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Weyl tensor decomposition to the formation of black hole

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Abstract: This work investigates the role of the Weyl tensor in the formation of a black hole. We discuss the development of the Weyl tensor and prove its existence in spacetime during the gravitational collapse of cosmic objects, utilizing the Riemannian curvature tensor, Ricci tensor, Kulkarni–Nomizu product, and Schouten tensor. By decomposing the Weyl tensor, we use theorems and proofs that satisfy the exact solutions of the Einstein field equations. We observe that the Riemann curvature tensor and Weyl tensor share the same symmetric identities, as $trW(\delta,.)\sigma = 0$ such that $W_{\delta\sigma\gamma\tau} = 0$ when Riemannian curvature tensor, $R_{\delta\sigma\gamma\tau} = 0$. Additionally, the Riemann curvature and Weyl scalar tensor invariants are conformally related to each other, as $R^{\delta\sigma\gamma\tau}R_{\delta\sigma\gamma\tau} = W^{\delta\sigma\gamma\tau}W_{\delta\sigma\gamma\tau} = \frac{48(GM)^2}{r^6}$ in the Schwarzschild metric. From the Einstein field equations, the Ricci tensor is $R_{\sigma\tau} = 0$; consequently, the stress-energy tensor, $T_{\sigma\tau} = 0$, indicating that the Einstein field equation is empty space. However, in the Schwarzschild black hole solution, the Ricci tensor vanishes, but the Weyl tensor does not. Additionally, it seems that divergence occurs around the event horizon in a stagnant and uncharged Schwarzschild black hole with proper acceleration. Furthermore, the investigation into the existence of the Weyl tensor in the Schwarzschild black hole reveals its presence. We also explore the Reissner–Nordström, Kerr, and Kerr–Newman black holes by examining the coupling between the Einstein-Maxwell field equations and the Weyl tensor, utilizing small Weyl corrections. We obtain the metric that reduces to the Kerr–Newman black hole solution in Boyer–Lindquist coordinates when $\alpha = 0$. The same metric equation obtained reduces to Kerr black hole solutions when the electric charge $q = 0$ and the coupling parameter $\alpha = 0$. Furthermore, when the parameter of the charged rotating black hole a vanishes, we obtain solutions for the static and spherically symmetric black hole with Weyl corrections. When the terms $a = q = 0$, the obtained metric reduces to the Schwarzschild black hole solution.

Key words: Gravitational collapse, Riemann curvature tensor, Schwarzschild black hole, Reissner–Nordström black hole, Kerr and Kerr–Newman black holes

1. Introduction

Gravitational collapse is the contraction of an astronomical object due to the influence of its gravity, which tends to pull matter towards the center of gravity [1–4]. The formation of black holes is the result of collapsing objects, allowing for the creation of a wide range of structures early on. A well-defined event horizon is a part of the spacetime region where the geometry behaves similarly

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to the external environment, leading to a highly constrained state of equilibrium with gravitational radiation emitting in various directions [2, 5–9]. In 1929, Herman Weyl introduced Weyl's equations as an alternative to the Dirac equation for describing subatomic particles such as nucleons [10–13]. Ordinary Dirac fermions can be derived from and are composed of two Weyl fermions [14]. Initially, neutrinos were thought to be Weyl fermions, but they were later found to possess mass. In General Relativity, the Weyl tensor provides the Riemannian curvature, while the Ricci tensor equals zero due to the nature of the Weyl tensor. The Ricci tensor, in the context of general relativity, represents the energy-momentum distribution of matter within confined fields. If there is no distributed matter, the Ricci tensor is also zero. However, even though the Riemannian curvature and Ricci tensor are zero in spacetime, it does not necessarily imply that space is flat [15]. While we understand the contribution of the Weyl tensor to the Riemannian curvature and Ricci tensor in spacetime, the gravitational field remains nonzero. This allows gravity to spread even where there is no matter or energy source. The Weyl tensor is the sole tensor where curvature exists in the vacuum of Einstein's region of space devoid of matter, particularly in the context of the Schwarzschild black hole metric. In general relativity, we learn the distribution of matter through the Riemannian metric tensor $R_{\delta\sigma\gamma\tau}$ and the Ricci tensor $R_{\sigma\tau}$ alongside the stress-energy momentum tensor in the Einstein field equations. Initially, we decompose the Riemannian metric tensor using the Einstein field equations.

$$R_{\sigma\tau} - \frac{1}{2}g_{\sigma\tau}R = \frac{8\pi G}{C^4}T_{\sigma\tau} \tag{1}$$

