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On the Extended Hecke Groups $\overline{H}(\lambda_q)$

Nihal Yılmaz Özgür, Recep Şahin*

Abstract

Hecke groups $H(\lambda_q)$ have been studied extensively for many aspects in the literature, [5], [8]. The Hecke group $H(\lambda_3)$, the modular group $PSL(2, \mathbb{Z})$, has especially been of great interest in many fields of mathematics, for example number theory, automorphic function theory and group theory. In this paper we consider the extended Hecke groups $\overline{H}(\lambda_q)$ which are defined analogously with the extended modular group. We find the conjugacy classes of torsion elements in $\overline{H}(\lambda_q)$. Using this we give some results about the normal subgroups and Fuchsian subgroups of $\overline{H}(\lambda_q)$.

Key Words: Extended Hecke group, conjugacy class, extended modular group.

1. Introduction

Hecke introduced an infinite class of discrete groups $H(\lambda_q)$ of linear fractional transformations preserving the upper-half plane [2]. The Hecke group $H(\lambda)$ is the group generated by

$$x(z) = -\frac{1}{z} \quad \text{and} \quad u(z) = z + \lambda,$$

where $\lambda = \lambda_q = 2 \cos \pi/q$, $q \geq 3$ integer, or $\lambda \geq 2$. We consider the former case. These Hecke groups $H(\lambda_q)$ are Fuchsian groups of the first kind. Let

$$y = xu = \frac{-1}{z + \lambda_q}.$$

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*Dedicated to Professor Turgut BASKAN on the occasion of his retirement.

Then $H(\lambda_q)$ has a presentation

$$H(\lambda_q) = \langle x, y : x^2 = y^q = 1 \rangle .$$

For $q = 3$, the resulting Hecke group $H(\lambda_3)$ is the modular group $PSL(2, \mathbb{Z})$. In a sense Hecke groups are generalizations of the modular group. Similar to the extended modular group case, the extended Hecke group, denoted by $\overline{H}(\lambda_q)$, is the group obtained by adding the reflection

$$r(z) = 1 / \bar{z}$$

to the generators of $H(\lambda_q)$, and the Hecke group $H(\lambda_q)$ is a subgroup of index 2 in $\overline{H}(\lambda_q)$. The extended Hecke group $\overline{H}(\lambda_q)$ has the presentation

$$\overline{H}(\lambda_q) = \langle x, y, r : x^2 = y^q = r^2 = (xr)^2 = (yr)^2 = 1 \rangle . \quad (1.1.1)$$

Again, for $q = 3$, we obtain the extended modular group studied in [3], [4].

In this paper, firstly we find the conjugacy classes of finite order elements of the extended Hecke groups $\overline{H}(\lambda_q)$. In studying normal subgroups (especially Fuchsian ones), it is important to find the conjugacy classes of torsion elements. Also we show that in $\overline{H}(\lambda_p)$, $p \geq 3$ prime number, every normal subgroup with torsion has finite index (in fact the index is 2, 4 or $2p$).

2. Conjugacy Classes

It is well known that the Hecke group $H(\lambda_q)$ is a free product of a cyclic group of order 2 and a cyclic group of order q , i.e. $H(\lambda_q) \cong \mathbb{Z}_2 * \mathbb{Z}_q$, [1]. The extended Hecke group $\overline{H}(\lambda_q)$ is also known to be a free product with amalgamation as $\overline{H}(\lambda_q) \cong D_2 *_{\mathbb{Z}_2} D_q$, [7], [4].

One of the consequences of the amalgam decomposition is a classification of the conjugacy classes of torsion elements in $\overline{H}(\lambda_q)$.

The following lemmas are trivial due to the presentation of $\overline{H}(\lambda_q)$ given in (1.1.1).

Lemma 2.1 *In $\overline{H}(\lambda_q)$, we have $y^s r = r y^{q-s}$, $1 \leq s \leq q - 1$.*

Lemma 2.2 *In $\overline{H}(\lambda_q)$, we have*

(i) $y^n r, 1 \leq n \leq q - 1$, is conjugate to r by $y^t r$ where $t = \frac{q \cdot k + n}{2}$ for some $k \in \mathbb{Z}$ satisfying the condition $t \in \mathbb{Z}$ (note that t is not unique).

(ii) $y^s, 1 \leq s \leq \frac{q-1}{2}$, is conjugate to y^{q-s} by r .

