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## Shape Operator $A_H$ for Slant Submanifolds in Generalized Complex Space Forms

*Adela Mihai\**

### Abstract

In this article, we establish an inequality between the sectional curvature function  $K$  and the shape operator  $A_H$  at the mean curvature vector for slant submanifolds in generalized complex space forms. Also a sharp relationship between the  $k$ -Ricci curvature and the shape operator  $A_H$  is proved.

**Key Words:** Shape operator, slant submanifolds, generalized complex space form,  $k$ -Ricci curvature.

### 1. Preliminaries

In the introduction of [2], B. Y. Chen recalls as one of the basic problems in submanifold theory:

“Find simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold”.

In the above mentioned paper, B. Y. Chen establishes a relationship between sectional curvature function  $K$  and the shape operator  $A_H$  for submanifolds in real space forms.

Also, in [3], B. Y. Chen proves a sharp inequality between the  $k$ -Ricci curvature and the shape operator  $A_H$ .

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In [6], we establish a relationship between the sectional curvature function  $K$  and the shape operator  $A_H$  and a sharp relationship between the  $k$ -Ricci curvature and the shape operator  $A_H$ , respectively, for slant submanifolds in complex space forms.

Let  $\widetilde{M}$  be an almost Hermitian manifold with almost complex structure  $J$  and Riemannian metric  $g$ . One denotes by  $\widetilde{\nabla}$  the operator of covariant differentiation with respect to  $g$  in  $\widetilde{M}$ .

**Definition.** If the almost complex structure  $J$  satisfies

$$(\widetilde{\nabla}_X J)Y + (\widetilde{\nabla}_Y J)X = 0,$$

for any vector fields  $X$  and  $Y$  on  $\widetilde{M}$ , then the manifold  $\widetilde{M}$  is called a *nearly-Kaehler manifold* [5], [11].

**Remark.** The above condition is equivalent to

$$(\widetilde{\nabla}_X J)X = 0, \quad \forall X \in \Gamma T\widetilde{M}.$$

For an almost complex structure  $J$  on the manifold  $\widetilde{M}$ , the *Nijenhuis tensor field* is defined by

$$N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y],$$

for any vector fields  $X, Y$  tangent to  $\widetilde{M}$ , where  $[, ]$  is the Lie bracket.

A necessary and sufficient condition for a nearly-Kaehler manifold to be Kaehler is the vanishing of the Nijenhuis tensor  $N_J$ .

Any 4-dimensional nearly-Kaehler manifold is a Kaehler manifold.

**Example.** Let  $S^6$  be the 6-dimensional unit sphere defined as follows:

Let  $\mathbf{E}^7$  be the set of all purely imaginary Cayley numbers. Then  $\mathbf{E}^7$  is a 7-dimensional subspace of the Cayley algebra  $C$ .

Let  $\{1, e_0, e_1, \dots, e_6\}$  be a basis of the Cayley algebra, 1 being the unit element of  $C$ .

If  $X = \sum_{i=0}^6 x^i e_i$  and  $Y = \sum_{i=0}^6 y^i e_i$  are two elements of  $\mathbf{E}^7$ , one defines the *scalar product* in  $\mathbf{E}^7$  by

$$\langle X, Y \rangle = \sum_{i=0}^6 x^i y^i,$$

and the *vector product* by

$$X \times Y = \sum_{i \neq j} x^i y^j e_i * e_j,$$

\* being the multiplication operation of  $C$ .

Consider the 6-dimensional unit sphere  $S^6$  in  $\mathbf{E}^7$ :

$$S^6 = \{X \in \mathbf{E}^7 \mid \langle X, X \rangle = 1\}.$$

The scalar product in  $\mathbf{E}^7$  induces the natural metric tensor field  $g$  on  $S^6$ .

The tangent space  $T_X S^6$  at  $X \in S^6$  can naturally be identified with the subspace of  $\mathbf{E}^7$  orthogonal to  $X$ .