The Weyl tensor is derived from the Riemann curvature tensor minus the Kulkarni–Nomizu product of the Schouten tensor. To obtain the Weyl tensor, we first establish the Riemann curvature tensor, Schouten tensor, and Kulkarni–Nomizu product. These tensors all involve the Ricci tensor and scalar curvature. Numerous researchers have studied the behavioral changes of black holes by employing Weyl corrections [16–18]. Investigations into rotating charged black holes within four-dimensional spherical symmetry, incorporating small Weyl corrections, have been conducted by [17–19]. Similarly, static and spherically symmetric charged black holes with Weyl tensor corrections have been explored by [17, 20–23]. The aim of this paper is to conduct a comprehensive investigation into the role of the Weyl tensor in black hole formation. We commence in Section 2, by deriving the Riemann curvature tensor, which serves as the key step in decomposing the Weyl tensor through defining and proving an appropriate theorem. In Section 3, we delve into the Schouten tensor, elucidating its crucial properties in Riemannian geometry towards the Weyl tensor decomposition which we obtain through calculations involving the Riemann curvature tensor. Section 4 is dedicated to the manipulation of the Kulkarni–Nomizu product, exploring its fundamental properties in differential geometry such as products, direct sums, wedge products, and symmetric products. Moving on to Section 5, we examine the algebraic properties of $R_{\delta\sigma\gamma\tau}$ using covariant derivatives to decompose Christoffel symbols, thereby capturing the essence of the algebraic properties of the Riemann curvature tensor, such as symmetric, antisymmetric, cyclic, and Bianchi identities. In Section 6, we investigate the Weyl tensor, particularly its implications for black hole formation, with a special emphasis on the Schwarzschild black hole. In Section 7, we delve into the role of the Weyl tensor in the formation of Reissner–Nordström (RN) black holes using a coupling method between Einstein field equations and Weyl corrections. In Section 8, we investigate the Weyl tensor in Kerr and Kerr–Newman (KN) black holes through the coupling of Einstein-Maxwell field equations with the Weyl tensor using minimal Weyl corrections. Moving forward to Section 9, we observe the implications of the study findings in both experimental and theoretical work in laboratory astrophysics concerning the gravitational collapse of cosmic objects. Finally, we conclude this paper with a summary in Section 10.

2. Riemann curvature tensor

Definition 1 *Given a Riemannian space (M, g), $p \in M$, two linear independent tangent vectors $u, v \in T_pM$, we define the sectional curvature [24, 25]:*

$$K(\delta, \sigma) = \frac{< R(\delta, \sigma)\sigma, \delta >}{< \delta, \delta >< \sigma, \sigma > - < \delta, \sigma >^2}, \tag{2}$$

where R is the Riemann curvature tensor defined by

$$R(\delta, \sigma)\gamma = \nabla_\delta \nabla_\sigma \gamma - \nabla_\sigma \nabla_\delta \gamma - \nabla_[\delta, \sigma]\gamma, \tag{3}$$

where δ and σ are linearly independent and in the expression (2), denominator is nonzero, if δ and σ are orthonormal, then the equation (2) will be in the form of

$$K(\delta, \sigma) =< R(\delta, \sigma)\sigma, \delta > . \tag{4}$$

Theorem 1 *A Riemannian manifold is a space form if it contains a constant value of sectional curvature k, then the curvature tensor can be written as:*

$$R(\delta, \sigma)\gamma = k(< \sigma, \gamma > \delta - < \delta, \gamma > \sigma), \forall \delta, \sigma, \gamma \in T_pM. \tag{5}$$

Proof We can use the first and second argument of polarization for $R(\delta, \sigma)\sigma$ and $R(\delta, \sigma)\gamma + R(\delta, \gamma)\sigma$, respectively. From these two polarizations with a combination of first Bianchi identity improves the given formula for $R(\delta, \sigma)\sigma$.
From the sectional curvature (2), we have

$$< R(\delta, \sigma)\sigma, \delta >= k(|\delta|^2|\sigma|^2 - < \delta, \sigma >^2), \tag{6}$$

where δ, σ are linearly independent. Let us give the arbitrary values of δ, σ, γ and compute $< R(\delta + \gamma, \sigma)\sigma, \delta + \gamma >$ in two different ways.