Now we can give the following theorem for the extended Hecke group $\overline{H}(\lambda_p)$ where $p \geq 3$ prime number. Note that, by the definition, $\overline{H}(\lambda_q)$ contains reflections.

Theorem 2.3 *Let $p \geq 3$ be prime number. There are exactly $\frac{p+5}{2}$ conjugacy classes of torsion elements in $\overline{H}(\lambda_p)$, three for those of order 2 and $\frac{p-1}{2}$ for those of order p . In particular, any elliptic transformation of order 2 is conjugate to $x : z \rightarrow -1/z$ and any reflection is conjugate to one of $r : z \rightarrow 1/\bar{z}$ or $xr : z \rightarrow -\bar{z}$ while any elliptic transformation of order p is conjugate to one of $y : z \rightarrow -1/(z + \lambda_p), y^2, \dots$ or $y^{\frac{p-1}{2}}$.*

Proof. We consider the following presentation of $\overline{H}(\lambda_p)$ given in (1.1.1):

$$\overline{H}(\lambda_p) = \langle x, y, r : x^2 = y^p = r^2 = (xr)^2 = (yr)^2 = 1 \rangle .$$

The decomposition of $\overline{H}(\lambda_p) \cong D_2 *_{\mathbb{Z}_2} D_p$ arose by letting

$$G_1 = \langle x, r : x^2 = r^2 = (xr)^2 = 1 \rangle \cong D_2$$

and

$$G_2 = \langle y, r : y^p = r^2 = (yr)^2 = 1 \rangle \cong D_p .$$

From a theorem of Kurosh, see [6], we know that any element of finite order in a generalized free product $A *_H B$ is conjugate to an element of finite order in one of the factors. Now let g be any element of finite order in $\overline{H}(\lambda_p)$. Then g must be conjugate to an element of finite order in one of the factors G_1 and G_2 . Therefore to find the conjugacy classes of finite order elements in $\overline{H}(\lambda_p)$, we must find the conjugacy classes in these factors.

G_1 has one conjugacy class of elliptic elements of order 2 with representative x , two conjugacy classes of reflections with representatives r and xr .

Similarly, in G_2 , there are p classes of reflections with representatives $r, yr, \dots, y^{p-1}r$ and $p - 1$ classes of elliptic elements of order p with representatives y, y^2, \dots, y^{p-1} . By Lemma 2.2 (i), $y^n r$ is conjugate to $r, 1 \leq n \leq p - 1$, then we have one conjugacy

class of reflections with representative r . Also by Lemma 2.2 (ii), y^s is conjugate to y^{p-s} and so we have $\frac{p-1}{2}$ conjugacy classes of elliptic elements of order p with representatives $y, y^2, \dots, y^{\frac{p-1}{2}}$.

Therefore, in $\overline{H}(\lambda_p)$ as a whole there are: one class of elliptic elements of order 2 with representative x ; two classes of reflections with representatives r, xr ; and, $\frac{p-1}{2}$ classes of elliptic elements of order p with representatives $y, y^2, \dots, y^{\frac{p-1}{2}}$. \square

The situation is more complex for any odd q which is not prime.

For odd and composite values of q , there are $\frac{\varphi(q)}{2}$ conjugacy classes of elliptic elements of order q with representatives $y, y^{r_1}, \dots, y^{r_{\varphi(q)/2}}$ where $(r_i, q) = 1, 1 \leq i \leq \frac{\varphi(q)}{2}$ and $\varphi(q)$ is the Euler function. There are: one conjugacy class of elliptic elements of order 2 with representative x ; two conjugacy classes of reflections with representatives r, xr ; and, in total $\frac{q-1}{2} - \frac{\varphi(q)}{2}$ conjugacy classes of elliptic elements of order t_i where $t_i | q$.

For even q , firstly let us start with $q = 4$ and $q = 6$.

