^{D_0}u^{E_0})(x^{A_1}y^{B_1}z^{C_1}t^{D_1}u^{E_1})(x^{A_2}y^{B_2}z^{C_2}t^{D_2}u^{E_2}) = x^{A_3}y^{B_3}z^{C_3}t^{D_3}u^{E_3} \\ & = (x^{A_0}y^{B_0}z^{C_0}x^{A_1}y^{B_1}z^{C_1}t^{D_0+D_1}u^{E_0+E_1+D_0A_1})(x^{A_2}y^{B_2}z^{C_2}t^{D_2}u^{E_2}) \\ & = (x^{A_0}y^{B_0}x^{A_1}y^{B_1}z^{C_0+C_1}t^{D_0+D_1+C_0A_1}u^{E_0+E_1+D_0A_1+C_0\binom{A_1}{2}})(x^{A_2}y^{B_2}z^{C_2}t^{D_2}u^{E_2}) \\ & = (x^{A_0+A_1}y^{B_0+B_1}z^{C_0+C_1+B_0A_1}t^{D_0+D_1+C_0A_1+B_0\binom{A_1}{2}} \\ & u^{E_0+E_1+D_0A_1+C_0\binom{A_1}{2}+b_0\binom{A_1}{3}})(x^{A_2}y^{B_2}z^{C_2}t^{D_2}u^{E_2}) \\ & = x^{A_0+A_1}y^{B_0+B_1}z^{C_0+C_1+B_0A_1}x^{A_2}y^{B_2}z^{C_2}t^{D_0+D_1+D_2+C_0A_1+B_0\binom{A_1}{2}} \\ & u^{E_0+E_1+E_2+D_0A_1+C_0\binom{A_1}{2}+B_0\binom{A_1}{3}+A_2(D_0+D_1+D_2+C_0A_1+B_0\binom{A_1}{2})} \\ & = x^{A_0+A_1}y^{B_0+B_1}x^{A_2}y^{B_2}z^{C_0+C_1+C_2+B_0A_1}t^{D_0+D_1+D_2+C_0A_1+B_0\binom{A_1}{2}}+A_2(C_0+C_1+B_0A) \\ & u^{E_0+E_1+E_2+D_0A_1+C_0\binom{A_1}{2}+B_0\binom{A_1}{3}+A_2(D_0+D_1+D_2+C_0A_1+B_0\binom{A_1}{2})}+(C_0+C_1+B_0A_1)\binom{A_2}{2} \\ & = x^{A_0+A_1+A_2}y^{B_0+B_1+B_2}z^{C_0+C_1+C_2+B_0A_1+(B_0+B_1)A_2}t^{D_0+D_1+D_2+C_0A_1+B_0\binom{A_1}{2}} \\ & t^{A_2(C_0+C_1+B_0A_1)+(B_0+B_1)\binom{A_2}{2}}u^{E_0+E_1+E_2+D_0A_1+C_0\binom{A_1}{2}+B_0\binom{A_1}{3}} \\ & u^{A_2(D_0+D_1+D_2+C_0A_1+B_0\binom{A_1}{2})+(C_0+C_1+B_0A_1)\binom{A_2}{2}+(B_0+B_1)\binom{A_2}{3}}. \end{aligned}$$

We can write

$$\begin{aligned} A_3 &= A_0 + A_1 + A_2 \\ B_3 &= B_0 + B_1 + B_2 \end{aligned}$$

$$C_3 = C_0 + C_1 + C_2 + B_0A_1 + (B_0 + B_1)A_2$$

$$D_3 = D_0 + D_1 + D_2 + C_0A_1 + B_0\binom{A_1}{2} + A_2(C_0 + C_1 + B_0A_1) + \binom{A_2}{2}(B_0 + B_1)$$

$$\begin{aligned} E_3 &= E_0 + E_1 + E_2 + D_0A_1 + C_0\binom{A_1}{2} + B_0\binom{A_1}{3} + A_2\left(D_0 + D_1 + C_0A_1 + B_0\binom{A_1}{2}\right) \\ &\quad + \binom{A_2}{2}(C_0 + C_1 + B_0A_1) + \binom{A_2}{3}(B_0 + B_1) \end{aligned}$$