i . By applying multilinearity, it equals

$$< R(\delta, \sigma)\sigma, \delta > + < R(\gamma, \sigma)\sigma, \gamma > + < R(\delta, \sigma)\sigma, \gamma > + < R(\gamma, \sigma)\sigma, \delta >, \tag{7}$$

ii . According to the equation (6), it equals

$$k(|\sigma|^2(|\delta|^2 + |\gamma|^2 + 2 < \delta, \gamma >) - < \delta, \sigma >^2 - < \gamma, \sigma >^2 - 2 < \delta, \sigma >< \gamma, \sigma >). \tag{8}$$

From equation (7), and recalling the Riemannian symmetry:

$$< R(\delta, \sigma)\sigma, \gamma >=< R(\gamma, \sigma)\sigma, \delta >, \tag{9}$$

then (7), with the help of (9), will be simplified to:

$$< R(\delta,\sigma)\sigma,\delta > + < R(\gamma,\sigma)\sigma,\gamma > +2 < R(\delta,\sigma)\sigma,\gamma > . \tag{10}$$

Also, equation (8) reduces to:

$$k(|\delta|^2|\sigma|^2 - < \delta,\sigma >^2) + k(|\gamma|^2|\sigma|^2 - < \gamma,\sigma >^2) + 2 < R(\delta,\sigma)\sigma,\gamma > . \tag{11}$$

Setting equation (10) and (11) equal, we have:

$$< R(\delta,\sigma)\sigma,\gamma >= k(|\sigma|^2 < \delta,\gamma > - < \delta,\sigma >< \gamma,\sigma >). \tag{12}$$

Since γ is arbitrary, then equation (12) will be:

$$R(\delta,\sigma)\sigma = k(|\sigma|^2\delta - < \delta,\sigma > \sigma). \tag{13}$$

Let δ,σ,γ be arbitrary and calculate

$$< R(\delta,\sigma+\gamma)(\sigma+\gamma),\delta > \tag{14}$$

in two different ways :

(1) By applying multilinearity, we get

$$< R(\delta,\sigma+\gamma)(\sigma+\gamma),\delta >= R(\delta,\sigma)\sigma + R(\delta,\gamma)\gamma + R(\delta,\sigma)\gamma + R(\delta,\gamma)\sigma, \tag{15}$$

(2) Using the formula given in equation (8), we get

$$\begin{gathered} < R(\delta,\sigma+\gamma)(\sigma+\gamma),\delta >= \\ k((|\sigma|^2 + |\gamma|^2 + 2 < \sigma,\gamma >)\delta - < \delta,\sigma > \sigma - < \delta,\gamma > \sigma - < \delta,\sigma > \gamma - < \delta,\gamma > \gamma). \end{gathered} \tag{16}$$

Simplifying, we get

$$= k(|\sigma|^2\delta - < \delta,\sigma > \sigma) + k(|\gamma|^2\delta - < \delta,\gamma > \gamma) + R(\delta,\sigma)\gamma + R(\delta,\gamma)\sigma. \tag{17}$$

Setting equations (16) and (17) equal, we have

$$R(\delta,\sigma)\gamma + R(\delta,\gamma)\sigma = k(2 < \sigma,\gamma > \delta - < \delta,\gamma > \sigma - < \delta,\sigma > \gamma). \tag{18}$$

Interchanging δ and σ and adding to Bianchi identity, we get

$$2R(\sigma,\delta)\gamma + R(\delta,\gamma)\sigma = k(2 < \delta,\gamma > \sigma - < \sigma,\gamma > \delta - < \delta,\sigma > \gamma). \tag{19}$$

Subtracting (18) from (19) in consideration of symmetry $R(\delta,\sigma)\gamma = -R(\sigma,\delta)\gamma$, we have:

$$R(\delta,\sigma)\gamma = k(< \sigma,\gamma > \delta - < \delta,\gamma > \sigma). \tag{20}$$

□

We know that the Riemannian metric is equivalent with respect to its Levi-Civita connection, which shows that the Riemannian tensor of any constant curvature in Einstein space is also equivalent (parallel). Then, the Ricci tensor and scalar curvature are given as $Ric = (n-1)kg$ and $n(n-1)k$, respectively. The Riemannian curvature tensor has some symmetries and identities, such as:

$$\begin{aligned} &R(\delta,\sigma) = -R(\sigma,\delta), \\ &< R(\delta,\sigma)\gamma,\tau >= - < R(\delta,\sigma)\tau,\gamma >, \\ &< R(\delta,\sigma)\gamma,\tau >=< R(\gamma,\tau)\delta,\sigma >, \\ &R(\delta,\sigma)\gamma + R(\sigma,\gamma)\delta + R(\gamma,\delta)\sigma = 0. \end{aligned} \tag{21}$$