Define the endomorphism  $J_X$  on  $T_X S^6$  by

$$J_X Y = X \times Y, \text{ for } Y \in T_X S^6.$$

It is easy to see that

$$g(J_X Y, J_X Z) = g(Y, Z), \text{ } Y, Z \in T_X S^6.$$

The correspondence  $X \mapsto J_X$  defines a tensor field  $J$  such that  $J^2 = -I$ .

Consequently,  $S^6$  admits an almost Hermitian structure  $(J, g)$ .

This structure is a non-Kaehlerian nearly-Kaehlerian structure (its Betti numbers of even order are 0).

We will consider a class of almost Hermitian manifolds, called *RK-manifolds*, which contains nearly-Kaehler manifolds.

**Definition** [10]. A *RK-manifold*  $(\widetilde{M}, J, g)$  is an almost Hermitian manifold for which the curvature tensor  $\widetilde{R}$  is invariant by  $J$ , i.e.

$$\widetilde{R}(JX, JY, JZ, JW) = \widetilde{R}(X, Y, Z, W),$$

for any  $X, Y, Z, W \in \Gamma T\widetilde{M}$ .

An almost Hermitian manifold  $\widetilde{M}$  is of *pointwise constant type* if, for any  $p \in \widetilde{M}$  and  $X \in T_p \widetilde{M}$ , we have

$$\lambda(X, Y) = \lambda(X, Z),$$

where

$$\lambda(X, Y) = \tilde{R}(X, Y, JX, JY) - \tilde{R}(X, Y, X, Y)$$

and  $Y$  and  $Z$  are unit tangent vectors on  $\tilde{M}$  at  $p$ , orthogonal to  $X$  and  $JX$ , i.e.

$$g(Z, Z) = g(Y, Y) = 1,$$

$$g(X, Y) = g(JX, Y) = g(X, Z) = g(JX, Z) = 0.$$

The manifold  $\tilde{M}$  is said to be of *constant type* if for any unit  $X, Y \in \Gamma T\tilde{M}$  with  $g(X, Y) = g(JX, Y) = 0$ ,  $\lambda(X, Y)$  is a constant function.

Recall the following result [10].

**Theorem.** *Let  $\tilde{M}$  be a RK-manifold. Then  $\tilde{M}$  is of pointwise constant type if and only if there exists a function  $\alpha$  on  $\tilde{M}$  such that*

$$\lambda(X, Y) = \alpha[g(X, X)g(Y, Y) - (g(X, Y))^2 - (g(X, JY))^2],$$

for any  $X, Y \in \Gamma T\tilde{M}$ .

Moreover,  $\tilde{M}$  is of constant type if and only if the above equality holds good for a constant  $\alpha$ .

In this case,  $\alpha$  is the *constant type* of  $\tilde{M}$ .

**Definition.** A *generalized complex space form* is a RK-manifold of constant holomorphic sectional curvature and of constant type.

We will denote a generalized complex space form by  $\tilde{M}(c, \alpha)$ , where  $c$  is the constant holomorphic sectional curvature and  $\alpha$  the constant type, respectively.

Each complex space form is a generalized complex space form. The converse statement is not true. The sphere  $S^6$  endowed with the standard nearly-Kaehler structure is an example of generalized complex space form which is not a complex space form.

Let  $\tilde{M}(c, \alpha)$  be a generalized complex space form of constant holomorphic sectional curvature  $c$  and of constant type  $\alpha$ . Then the curvature tensor  $\tilde{R}$  of  $\tilde{M}(c, \alpha)$  has the following expression [10]:

$$\tilde{R}(X, Y)Z = \frac{c + 3\alpha}{4}[g(Y, Z)X - g(X, Z)Y] + \tag{1.1}$$

$$+\frac{c-\alpha}{4}[g(X, JZ)JY - g(Y, JZ)JX + 2g(X, JY)JZ].$$

Let  $M$  be an  $n$ -dimensional submanifold of an  $2m$ -dimensional generalized complex space form  $\widetilde{M}(c, \alpha)$ . We denote by  $K(\pi)$  the *sectional curvature* of  $M$  associated with a plane section  $\pi \subset T_pM, p \in M$ . Let  $\nabla$  and  $h$  be the Levi-Civita connection of  $M$  and the second fundamental form, respectively.