Theorem 2.4 (i) *In $\overline{H}(\sqrt{2})$, there are two conjugacy classes of elliptic elements of order 2 with representatives x, y^2 , two conjugacy classes of reflections with representatives r, xr and one class of elliptic elements of order 4 with representative y .*

(ii) *In $\overline{H}(\sqrt{3})$, there are two conjugacy classes of elliptic elements of order 2 with representatives x, y^3 , two conjugacy classes of reflections with representatives r, xr and one class of elliptic elements of order 3 with representative y^2 and one class of elliptic elements of order 6 with representative y .*

For any even $q > 6$, we can only say that there are $\frac{\varphi(q)}{2}$ conjugacy classes of elliptic elements of order q with representatives $y, y^{r_1}, \dots, y^{r_{\varphi(q)/2}}$ where $(r_i, q) = 1$. Also, there are two conjugacy classes of elliptic elements of order 2 with representatives $x, y^{\frac{q}{2}}$ and two conjugacy classes of reflections with representatives r, xr . Furthermore, there are totally $\frac{q}{2} - \frac{\varphi(q)}{2} - 1$ conjugacy classes of elliptic elements of order t_i where $t_i | q$ and $t_i \neq 2$.

Here we mention briefly conjugacy classes of torsion elements in Hecke groups $H(\lambda_q)$. Note that most of these results can be obtained by use of the notion of fundamental region of Hecke groups. Similar to the extended Hecke group case, for any even q , we can only say that there are $\varphi(q)$ conjugacy classes of elliptic elements of order q with

representative $y, y^{r^1}, \dots, y^{r^{\varphi(q)}}$ where $(r_i, q) = 1$, two classes of elliptic elements of order 2 with representatives $x, y^{\frac{q}{2}}$. Furthermore, there are in total $q - \varphi(q) - 2$ conjugacy classes of elliptic elements of order t_i where $t_i \mid q, t_i \neq 2$. If q is prime, then there are only q conjugacy classes of torsion elements in $H(\lambda_q)$, one for those of order 2 and $q - 1$ for those of order q . In particular, any elliptic transformation of order 2 is conjugate to $x : z \rightarrow -1/z$ and any elliptic transformation of order q is conjugate to one of $y : z \rightarrow -1/(z + \lambda_p), y^2, \dots, y^{q-1}$. For odd and composite values of q , there are $\varphi(q)$ conjugacy classes of elliptic elements of order q with representatives $y, y^{r^1}, \dots, y^{r^{\varphi(q)}}$ where $(r_i, q) = 1, 1 \leq i \leq \varphi(q)$. There are one conjugacy classes of elliptic elements of order 2 with representative x and totally $q - 1 - \varphi(q)$ conjugacy classes of elliptic elements of order t_i where $t_i \mid q$.

Using Theorem 2.3 we get the following.

Theorem 2.5 *If G is a normal subgroup of $\overline{H}(\lambda_p)$, p prime, and G has torsion, then the index $|\overline{H}(\lambda_p) : G|$ is finite.*

Proof. Suppose $g \in \overline{H}(\lambda_p)$ and g has finite order. Since $G \triangleleft \overline{H}(\lambda_p)$, if $g \in G$ then $N(g) \subseteq G$ implies $|\overline{H}(\lambda_p) : G| \mid |\overline{H}(\lambda_p) : N(g)|$ where $N(g)$ is the normal closure of g in $\overline{H}(\lambda_p)$.

Since $|\overline{H}(\lambda_p) : N(g)| = |\overline{H}(\lambda_p) : N(g^*)|$ where g^* is any conjugate of g , we complete the argument by showing that $|\overline{H}(\lambda_p) : N(h)|$ is finite. Here h is any of the class representatives of torsion elements listed in Theorem 2.3. Now $h = x, r, xr, y, y^2, \dots, y^{\frac{p-1}{2}}$. The quotient group $\overline{H}(\lambda_p) / N(h)$ is the group obtained by adding the relation $h = 1$ to the relations of $\overline{H}(\lambda_p)$, [6].

(1) Suppose $h = x$. Then

$$\overline{H}(\lambda_p)/N(x) \cong \langle x, y, r : x^2 = y^p = r^2 = (xr)^2 = (yr)^2 = 1, x = 1 \rangle$$

$$\cong \langle y, r : y^p = r^2 = (yr)^2 = 1 \rangle \cong D_p.$$

Therefore $|\overline{H}(\lambda_p) : N(x)| = 2p$.

(2) Suppose $h = r$. Then we find

$$\overline{H}(\lambda_p)/N(r) \cong \langle x, y : x^2 = y^p = r^2 = (xr)^2 = (yr)^2 = 1, r = 1 \rangle$$

$$\cong \langle x : x^2 = 1 \rangle \cong C_2$$

since $y^2 = y^p = 1$. Therefore $|\overline{H}(\lambda_p) : N(r)| = 2$.