We write Riemann curvature tensor in terms of indices as follows:

$$\begin{aligned} &R_{\delta\sigma\gamma\tau} = -R_{\delta\sigma\tau\gamma} \iff R_{\delta\sigma(\gamma\tau)} = 0, \\ &R_{\delta\sigma\gamma\tau} = -R_{\sigma\delta\gamma\tau} \iff R_{(\delta\sigma)\gamma\tau} = 0, \\ &R_{\delta\sigma\gamma\tau} + R_{\delta\gamma\sigma\tau} + R_{\delta\tau\sigma\gamma} = 0, \\ &R_{\delta\sigma\gamma\tau} = R_{\gamma\tau\delta\sigma}. \end{aligned} \tag{22}$$

Weyl tensor has the same symmetries and identities as the Riemann curvature tensor as follows:

$$\begin{aligned} &W(\delta,\sigma) = -W(\sigma,\delta), \\ &< W(\delta,\sigma)\gamma,\tau >= - < W(\delta,\sigma)\tau,\gamma >, \\ &< W(\delta,\sigma)\gamma,\tau >=< W(\gamma,\tau)\delta,\sigma >, \\ &W(\delta,\sigma)\gamma + W(\sigma,\gamma)\delta + W(\gamma,\delta)\sigma = 0. \end{aligned} \tag{23}$$

We know that the Weyl tensor is trace-free as:

$$trW(\delta,.)\sigma = 0, \quad \forall \delta,\sigma \in T_pM. \tag{24}$$

We write this in indices as:

$$W_{\delta\sigma\gamma\tau} = -W_{\sigma\delta\gamma,\tau} = -W_{\delta\sigma\tau\gamma}, \tag{25}$$

and

$$W_{\delta\sigma\gamma\tau} + W_{\delta\gamma\tau\sigma} + W_{\delta\tau\sigma\gamma} = 0. \tag{26}$$

Therefore,

$$W^{\delta}_{\sigma\delta\gamma} = 0. \tag{27}$$

3. Schouten tensor

Schouten tensor in Riemannian geometry is the second order covariant tensor introduced by Schouten [26, 27], defined for $n \geq 3$. We know that the Weyl tensor is obtained through the Riemann curvature tensor by subtracting several traces. These calculations are done by decomposing the Riemann tensor in $(0,4)$ valence tensor by subtracting with metric as [28]:

$$\zeta = R_m - \frac{1}{n-2}\Big(Ric - \frac{R}{n}g\Big) \owedge g - \frac{R}{2n(n-1)} g \owedge g, \tag{28}$$

where n is the dimension of the collapsing object, g is the metric, R_m is the Riemann curvature tensor, Ric is the Ricci tensor, R is the scalar curvature, and $u \owedge v$ denote the Kulkarni–Nomizu product of two symmetrical tensors of type $(0,2)$; in the next section, we will see its decompositions.

$$u \owedge v(e_1, e_2, e_3, e_4) = u(e_1, e_3)v(e_2, e_4) + u(e_2, e_4)v(e_1, e_3) - u(e_1, e_4)v(e_2, e_3) - u(e_2, e_3)v(e_1, e_4), \quad (29)$$

and we can write this in tensor components as:

$$\zeta_{\delta\sigma\gamma\tau} = R_{\delta\sigma\gamma\tau} + \frac{1}{n-2}(R_{\delta\tau}g_{\sigma\gamma} - R_{\delta\gamma}g_{\sigma\tau} + R_{\sigma\gamma}g_{\delta\tau} - R_{\sigma\tau}g_{\delta\gamma} + \frac{1}{(n-1)(n-2)}R(g_{\delta\gamma}g_{\sigma\tau} - g_{\delta\tau}g_{\sigma\gamma}). \quad (30)$$

The Weyl tensor with ordinary valence like $(1,3)$ can be given with the contraction of the above equation (30) with inverse metric. Then, the decomposition that expresses the Riemann curvature tensor as an orthonormal direct sum is given as:

$$|R_m|^2 = |\zeta|^2 + \Big|\frac{1}{n-2}(Ric - \frac{R}{n}g) \owedge g\Big|^2 + \Big|\frac{R}{2n(n-1)}g \owedge g\Big|^2. \quad (31)$$