Then the equation of Gauss is given by

$$\begin{aligned} \widetilde{R}(X, Y, Z, W) &= R(X, Y, Z, W) + \\ &+g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)), \end{aligned} \tag{1.2}$$

for any vectors  $X, Y, Z, W$  tangent to  $M$ , where  $R$  is the *Riemann curvature tensor* of  $M$ .

We denote by  $H$  the *mean curvature vector* at  $p \in M$ , i.e.

$$H(p) = \frac{1}{n} \sum_{i=1}^n h(e_i, e_i), \tag{1.3}$$

where  $\{e_1, \dots, e_{2m}\}$  is an orthonormal basis of the tangent space  $T_p\widetilde{M}(c, \alpha)$ , such that  $\{e_1, \dots, e_n\}$  are tangent to  $M$ .

Also, we set

$$h_{ij}^r = g(h(e_i, e_j), e_r), \quad i, j = 1, \dots, n; \quad r = n + 1, \dots, 2m, \tag{1.4}$$

and

$$\|h\|^2 = \sum_{i,j=1}^n g(h(e_i, e_j), h(e_i, e_j)). \tag{1.5}$$

For any  $p \in M$  and for any  $X \in T_pM$ , we put  $JX = PX + FX$ , where  $PX \in T_pM, FX \in T_p^\perp M$ .

We put

$$\|P\|^2 = \sum_{i,j=1}^n g^2(Pe_i, e_j). \tag{1.6}$$

Suppose  $L$  is a  $k$ -plane section of  $T_pM$  and  $X$  a unit vector in  $L$ . We choose an orthonormal basis  $\{e_1, \dots, e_k\}$  of  $L$  such that  $e_1 = X$ .

Define the *Ricci curvature*  $Ric_L$  of  $L$  at  $X$  by

$$Ric_L(X) = K_{12} + K_{13} + \dots + K_{1k}, \tag{1.7}$$

where  $K_{ij}$  denotes the *sectional curvature* of the 2-plane section spanned by  $e_i, e_j$ . We simply called such a curvature a *k-Ricci curvature*.

The scalar curvature  $\tau$  of the  $k$ -plane section  $L$  is given by

$$\tau(L) = \sum_{1 \leq i < j \leq k} K_{ij}. \tag{1.8}$$

For each integer  $k, 2 \leq k \leq n$ , the Riemannian invariant  $\Theta_k$  on an  $n$ -dimensional Riemannian manifold  $M$  is defined by

$$\Theta_k(p) = \frac{1}{k-1} \inf_{L, X} Ric_L(X), \quad p \in M, \tag{1.9}$$

where  $L$  runs over all  $k$ -plane sections in  $T_pM$  and  $X$  runs over all unit vectors in  $L$ .

Recall that for a submanifold  $M$  in a Riemannian manifold, the *relative null space* of  $M$  at a point  $p \in M$  is defined by

$$N(p) = \{X \in T_pM \mid h(X, Y) = 0, \forall Y \in T_pM\}. \tag{1.10}$$

## 2. Sectional curvature and shape operator

The notion of a slant submanifold of an almost Hermitian manifold was introduced by B. Y. Chen [1].

**Definition.** A submanifold  $M$  of an almost Hermitian manifold  $\widetilde{M}$  is said to be a *slant submanifold* if for any  $p \in M$  and any nonzero vector  $X \in T_pM$ , the angle between  $JX$  and the tangent space  $T_pM$  is constant ( $= \theta$ ).

We prove an inequality for an  $n$ -dimensional slant submanifold  $M$  into a  $2m$ -dimensional generalized complex space form  $\widetilde{M}(c, \alpha)$  of constant holomorphic sectional curvature  $c$  and of constant type  $\alpha$ .