Thus, we can write

$$\begin{aligned} A_3 &= la_0 + ma_1 + na_2 \\ B_3 &= lb_0 + mb_1 + nb_2 \end{aligned}$$

$$C_3 = lc_0 + \binom{l}{2}b_0a_0 + mc_1 + \binom{m}{2}b_1a_1 + nc_2 + \binom{n}{2}b_2a_2 + lmb_0a_1 + (lb_0 + mb_1)na_2$$

$$\begin{aligned} D_3 &= ld_0 + \binom{l}{2}c_0a_0 + \binom{l}{2}b_0\binom{a_0}{2} + \binom{l}{3}b_0a_0^2 + md_1 + \binom{m}{2}c_1a_1 \\ &\quad + \binom{m}{2}b_1\binom{a_1}{2} + \binom{m}{3}b_1a_1^2 + nd_2 + \binom{n}{2}c_2a_2 + \binom{n}{2}b_2\binom{a_2}{2} + \binom{n}{3}b_2a_2^2 \\ &\quad + (lc_0 + \binom{l}{2}b_0a_0)ma_1 + lb_0\binom{ma_1}{2} + na_2(lc_0 + \binom{l}{2}b_0a_0) \\ &\quad + mc_1 + \binom{m}{2}b_1a_1 + lb_0ma_1 + \binom{na_2}{2}(lb_0 + mb_1) \end{aligned}$$



$$\begin{aligned}
 E_3 = & \quad l e_0 + \binom{l}{2} d_0 a_0 + \binom{l}{2} c_0 \binom{a_0}{2} + \binom{l}{3} b_0 \binom{a_0}{3} + \binom{l}{3} c_0 a_0^2 + \binom{l}{4} b_0 a_0^3 \\
 & + 2 \binom{l}{3} b_0 a_0 \binom{a_0}{2} + m e_1 + \binom{m}{2} d_1 a_1 + \binom{m}{2} c_1 \binom{a_1}{2} + \binom{m}{2} b_1 \binom{a_1}{3} \\
 & + \binom{m}{3} c_1 a_1^2 + \binom{m}{4} b_1 a_1^3 + 2 \binom{m}{3} b_1 a_1 \binom{a_1}{2} + n e_2 + \binom{n}{2} d_2 a_2 + \binom{n}{2} c_2 \binom{a_2}{2} \\
 & + \binom{n}{2} b_2 \binom{a_2}{3} + \binom{n}{3} c_2 a_2^2 + \binom{n}{4} b_2 a_2^3 + 2 \binom{n}{3} b_2 a_2 \binom{a_2}{2} \\
 & + (l d_0 + \binom{l}{2} c_0 a_0 + \binom{l}{2} b_0 \binom{a_0}{2} + \binom{l}{3} b_0 a_0^2) m a_1 \\
 & + (l c_0 + \binom{l}{2} b_0 a_0) \binom{m a_1}{2} + l b_0 \binom{m a_1}{3} \\
 & + n a_2 (l d_0 + \binom{l}{2} c_0 a_0 + \binom{l}{2} b_0 \binom{a_0}{2} + \binom{l}{3} b_0 a_0^2 + m d_1 + \binom{m}{2} c_1 a_1 \\
 & + \binom{m}{2} b_1 \binom{a_1}{2} + \binom{m}{3} b_1 a_1^2 + (l c_0 + \binom{l}{2} b_0 a_0) m a_1 + l b_0 \binom{m a_1}{2}) \\
 & + \binom{n a_2}{2} (l c_0 + \binom{l}{2} b_0 a_0 + m c_1 + \binom{m}{2} b_1 a_1 + l b_0 m a_1) + \binom{n a_2}{3} (l b_0 + m b_1)
 \end{aligned}$$

We shall use vector notation to calculate the sequence and define a bi-infinite sequence  $(r_i) = (a_i, b_i, c_i, d_i, e_i)$  via the 3-step general recurrence. We must consider two types of initial data for loops in  $G$ . We have a loop  $v$  of type I with initial data

$$\begin{aligned}
 v_0 &= (0, 0, 0, 0, 0) \\
 v_1 &= (1, 0, 0, 0, 0) \\
 v_2 &= (0, 1, 0, 0, 0).
 \end{aligned}$$

and another  $w$  of type II with initial data

$$\begin{aligned}
 w_0 &= (1, 0, 0, 0, 0) \\
 w_1 &= (0, 1, 0, 0, 0) \\
 w_2 &= (0, 0, 0, 0, 0)
 \end{aligned}$$

The analysis of the type II loop is entirely similar to that of type I. Thus the type I loop

begins

$$\begin{aligned} v_0 &= (t_0, s_0, 0, 0, 0) \\ v_1 &= (t_1, s_1, 0, 0, 0) \end{aligned}$$

and

$$v_2 = (t_2, s_2, 0, 0, 0)$$

so that  $(a_i) = (t_i)$  and  $(b_i) = (s_i)$ .