This is a Ricci decomposition. Therefore, now we can write the equation of Schouten tensor defined for $n \geq 3$ as [26–28]:

$$Q = \frac{1}{n-2}\Big(Ric - \frac{R}{2(n-1)}g\Big) \iff Ric = (n-2)Q + Jg, \quad (32)$$

where Ric is a Ricci tensor, and it can be defined by contracting the first and the third indices of the Riemannian tensor, R is the scalar curvature, g is the Riemannian metric, $J = \frac{1}{n-2}R$ is the trance of Q, and n is the dimension of the manifold.
Therefore, through the expressions of the Schouten tensor, we can write the Weyl tensor as a trace-adjusted multiple with Ricci tensor, as we can see in equation (32) as:

$$Q = \frac{1}{n-2}\Big(Ric - \frac{R}{2(n-1)}g\Big). \quad (33)$$

From equation (31), we express $\zeta = R - Q \owedge g$, then, through indices, we obtain the Weyl tensor as:

$$W_{\delta\sigma\gamma\tau} = R_{\delta\sigma\gamma\tau} + \frac{2}{n-2}\Big(g_{\delta[\gamma}R_{\tau]} - g_{\sigma[\gamma}R_{\tau]\delta}\Big) + \frac{2}{(n-1)(n-2)}Rg_{\delta[\gamma}g_{\tau]\sigma}, \quad (34)$$

where $R_{\delta\sigma\gamma\tau}$ is the Riemann tensor, $R_{\delta\sigma}$ is the Ricci tensor, R is the scalar curvature and the brackets around the indices are antisymmetric part of the equation. Consistently,

$$W_{\delta\sigma}^{\gamma\tau} = -4S[^{\gamma}_{\delta}\Delta^{\tau}_{\sigma}], \quad (35)$$

where S is the Schouten tensor.

4. Kulkarni–Nomizu product

The Kulkarni–Nomizu product, sometimes known as the $\mathcal{K}-\mathcal{N}$ product, is very famous in differential geometry for finding $(0,2)$ tensors and gives the results of $(0,4)$-tensor; this method was discovered by Kulkarni and Nomizu [29].

Definition 2 *If* $\mathcal{K}$ *and* $\mathcal{N}$ *are symmetric* $(0,2)-tensors$*, then*

$$\mathcal{K} \owedge \mathcal{N}(e_1,e_2,e_3,e_4) = \mathcal{K}(e_1,e_3)\mathcal{N}(e_2,e_4) + \mathcal{K}(e_2,e_4)\mathcal{N}(e_1,e_3) - \mathcal{K}(e_1,e_4)\mathcal{N}(e_2,e_3) - \mathcal{K}(e_2,e_3)\mathcal{N}(e_1,e_4)$$

$$= \begin{vmatrix} \mathcal{K}(e_1,e_3) & \mathcal{K}(e_1,e_4) \\ \mathcal{N}(e_2,e_3) & \mathcal{N}(e_2,e_4) \end{vmatrix} + \begin{vmatrix} \mathcal{N}(e_1,e_3) & \mathcal{N}(e_1,e_4) \\ \mathcal{N}(e_2,e_3) & \mathcal{K}(e_2,e_4) \end{vmatrix}, \tag{36}$$

where e_i *are tangent vectors and* $|..|$ *are the determinants of matrix. From equation (36), we see that* $\mathcal{K} \owedge \mathcal{N} = \mathcal{N} \owedge \mathcal{K}$.

The tangent space with respect to basis $\{\partial_i\}$ can be compressed into the following form:

$$(\mathcal{K} \owedge \mathcal{N})_{ijlm} = (\mathcal{K} \owedge \mathcal{N})(\partial_i, \partial_j, \partial_l, \partial_m) = 2\mathcal{K}_{i[l}\mathcal{N}_{m]j} + 2\mathcal{K}_{j[m}\mathcal{N}_{l]i}, \tag{37}$$

where $[...]$ denotes the total antisymmetrical symbol which shows the alternating signs $(+/-)$. The Kulkarni–Nomizu product is the graded algebra with the particular case of products like direct sum, wedge product, and symmetric product as follows:

$$\bigoplus_{\lim_{p=1}}^{n} S^2(\Upsilon^p M) \tag{38}$$

means that $(\alpha.\beta) \owedge (\gamma.\delta) = (\alpha \wedge \gamma) \odot (\beta \wedge \delta)$, where $\odot$ is a symmetric product [30]. We know that the Riemann curvature tensor also has the expressions in terms of the Kulkarni–Nomizu product of the metric:

$$g = g_{\delta\sigma} dv^\delta \otimes dv^\sigma, \tag{39}$$

with itself.
If we denote by

$$R((\partial_\delta, \partial_\sigma)\partial_\gamma = R^\tau_{\delta\sigma\gamma}\partial_l, \tag{40}$$