**Theorem 2.1.** *Let  $x : M \rightarrow \widetilde{M}(c, \alpha)$  be an isometric immersion of an  $n$ -dimensional  $\theta$ -slant submanifold into a  $2m$ -dimensional generalized complex space form  $\widetilde{M}(c, \alpha)$  of constant holomorphic sectional curvature  $c > \alpha > 0$ . If there exists a point  $p \in M$  and a number  $b > \frac{c+3\alpha}{4} + 3\frac{c-\alpha}{2n} \cos^2 \theta$  such that  $K \geq b$  at  $p$ , then the shape operator at the*

mean curvature vector satisfies

$$A_H > \frac{n-1}{n} \left[ b - \frac{c+3\alpha}{4} - 3 \frac{c-\alpha}{4(n-1)} \cos^2 \theta \right] I_n, \text{ at } p, \quad (2.1)$$

where  $I_n$  is the identity map.

**Proof.** Let  $p \in M$  and a number  $b > \frac{c+3\alpha}{4} + 3 \frac{c-\alpha}{2u} \cos^2 \theta$  such that  $K \geq b$  at  $p$ . We choose an orthonormal basis  $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2m}\}$  at  $p$  such that  $e_{n+1}$  is parallel to the mean curvature vector  $H$  and  $e_1, \dots, e_n$  diagonalize the shape operator  $A_{n+1}$ .

Then we have

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{pmatrix}, \quad (2.2)$$

$$A_r = (h_{ij}^r), i, j = 1, \dots, n, r = n+2, \dots, 2m, \text{ trace } A_r = \sum_{i=1}^n h_{ii}^r = 0. \quad (2.3)$$

For  $i \neq j$ , we denote by

$$u_{ij} = a_i a_j. \quad (2.4)$$

From Gauss equation for  $X = Z = e_i, Y = W = e_j$ , we get

$$u_{ij} \geq b - \frac{c+3\alpha}{4} - 3 \frac{c-\alpha}{4} g^2(e_i, J e_j) - \sum_{r=n+2}^{2m} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2]. \quad (2.5)$$

We prove that  $u_{ij}$  have the following properties:

1. For any fixed  $i \in \{1, \dots, n\}$ , we have

$$\sum_{i \neq j} u_{ij} \geq (n-1) \left( b - \frac{c+3\alpha}{4} \right) - 3 \frac{c-\alpha}{4} \cos^2 \theta > 0.$$

2.  $u_{ij} \neq 0$ , for  $i \neq j$ .

3. For distinct  $i, j, k \in \{1, \dots, n\}, a_i^2 = \frac{u_{ij} u_{ik}}{u_{jk}}$ .



4. We denote by  $S_k = \{B \subset \{1, \dots, n\}; |B| = k\}$  and for any  $B \in S_k$  we denote by  $\bar{B} = \{1, \dots, n\} \setminus B$ . Then, for a fixed  $k, 1 \leq k \leq \lfloor \frac{n}{2} \rfloor$  and each  $B \in S_k$ , we have

$$\sum_{j \in B} \sum_{t \in \bar{B}} u_{jt} > 0.$$

5. For distinct  $i, j \in \{1, \dots, n\}, u_{ij} > 0$ .

1. From (2.3), (2.4) and (2.5), we have:

$$\begin{aligned} \sum_{j \neq i} u_{ij} &\geq (n-1)\left(b - \frac{c+3\alpha}{4}\right) - 3\frac{c-\alpha}{4} \|Pe_i\|^2 - \sum_{r=n+2}^{2m} [h_{ii}^r (\sum_{j \neq i} h_{jj}^r) - \sum_{j \neq i} (h_{ij}^r)^2] = \\ &= (n-1)\left(b - \frac{c+3\alpha}{4}\right) - 3\frac{c-\alpha}{4} \cos^2 \theta - \sum_{r=n+2}^{2m} [h_{ii}^r (-h_{ii}^r) - \sum_{j \neq i} (h_{ij}^r)^2] = \\ &= (n-1)\left(b - \frac{c+3\alpha}{4}\right) - 3\frac{c-\alpha}{4} \cos^2 \theta + \sum_{r=n+2}^{2m} \sum_{j=1}^n (h_{ij}^r)^2 \geq \\ &\geq (n-1)\left(b - \frac{c+3\alpha}{4}\right) - 3\frac{c-\alpha}{4} \cos^2 \theta > 0. \end{aligned}$$

2. If  $u_{ij} = 0$ , for  $i \neq j$ , then  $a_i = 0$  or  $a_j = 0$ .  $a_i = 0$  implies that  $u_{it} = a_i a_t = 0, \forall t \in \{1, \dots, n\}, t \neq i$ .