It can easily be seen that the sequence  $t_i$  can be written in terms of  $s_i$  as  $t_i = s_{i+1} - ns_i$ . We shall need two formulas for  $C_\alpha$  and  $D_\alpha$  to work out the formula for  $E_\alpha$ ,  $\alpha \geq 0$ . Now, let the notation  $\sum$  denote the notation  $\sum_{i=0}^{\alpha-1}$ . Then by induction,

$$\begin{aligned} C_\alpha &= \binom{l}{2} \sum s_{\alpha-i-1} s_i t_i + \binom{m}{2} \sum s_{\alpha-i-1} s_{i+1} t_{i+1} + \binom{n}{2} \sum s_{\alpha-i-1} s_{i+2} t_{i+2} \\ &\quad + lm \sum s_{\alpha-i-1} s_i t_{i+1} + nl \sum s_{\alpha-i-1} s_i t_{i+2} + mn \sum s_{\alpha-i-1} s_{i+1} t_{i+2} \end{aligned}$$

and

$$\begin{aligned} D_\alpha &= \binom{l}{2} \sum s_{\alpha-i-1} c_i t_i + \binom{l}{2} \sum s_{\alpha-i-1} s_i \binom{t_i}{2} + \binom{l}{3} \sum s_{\alpha-i-1} s_i t_i^2 \\ &\quad + \binom{m}{2} \sum s_{\alpha-i-1} c_{i+1} t_{i+1} + \binom{m}{2} \sum s_{\alpha-i-1} s_{i+1} \binom{t_{i+1}}{2} + \binom{m}{3} \sum s_{\alpha-i-1} s_{i+1} t_{i+1}^2 \\ &\quad + \binom{n}{2} \sum s_{\alpha-i-1} c_{i+2} t_{i+2} + \binom{n}{2} \sum s_{\alpha-i-1} s_{i+2} \binom{t_{i+2}}{2} + \binom{n}{3} \sum s_{\alpha-i-1} s_{i+2} t_{i+2}^2 \\ &\quad + lm \sum s_{\alpha-i-1} c_i t_{i+1} + m \binom{l}{2} \sum s_{\alpha-i-1} s_i t_i t_{i+1} + l \sum s_{\alpha-i-1} s_i \binom{m t_{i+1}}{2} \\ &\quad + nl \sum s_{\alpha-i-1} c_i t_{i+2} + n \binom{l}{2} \sum s_{\alpha-i-1} s_i t_i t_{i+2} + nm \sum s_{\alpha-i-1} c_{i+1} t_{i+2} \\ &\quad + n \binom{m}{2} \sum s_{\alpha-i-1} s_{i+1} t_{i+1} t_{i+2} + nlm \sum s_{\alpha-i-1} s_i t_{i+1} t_{i+2} + l \sum s_{\alpha-i-1} s_i \binom{n t_{i+2}}{2} \\ &\quad + m \sum s_{\alpha-i-1} s_{i+1} \binom{n t_{i+2}}{2} \end{aligned}$$

for  $\alpha \geq 0$ . These enable us, via a similar process, to describe  $E_\alpha$  for  $\alpha \geq 0$  as