then, the $(1,3)-$Riemann curvature tensor is given by

$$R_m = R_{\delta\sigma\gamma\tau} dv^\delta \otimes dv^\sigma \otimes dv^\gamma \otimes dv^\tau, \tag{41}$$

with $R_{\delta\sigma\gamma\tau} = g_{\delta k} R^\gamma_{\sigma\gamma\tau}$, then the Riemann curvature tensor will be reduced to

$$R_m = \frac{Scal}{4} g \owedge g, \tag{42}$$

where $Scal = tracgRic = R^\delta_\delta$ is the scalar curvature and $Ric(\delta\sigma) = trac_g\{\delta \to R(\delta\sigma)\gamma\}$ is the Ricci tensor which reads as $R_{\delta\sigma} = R^\gamma_{\delta\gamma\sigma}$.
Therefore, in the expansion of Kulkarni–Nomizu product $g \owedge g$ by using the definition (36), we get:

$$R^\delta_{\sigma\gamma\tau} = \frac{Scal}{4} g_{\delta[\gamma} g_{\tau]\sigma} = \frac{Scal}{2}(g_{\delta\gamma}g_{\sigma\tau} - g_{\delta\tau}g_{\sigma\gamma}), \tag{43}$$

which is the same as the Riemann curvature tensor as stated above. We know that the Riemannian manifolds have constant sectional curvature k if and only if the Riemann curvature tensor has the following form:

$$\frac{k}{2} g \owedge g, \tag{44}$$

where g is the metric tensor. Note that the $(0,4)-tensor$ of the Kulkarni–Nomizu product satisfies the Bianchi identities and the skew-symmetric property. Therefore, the Weyl tensor can be obtained from the Kulkarni–Nomizu product through the Riemann curvature tensor of type $(0,4)$ by subtracting several traces as in equation (28).

5. Algebraic properties of $R_{\delta\sigma\gamma\tau}$

Instead of using the algebraic properties of curvature tensor $R^{\delta}_{\sigma\gamma\tau}$, we prefer using the covariant form as:

$$R_{\delta\sigma\gamma\tau} \equiv g_{\delta\beta} R^{\beta}_{\sigma\gamma\tau}. \tag{45}$$

Through the decomposition of second-kind Christoffel symbols, we get the algebraic properties of the Riemann curvature tensor as follows:

$$R^{\delta}_{\sigma\gamma\tau} \equiv \frac{\partial \Gamma^{\delta}_{\sigma\gamma}}{\partial x^{\tau}} - \frac{\partial \Gamma^{\delta}_{\sigma\tau}}{\partial x^{\gamma}} + \Gamma^{\eta}_{\sigma\gamma}\Gamma^{\delta}_{\tau\eta} - \Gamma^{\eta}_{\sigma\tau}\Gamma^{\delta}_{\gamma\eta}, \tag{46}$$

and

$$\Gamma^{\beta}_{\delta\sigma} = \frac{1}{2} g^{\gamma\beta} \{ \frac{\partial g_{\sigma\gamma}}{\partial x^{\delta}} + \frac{\partial g_{\delta\gamma}}{\partial x^{\sigma}} - \frac{\partial g_{\sigma\delta}}{\partial x^{\gamma}} \}. \tag{47}$$

Using (46) and (47) in (45), we get

$$R_{\delta\sigma\gamma\tau} = \frac{1}{2} g_{\delta\beta} \frac{\partial g^{\beta p}}{\partial x^{\tau}} \{ \frac{\partial g_{p\sigma}}{\partial x^{\gamma}} + \frac{\partial g_{p\gamma}}{\partial x^{\sigma}} - \frac{\partial g_{\sigma\gamma}}{\partial x^{p}} \} - \frac{1}{2} g_{\delta\beta} \frac{\partial g^{\beta p}}{\partial x^{\gamma}} \{ \frac{\partial g_{p\sigma}}{\partial x^{\tau}} + \frac{\partial g_{p\tau}}{\partial x^{\sigma}} - \frac{\partial g_{\sigma\tau}}{\partial x^{p}} \} + g_{\delta\beta} \{ \Gamma^{\eta}_{\sigma\gamma}\Gamma^{\beta}_{\tau\eta} - \Gamma^{\eta}_{\sigma\tau}\Gamma^{\beta}_{\gamma\eta} \}. \tag{48}$$