It follows that

$$\sum_{j \neq i} u_{ij} = 0,$$

in contradiction with 1.

3.  $\frac{u_{ij}u_{ik}}{u_{jk}} = \frac{a_i a_j a_i a_k}{a_j a_k} = a_i^2$ .

4. Since we can change the order of  $e_1, \dots, e_n$ , we may assume  $B = \{1, \dots, k\}$  and  $\bar{B} = \{k+1, \dots, n\}$ . Then

$$\begin{aligned} \sum_{j \in B} \sum_{t \in \bar{B}} u_{jt} &= k(n-k)\left(b - \frac{c+3\alpha}{4}\right) - 3\frac{c-\alpha}{4} \sum_{j=1}^k \sum_{t=k+1}^n g^2(Je_j, e_t) - \\ &- \sum_{r=n+2}^{2m} \left\{ \sum_{j=1}^k \sum_{t=k+1}^n [h_{jj}^r h_{tt}^r - (h_{jt}^r)^2] \right\} \geq \end{aligned}$$

$$\begin{aligned} &\geq k(n-k)\left(b - \frac{c+3\alpha}{4}\right) - 3k\frac{c-\alpha}{4}\cos^2\theta + \\ &\quad + \sum_{r=n+2}^{2m} \left[ \sum_{j=1}^k \sum_{t=k+1}^n (h_{jt}^r)^2 + \sum_{j=1}^k (h_{jj}^r)^2 \right] \geq \\ &\geq k(n-k)\left(b - \frac{c+3\alpha}{4}\right) - 3k\frac{c-\alpha}{4}\cos^2\theta > 0. \end{aligned}$$

5. Assume  $u_{1n} < 0$ . From 3, we get  $u_{1i}u_{in} < 0$ , for  $1 < i < n$ .

Without loss of generality, we may assume

$$\begin{cases} u_{12}, \dots, u_{1l}, u_{(l+1)n}, \dots, u_{(n-1)n} > 0, \\ u_{1(l+1)}, \dots, u_{1n}, u_{2n}, \dots, u_{ln} < 0, \end{cases} \quad (2.6)$$

for some  $\lfloor \frac{n+1}{2} \rfloor \leq l \leq n-1$ .

If  $l = n-1$ , then  $u_{1n} + u_{2n} + \dots + u_{(n-1)n} < 0$ , which contradicts to 1. Thus,  $l < n-1$ .

From 3, we get

$$a_n^2 = \frac{u_{in}u_{tn}}{u_{it}} > 0, \quad (2.7)$$

where  $2 \leq i \leq l, l+1 \leq t \leq n-1$ . By (2.6) and (2.7), we obtain  $u_{it} < 0$ , which implies

$$\sum_{i=1}^l \sum_{t=l+1}^n u_{it} = \sum_{i=2}^l \sum_{t=l+1}^{n-1} u_{it} + \sum_{i=1}^l u_{in} + \sum_{t=l+1}^n u_{1t} < 0.$$

This contradicts to 4.

Now, we return to the proof of Theorem 2.1.

From 5, it follows that  $a_1, \dots, a_n$  have the same sign. Assume  $a_j > 0, \forall j \in \{1, \dots, n\}$ .

Then

$$\sum_{j \neq i} u_{ij} = a_i(a_1 + \dots + a_n) - a_i^2 \geq (n-1)\left(b - \frac{c+3\alpha}{4}\right) - 3\frac{c-\alpha}{4}\cos^2\theta.$$

From the above relation and from (2.2), we have

$$a_i n \|H\| \geq (n-1)\left(b - \frac{c+3\alpha}{4}\right) - 3\frac{c-\alpha}{4}\cos^2\theta + a_i^2 >$$

$$> (n - 1)\left(b - \frac{c + 3\alpha}{4}\right) - 3\frac{c - \alpha}{4} \cos^2 \theta.$$

This equation implies

$$a_i \|H\| > \frac{n - 1}{n} \left[ b - \frac{c + 3\alpha}{4} - 3\frac{c - \alpha}{4(n - 1)} \cos^2 \theta \right],$$

and consequently (2.1). ■

In particular, for  $\alpha = 0$ , we rekind Theorem 3.1 from [6].