$$\begin{aligned}
 E_\alpha = & \binom{l}{2} \sum s_{\alpha-i-1} d_i t_i + \binom{l}{2} \sum s_{\alpha-i-1} c_i \binom{t_i}{2} + \binom{l}{2} \sum s_{\alpha-i-1} s_i \binom{t_i}{3} \\
 & + \binom{l}{3} \sum s_{\alpha-i-1} c_i t_i^2 + \binom{l}{4} \sum s_{\alpha-i-1} s_i t_i^3 + 2 \binom{l}{3} \sum s_{\alpha-i-1} s_i t_i \binom{t_i}{2} \\
 & + \binom{m}{2} \sum s_{\alpha-i-1} d_{i+1} t_{i+1} + \binom{m}{2} \sum s_{\alpha-i-1} c_{i+1} \binom{t_{i+1}}{2} \\
 & + \binom{m}{2} \sum s_{\alpha-i-1} s_{i+1} \binom{t_{i+1}}{3} + \binom{m}{3} \sum s_{\alpha-i-1} c_{i+1} t_{i+1}^2 \\
 & + \binom{n}{2} \sum s_{\alpha-i-1} d_{i+2} t_{i+2} + \binom{n}{2} \sum s_{\alpha-i-1} c_{i+2} \binom{t_{i+2}}{2} \\
 & + \binom{n}{2} \sum s_{\alpha-i-1} s_{i+2} \binom{t_{i+2}}{3} + \binom{n}{3} \sum s_{\alpha-i-1} c_{i+2} t_{i+2}^2 \\
 & + \binom{n}{4} \sum s_{\alpha-i-1} s_{i+2} t_{i+2}^3 + m \binom{l}{2} \sum s_{\alpha-i-1} s_i t_{i+1} \binom{t_i}{2} \\
 & + m \binom{l}{2} \sum s_{\alpha-i-1} s_i t_{i+1} t_i^2 + l \sum s_{\alpha-i-1} c_i \binom{m t_{i+1}}{2} \\
 & + \binom{l}{2} \sum s_{\alpha-i-1} s_i t_i \binom{m t_{i+1}}{2} + l \sum s_{\alpha-i-1} s_i \binom{m t_{i+1}}{3} + n l \sum s_{\alpha-i-1} d_i t_{i+2} \\
 & + n \binom{l}{2} \sum s_{\alpha-i-1} c_i t_i t_{i+2} + n \binom{l}{2} \sum s_{\alpha-i-1} s_i t_{i+2} \binom{t_i}{2} \\
 & + n \binom{l}{3} \sum s_{\alpha-i-1} s_i t_{i+2} t_i^2 + n m \sum s_{\alpha-i-1} d_{i+1} t_{i+2} \\
 & + n \binom{m}{2} \sum s_{\alpha-i-1} c_{i+1} t_{i+1} t_{i+2} + n \binom{m}{2} \sum s_{\alpha-i-1} s_{i+1} t_{i+2} \binom{t_{i+1}}{2} \\
 & + n \binom{m}{3} \sum s_{\alpha-i-1} s_{i+1} t_{i+2} t_{i+1}^2 + n l m \sum s_{\alpha-i-1} c_i t_{i+1} t_{i+2} \\
 & + n m \binom{l}{2} \sum s_{\alpha-i-1} s_i t_i t_{i+1} t_{i+2} + n l \sum s_{\alpha-i-1} s_i t_{i+2} \binom{m t_{i+1}}{2} \\
 & + l \sum s_{\alpha-i-1} c_i \binom{n t_{i+2}}{2} + \frac{l}{2} \sum s_{\alpha-i-1} s_i t_i \binom{n t_{i+1}}{2}
 \end{aligned}$$

$$\begin{aligned}
 &+m \sum s_{\alpha-i-1}c_{i+1} \binom{nt_{i+2}}{2} + \binom{m}{2} \sum s_{\alpha-i-1}s_{i+1}t_{i+1} \binom{nt_{i+2}}{2} \\
 &+lm \sum s_{\alpha-i-1}s_it_{i+1} \binom{nt_{i+2}}{2} + l \sum s_{\alpha-i-1}s_i \binom{nt_{i+2}}{3} + m \sum s_{\alpha-i-1}s_{i+1} \binom{nt_{i+2}}{3}.
 \end{aligned}$$

In the table,  $A(i) = T(i)$  and  $B(i) = S(i)$ .

Now we use these formulas to calculate the Fibonacci loop  $l(x, y, z, \dots)$ . The first few terms are as follows:

$$(0, 0, 0, 0, 0)$$

$$(1, 0, 0, 0, 0)$$

$$(0, 1, 0, 0, 0)$$

$$(m, n, 0, 0, 0)$$

$$\begin{aligned}
 &(l + mn, m + n^2, \binom{n}{2}mn + nm^2, \binom{n}{2}n \binom{m}{2} + \binom{n}{3}nm^2 + m \binom{n.m}{2}, \binom{n}{2}nm + \\
 &\binom{n}{4}nm^3 + 2 \binom{n}{3}nm \binom{m}{2} + \binom{n.m}{3}m).
 \end{aligned}$$

The following table shows elements of the sequence for various values of  $l, m$  and  $n$ . The elements can be obtained by hand and computer program. Moreover, we have more data for various values of  $l, m$  and  $n$ . Computer codes and their results are available on a request.