Using the relation

$$g_{\delta\beta} \frac{\partial}{\partial x^{\tau}} g^{\beta p} = -g^{\beta p} \frac{\partial}{\partial x^{\tau}} g_{\delta\beta} = -g^{\beta p} \Big(\Gamma^{\eta}_{\tau\delta} g_{\eta\beta} + \Gamma^{\eta}_{\tau\beta} g_{\eta\delta} \Big), \tag{49}$$

we obtain

$$\begin{aligned} R_{\delta\sigma\gamma\tau} = \frac{1}{2} \Big[\frac{\partial^2 g_{\delta\gamma}}{\partial x^{\tau}\partial x^{\sigma}} - \frac{\partial^2 g_{\sigma\gamma}}{\partial x^{\tau}\partial x^{\delta}} - \frac{\partial^2 g_{\delta\tau}}{\partial x^{\gamma}\partial x^{\sigma}} + \frac{\partial^2 g_{\sigma\tau}}{\partial x^{\gamma}\partial x^{\delta}} \Big] - \Big[\Gamma^{\eta}_{\tau\delta} g_{\eta\beta} + \Gamma^{\eta}_{\tau\beta} g_{\eta\delta} \Big] \Gamma^{\beta}_{\sigma\gamma} \\ + \Big[\Gamma^{\eta}_{\gamma\delta} g_{\eta\beta} + \Gamma^{\eta}_{\gamma\beta} g_{\eta\delta} \Big] \Gamma^{\beta}_{\sigma\tau} + g_{\delta\beta} \Big[\Gamma^{\eta}_{\sigma\gamma}\Gamma^{\beta}_{\tau\eta} - \Gamma^{\eta}_{\sigma\tau}\Gamma^{\beta}_{\gamma\eta} \Big]. \end{aligned} \tag{50}$$

Most of the terms cancel and leave us with

$$R_{\delta\sigma\gamma\tau} = \frac{1}{2} \Big[\frac{\partial^2 g_{\delta\gamma}}{\partial x^{\tau}\partial x^{\sigma}} - \frac{\partial^2 g_{\sigma\gamma}}{\partial x^{\tau}\partial x^{\delta}} - \frac{\partial^2 g_{\delta\tau}}{\partial x^{\gamma}\partial x^{\sigma}} + \frac{\partial^2 g_{\sigma\tau}}{\partial x^{\gamma}\partial x^{\delta}} \Big] + g_{\eta\beta} \Big[\Gamma^{\eta}_{\gamma\delta}\Gamma^{\beta}_{\sigma\tau} - \Gamma^{\eta}_{\tau\delta}\Gamma^{\beta}_{\sigma\gamma} \Big]. \tag{51}$$

From (51), we can see the algebraic properties of the curvature tensor the same as we see it from (23) as:

- Symmetry:

$$R_{\delta\sigma\gamma\tau} = R_{\gamma\tau\delta\sigma} \tag{52}$$

- Antisymmetry:

$$R_{\delta\sigma\gamma\tau} = -R_{\sigma\delta\gamma\tau} - R_{\delta\sigma\tau\gamma} = +R_{\sigma\delta\tau\gamma} \tag{53}$$

- Cyclicity:

$$R_{\delta\sigma\gamma\tau} + R_{\delta\tau\sigma\gamma} + R_{\delta\gamma\tau\sigma} = 0 \tag{54}$$

- Bianchi identities:

$$R_{\delta\sigma\gamma\tau;\eta} + R_{\delta\sigma\eta\gamma;\tau} + R_{\delta\sigma\tau\eta;\gamma} = 0 \tag{55}$$

The Bianchi identities are obtained by permuting γ, τ, and η cyclically. Equation (55) is covariant and holds in a locally inertial system. Also, we can see the contracted form of (55) by recalling the covariant derivatives of $g^{\delta\gamma}$ vanish, we find on contraction of δ with γ that [31, 32]:

$$R_{\sigma\tau;\eta} - R_{\sigma\eta;\tau} + R^{\gamma}_{\sigma\tau\eta;\gamma} = 0, \tag{56}$$

contracting again gives:

$$\left(R^{\sigma}\eta - \frac{1}{2}\Delta^{\sigma}_{\eta}R\right); \sigma = 0, \tag{57}$$

which is equivalent to

$$\left(R^{\sigma\gamma} - \frac{1}{2}g^{\sigma\gamma}R\right); \sigma = 0. \tag{58}$$

We mention that, the $R_{\delta\sigma\gamma\tau}$ may be contracted to give the Ricci tensor as:

$$R_{\sigma\tau} = g^{\delta\gamma}R_{\delta\sigma\gamma\tau}. \tag{59}$$

Also, using the antisymmetry property(53), we see that there is only one way of contracting $R_{\delta\sigma\gamma\tau}$ to construct the scalar curvature:

$$\begin{aligned} R &\equiv g^{\delta\gamma}g^{\sigma\tau}R_{\delta\sigma\gamma\tau} = -g^{\delta\gamma}g^{\sigma\tau}R_{\sigma\delta\gamma\tau}, \\ 0 &= g^{\delta\sigma}g^{\gamma\tau}R_{\delta\sigma\gamma\tau}. \end{aligned} \tag{60}$$