For totally real submanifolds, we have the following

**Corollary 2.2.** *Let  $x : M \rightarrow \widetilde{M}(c, \alpha)$  be an isometric immersion of an  $n$ -dimensional totally real submanifold into a  $2m$ -dimensional generalized complex space form  $\widetilde{M}(c, \alpha)$ . If there exists a point  $p \in M$  and a number  $b > \frac{c+3\alpha}{4}$  such that  $K \geq b$  at  $p$ , then the shape operator at the mean curvature vector satisfies*

$$A_H > \frac{n - 1}{n} \left( b - \frac{c + 3\alpha}{4} \right) I_n, \text{ at } p,$$

where  $I_n$  is the identity map.

### 3. $k$ -Ricci curvature and shape operator

We prove an inequality for a slant submanifold  $M$  of a  $2m$ -dimensional generalized complex space form  $\widetilde{M}(c, \alpha)$  of constant holomorphic sectional curvature  $c$  and of constant type  $\alpha$ .

**Theorem 3.1.** *Let  $x : M \rightarrow \widetilde{M}(c, \alpha)$  be an isometric immersion of an  $n$ -dimensional  $\theta$ -slant submanifold  $M$  into a  $2m$ -dimensional generalized complex space form  $\widetilde{M}(c, \alpha)$ . Then, for any integer  $k, 2 \leq k \leq n$ , and any point  $p \in M$ , we have:*

*i) If  $\Theta_k(p) \neq \frac{c+3\alpha}{4} + 3\frac{c-\alpha}{4(n-1)} \cos^2 \theta$ , then the shape operator at the mean curvature satisfies*

$$A_H > \frac{n - 1}{n} \left[ \Theta_k(p) - \frac{c + 3\alpha}{4} - 3\frac{c - \alpha}{4(n - 1)} \cos^2 \theta \right] I_n, \text{ at } p, \tag{3.1}$$

where  $I_n$  denotes the identity map of  $T_p M$ .

ii) If  $\Theta_k(p) = \frac{c+3\alpha}{4} + 3\frac{c-\alpha}{4(n-1)} \cos^2 \theta$ , then  $A_H \geq 0$  at  $p$ .

iii) A unit vector  $X \in T_p M$  satisfies

$$A_H X = \frac{n-1}{n} [\Theta_k(p) - \frac{c+3\alpha}{4} - 3\frac{c-\alpha}{4(n-1)} \cos^2 \theta] X \quad (3.2)$$

if and only if  $\Theta_k(p) = \frac{c+3\alpha}{4} + 3\frac{c-\alpha}{4(n-1)} \cos^2 \theta$  and  $X \in N(p)$ .

iv)  $A_H = \frac{n-1}{n} [\Theta_k(p) - \frac{c+3\alpha}{4} - 3\frac{c-\alpha}{4(n-1)} \cos^2 \theta] I_n$  at  $p$  if and only if  $p$  is a totally geodesic point.

**Proof.** i) Let  $\{e_1, \dots, e_n\}$  be an orthonormal basis of  $T_p M$ . Denote by  $L_{i_1 \dots i_k}$  the  $k$ -plane section spanned by  $e_{i_1}, \dots, e_{i_k}$ . It is easily seen by the definitions

$$\tau(L_{i_1 \dots i_k}) = \frac{1}{2} \sum_{i \in \{i_1, \dots, i_k\}} Ric_{L_{i_1 \dots i_k}}(e_i), \quad (3.3)$$

$$\tau(p) = \frac{1}{C_{n-2}^{k-2}} \sum_{1 \leq i_1 < \dots < i_k \leq n} \tau(L_{i_1 \dots i_k}). \quad (3.4)$$