(3) Let $h = xr$. Then

$$\overline{H}(\lambda_p)/N(xr) \cong \langle x, y, r : x^2 = y^p = r^2 = (xr)^2 = (yr)^2 = 1, xr = 1 \rangle.$$

As $xr = 1$ we see that $x = r$ since $r^2 = 1$. Therefore we get

$$\overline{H}(\lambda_p)/N(xr) \cong \langle x, y : x^2 = y^p = (xy)^2 = 1 \rangle \cong D_p,$$

so $|\overline{H}(\lambda_p) : N(xr)| = 2p$.

(4) Let $h = y$. Then

$$\overline{H}(\lambda_p)/N(y) \cong \langle x, r : x^2 = r^2 = (xr)^2 = 1 \rangle \cong D_2,$$

so $|\overline{H}(\lambda_p) : N(y)| = 4$.

(5) Similarly, if $h = y^a$, $2 \leq a \leq \frac{p-1}{2}$, then $(a, p) = 1$ and so we have

$$\overline{H}(\lambda_p)/N(y^a) \cong \langle x, r : x^2 = r^2 = (xr)^2 = 1 \rangle \cong D_2.$$

Hence $|\overline{H}(\lambda_p) : N(y^a)| = 4$.

Thus in all cases, the index is finite. □

We can restate this as in the following corollary.

Corollary 2.6 *If $G \triangleleft \overline{H}(\lambda_p)$ and G has an elliptic element or reflection then $|\overline{H}(\lambda_p) : G|$ divides $4p$ (divides 2, 4, or $2p$, depending on elliptic element or reflection).*

Since the index of Hecke group $H(\lambda_q)$ in $\overline{H}(\lambda_q)$ is 2 and since if G is normal in $H(\lambda_q)$, then is also normal in $\overline{H}(\lambda_q)$, we have

Corollary 2.7 *If $G \triangleleft H(\lambda_p)$, $p \geq 3$ prime number, and G has an elliptic element then the index $|H(\lambda_p) : G|$ divides $2p$ (divides 2 or p , depending on elliptic element).*

3. Fuchsian Subgroups

It is well-known that $H(\lambda_q)$ is discontinuous in the upper half-plane and Fuchsian with the real line as a fixed circle.

If C is a circle, we let $P(C)$ be the Fuchsian stabilizer of C in $\overline{H}(\lambda_q)$, i.e. the subgroup of $\overline{H}(\lambda_q)$ which maps both C and the interior of C on itself and $P_N(C)$ the normal closure of $P(C)$ in $\overline{H}(\lambda_q)$.

Now we can give the following corollary for the extended Hecke group $\overline{H}(\lambda_p)$ for prime number $p \geq 3$.

Corollary 3.1 *If $P(C)$ contains any elliptic element in $\overline{H}(\lambda_p)$ then the index $|\overline{H}(\lambda_p) : P_N(C)|$ is finite. In particular, $|\overline{H}(\lambda_p) : P_N(C)|$ divides 2, 4, $2p$.*

Let $L(C)$ be the general stabilizer of the circle C in $\overline{H}(\lambda_p)$, i.e. the subgroup of $\overline{H}(\lambda_p)$ which maps C on itself and let $L_N(C)$ be its normal closure. Then we have

Theorem 3.2 *If the circle C is fixed by either an elliptic or parabolic element or a reflection in $\overline{H}(\lambda_p)$ then $|\overline{H}(\lambda_p) : L_N(C)|$ is finite (in fact $|\overline{H}(\lambda_p) : L_N(C)| \mid 4p$).*

Proof. If the circle C is fixed by an elliptic element or reflection, then the result follows from Theorem 2.5. Now let C is fixed by a parabolic map t . Any parabolic element in $\overline{H}(\lambda_p)$ is conjugate to a translation $t' : z \rightarrow z + \alpha$. So, if $vtv^{-1} = t'$, then $v(C)$ is a fixed circle of t' . The real line is fixed by t' . Further, the real line is fixed by $x : z \rightarrow -1/z$. Then $v^{-1}xv$ fixes C , so $L(C)$ contains an elliptic map. From Theorem 2.5, $|\overline{H}(\lambda_p) : L_N(C)|$ is finite and in fact it divides $4p$. \square

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