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Table

$l = 1$	A(3)	1	B(3)	1	C(3)	0	D(3)	0	E(3)	0
	A(4)	2	B(4)	2	C(4)	1	D(4)	0	E(4)	0
$m = 1$	A(5)	3	B(5)	4	C(5)	6	D(5)	4	E(5)	1
	A(6)	6	B(6)	7	C(6)	18	D(6)	23	E(6)	16
$n = 1$	A(7)	11	B(7)	13	C(7)	67	D(7)	16	E(7)	415
	A(8)	20	B(8)	24	C(8)	236	D(8)	1460	E(8)	6434
$l = 1$	A(3)	2	B(3)	2	C(3)	0	D(3)	0	E(3)	0
	A(4)	5	B(4)	6	C(4)	12	D(4)	14	E(4)	8
$m = 2$	A(5)	14	B(5)	17	C(5)	112	D(5)	462	E(5)	1310
	A(6)	40	B(6)	48	C(6)	928	D(6)	11616	E(6)	106474
$n = 2$	A(7)	113	B(7)	136	C(7)	7618	D(7)	281329	E(7)	7773408
	A(8)	320	B(8)	385	C(8)	61390	D(8)	6480888	E(8)	515371456
$l=1$	A(3)	2	B(3)	4	C(3)	0	D(3)	0	E(3)	0
	A(4)	9	B(4)	18	C(4)	64	D(4)	144	E(4)	208
$m=2$	A(5)	40	B(5)	81	C(5)	1564	D(5)	19874	E(5)	184202
	A(6)	180	B(6)	364	C(6)	32458	D(6)	1920918	E(6)	84789040
$n=4$	A(7)	809	B(7)	1636	C(7)	660424	D(7)	177539344	E(7)	1409806336
	A(8)	3636	B(8)	7353	C(8)	13361924	D(8)	-999957504	E(8)	-1920991232
$l=2$	A(3)	3	B(3)	5	C(3)	0	D(3)	0	E(3)	0
	A(4)	17	B(4)	28	C(4)	195	D(4)	915	E(4)	2990
$m=3$	A(5)	94	B(5)	157	C(5)	7243	D(5)	222964	E(5)	5090233
	A(6)	527	B(6)	879	C(6)	230513	D(6)	40249532	E(6)	967275520
$n=5$	A(7)	2951	B(7)	4922	C(7)	7256201	D(7)	-460688640	E(7)	-108789760
	A(8)	16524	B(8)	27561	C(8)	227678144	D(8)	-574029824	E(8)	-2013265920
$l=2$	A(3)	4	B(3)	1	C(3)	0	D(3)	0	E(3)	0
	A(4)	6	B(4)	5	C(4)	16	D(4)	24	E(4)	16
$m=4$	A(5)	22	B(5)	11	C(5)	108	D(5)	790	E(5)	4416
	A(6)	54	B(6)	33	C(6)	888	D(6)	14520	E(6)	176863
$n=1$	A(7)	154	B(7)	87	C(7)	6630	D(7)	33399	E(7)	13348748
	A(8)	414	B(8)	241	C(8)	49800	D(8)	6585600	E(8)	6879074592
$l=4$	A(3)	1	B(3)	2	C(3)	0	D(3)	0	E(3)	0
	A(4)	6	B(4)	5	C(4)	4	D(4)	1	E(4)	0
$m=1$	A(5)	13	B(5)	16	C(5)	114	D(5)	557	E(5)	1750
	A(6)	36	B(6)	45	C(6)	838	D(6)	9986	E(6)	84425
$n=2$	A(7)	109	B(7)	126	C(7)	6458	D(7)	217889	E(7)	5458466
	A(8)	306	B(8)	361	C(8)	55258	D(8)	5660503	E(8)	433183040

Conjecture

One can prove theorems similar to those in [4]. That is, there is a relation between the fundamental period (Wall number) of a 3-step general Fibonacci sequence and the length of a fundamental period of any loop satisfying the recurrence.

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