Combining (3.3) and (3.4), we find

$$\tau(p) \geq \frac{n(n-1)}{2} \Theta_k(p). \quad (3.5)$$

From the equation of Gauss for  $X = Z = e_i, Y = W = e_j$ , by summing, we obtain

$$n^2 \|H\|^2 = 2\tau + \|h\|^2 - \frac{c+3\alpha}{4} n(n-1) - 3\frac{c-\alpha}{4} \|P\|^2. \quad (3.6)$$

We choose an orthonormal basis  $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2m}\}$  at  $p$  such that  $e_{n+1}$  is parallel to the mean curvature vector  $H(p)$  and  $e_1, \dots, e_n$  diagonalize the shape operator  $A_{n+1}$ . Then we have the relations (2.2) and (2.3).

From (3.6), we get

$$n^2 \|H\|^2 = 2\tau + \sum_{i=1}^n a_i^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^n (h_{ij}^r)^2 - \frac{c+3\alpha}{4} n(n-1) - 3\frac{c-\alpha}{4} \|P\|^2. \quad (3.7)$$

On the other hand, since

$$0 \leq \sum_{i < j} (a_i - a_j)^2 = (n - 1) \sum_i a_i^2 - 2 \sum_{i < j} a_i a_j,$$

we obtain

$$n^2 \|H\|^2 = \left(\sum_{i=1}^n a_i\right)^2 = \sum_{i=1}^n a_i^2 + 2 \sum_{i < j} a_i a_j \leq n \sum_{i=1}^n a_i^2, \tag{3.8}$$

which implies

$$\sum_{i=1}^n a_i^2 \geq n \|H\|^2.$$

We have from (3.7)

$$n^2 \|H\|^2 \geq 2\tau + n \|H\|^2 - \frac{c + 3\alpha}{4} n(n - 1) - 3 \frac{c - \alpha}{4} \|P\|^2, \tag{3.9}$$

or, equivalently,

$$\|H\|^2 \geq \frac{2\tau}{n(n - 1)} - \frac{c + 3\alpha}{4} - 3 \frac{c - \alpha}{4n(n - 1)} \|P\|^2. \tag{3.10}$$

Since  $M$  is a slant submanifold, from (3.5) and (3.10), we obtain

$$\begin{aligned} \|H\|^2(p) &\geq \Theta_k(p) - \frac{c + 3\alpha}{4} - 3 \frac{c - \alpha}{4n(n - 1)} \|P\|^2 = \\ &= \Theta_k(p) - \frac{c + 3\alpha}{4} - 3 \frac{c - \alpha}{4(n - 1)} \cos^2 \theta. \end{aligned} \tag{3.11}$$

This shows that  $H(p) = 0$  may occurs only when  $\Theta_k(p) \leq \frac{c+3\alpha}{4} + 3 \frac{c-\alpha}{4(n-1)} \cos^2 \theta$ . Consequently, if  $H(p) = 0$ , statements i) and ii) hold automatically. Therefore, without loss of generality, we may assume  $H(p) \neq 0$ .

From the equation of Gauss we get

$$a_i a_j = K_{ij} - \frac{c + 3\alpha}{4} - 3 \frac{c - \alpha}{4} g^2(e_i, J e_j) - \sum_{r=n+2}^{2m} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2]. \tag{3.12}$$

By (3.12), we obtain

$$a_1(a_{i_2} + \dots + a_{i_k}) = Ric_{L_{i_2 \dots i_k}}(e_1) - (k-1)\frac{c+3\alpha}{4} - \tag{3.13}$$

$$-3\frac{c-\alpha}{4} \sum_{j=2}^k g^2(e_1, J e_{i_j}) - \sum_{r=n+2}^{2m} \sum_{j=2}^k [h_{11}^r h_{i_j i_j}^r - (h_{1i_j}^r)^2],$$

which yields

$$a_1(a_2 + \dots + a_n) = \frac{1}{C_{n-2}^{k-2}} \sum_{2 \leq i_2 < \dots < i_k \leq n} Ric_{L_{i_2 \dots i_k}}(e_1) - \tag{3.14}$$

$$-(n-1)\frac{c+3\alpha}{4} - 3\frac{c-\alpha}{4} \sum_{j=2}^n g^2(e_1, J e_j) + \sum_{r=n+2}^{2m} \sum_{j=1}^n (h_{1j}^r)^2.$$

We find

$$a_1(a_2 + \dots + a_n) \geq (n-1)[\Theta_k(p) - \frac{c+3\alpha}{4} - 3\frac{c-\alpha}{4(n-1)} \cos^2 \theta]. \tag{3.15}$$

Then

$$a_1(a_1 + a_2 + \dots + a_n) = a_1^2 + a_1(a_2 + \dots + a_n) \geq \tag{3.16}$$

$$\geq a_1^2 + (n-1)[\Theta_k(p) - \frac{c+3\alpha}{4} - 3\frac{c-\alpha}{4(n-1)} \cos^2 \theta] \geq$$

$$\geq (n-1)[\Theta_k(p) - \frac{c+3\alpha}{4} - 3\frac{c-\alpha}{4(n-1)} \cos^2 \theta].$$

Since  $n \|H\| = a_1 + \dots + a_n$ , the above equation implies

$$A_H \geq \frac{n-1}{n} [\Theta_k(p) - \frac{c+3\alpha}{4} - 3\frac{c-\alpha}{4(n-1)} \cos^2 \theta] I_n.$$

The equality does not hold, because in our case  $H(p) \neq 0$ .

The assertion ii) is obvious.

iii) Let  $X \in T_p M$  a unit vector satisfying (3.2). By (3.16) and (3.14) one has  $a_1 = 0$  and  $h_{1j}^r = 0, \forall j \in \{1, \dots, n\}, r \in \{n+2, \dots, 2m\}$ , respectively. The above conditions imply  $\Theta_k(p) = \frac{c+3\alpha}{4} + 3\frac{c-\alpha}{4(n-1)} \cos^2 \theta$  and  $X \in N(p)$ .

The converse is clear.

iv) The equality (3.2) holds for any  $X \in T_p M$  if and only if  $N(p) = T_p M$ , i.e.  $p$  is a totally geodesic point. ■

**Remark.** If we denote by  $\lambda_i$  the eigenvalues of  $A_H$ , i.e.  $\lambda_i = a_i \|H\|$ ,  $i \in \{1, \dots, n\}$ , we obtain the following inequality for arbitrary submanifolds of generalized complex space forms:

$$\lambda_i \geq \frac{n-1}{n} \left[ \Theta_k(p) - \frac{c+3\alpha}{4} - 3 \frac{c-\alpha}{4(n-1)} \|Pe_i\|^2 \right].$$

In particular, for  $\alpha = 0$ , we obtain Theorem 4.1 from [6].

**Corollary 3.2.** *Let  $x : M \rightarrow \widetilde{M}(c, \alpha)$  be an isometric immersion of an  $n$ -dimensional totally real submanifold  $M$  into a generalized complex space form  $\widetilde{M}(c, \alpha)$ . Then, for any integer  $k, 2 \leq k \leq n$ , and any point  $p \in M$ , we have:*

i) *If  $\Theta_k(p) \neq \frac{c+3\alpha}{4}$ , then the shape operator at the mean curvature vector satisfies*

$$A_H > \frac{n-1}{n} \left[ \Theta_k(p) - \frac{c+3\alpha}{4} \right] I_n, \text{ at } p,$$

where  $I_n$  denotes the identity map of  $T_p M$ .

ii) *If  $\Theta_k(p) = \frac{c+3\alpha}{4}$ , then  $A_H \geq 0$  at  $p$ .*

iii) *A unit vector  $X \in T_p M$  satisfies*

$$A_H X = \frac{n-1}{n} \left[ \Theta_k(p) - \frac{c+3\alpha}{4} \right] X$$

*if and only if  $\Theta_k(p) = \frac{c+3\alpha}{4}$  and  $X \in N(p)$ .*

iv)  *$A_H = \frac{n-1}{n} \left[ \Theta_k(p) - \frac{c+3\alpha}{4} \right] I_n$  at  $p$  if and only if  $p$  is a totally geodesic point.*

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