Therefore, the cyclicity property mentioned in equation (54) ultimately eliminates any other scalar curvature that could have formed in four dimensions, that is:

$$\frac{1}{\sqrt{g}}e^{\delta\sigma\gamma\tau}R_{\delta\sigma\gamma\tau} = 0. \tag{61}$$

Also, we consider the curvature in N-dimensional spacetime to find the number of algebraically independent components of $R_{\delta\sigma\gamma\tau}$. We can adopt the Petrov notation [33] and consider $R_{\delta\sigma\gamma\tau}$ as a matrix of $R_{(\delta\sigma)(\gamma\tau)}$ with indices $(\delta\sigma)$ and $(\gamma\tau)$. From (53), each index takes some independent values equal to the number of independent elements of an antisymmetric matrix in dimension N as:

$$\frac{1}{2}N(N-1). \tag{62}$$

From (52), we see that $R_{(\delta\sigma)(\gamma\tau)}$ is symmetric in these indices. Therefore, when (52) and (53) stand alone, $R_{\delta\sigma\gamma\tau}$ will have independent components equal to the number of independent elements of a symmetric matrix, as stated in equation (62) [31–33].

$$\frac{1}{2}\left[\frac{1}{2}N(N-1)\right]\left[\frac{1}{2}N(N-1)+1\right]=\frac{1}{8}N(N-1)(N^2-N+2). \tag{63}$$

Equation (52) and (53) also makes the cyclic sum of

$$R_{\delta\sigma\gamma\tau}+R_{\delta\tau\sigma\gamma}+R_{\delta\gamma\tau\sigma}, \tag{64}$$

which is completely antisymmetric. Thus, equation (54) introduces further constraints, adding:

$$\frac{N(N-1)(N-2)(N+3)}{4!}, \tag{65}$$

which leads $R_{\delta\sigma\gamma\tau}$ to have a number of independent components equal to

$$C_N=\frac{1}{12}N^2(N^2-1). \tag{66}$$

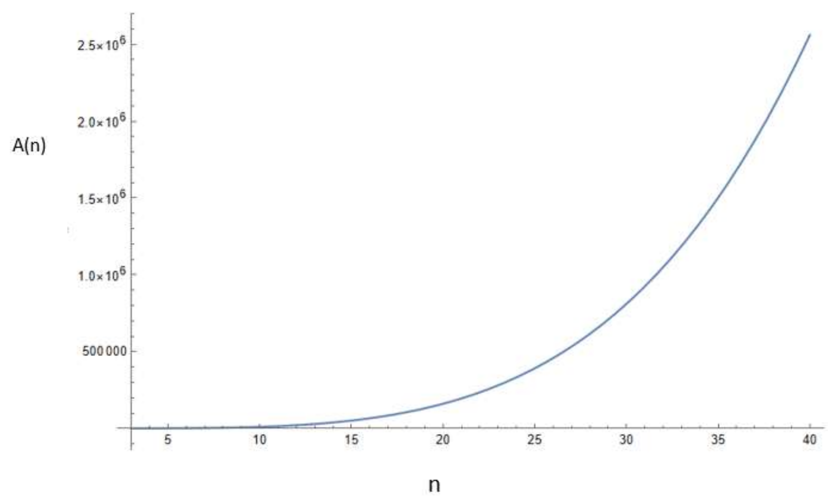


Figure 1. Riemannian curvature tensor.

We observe that the curvature tensor in one dimension is R_{1111}, which always vanishes. This can be seen from either equation (53) or (54), or from the fact that equation (66) yields $C_1=0$, indicating zero independent components. This observation is also depicted in Figure 1. It only reveals the internal characteristics of space, reflecting through $R_{\delta\sigma\gamma\tau}$, without considering its embedding in additional spacetime dimensions. In one dimension, the transformation rules for the metric tensor are given by

$$g_{11}^{'}=\left(\frac{dx}{dx'}\right)^2 g_{11}, \tag{67}$$

so that g'_{11} can be completely equal to ± 1 everywhere by choosing $x' = \int \sqrt{\pm g_{11}}dx$. In two dimensions, equation (66) indicates that $R_{\delta\sigma\gamma\tau}$ has only one independent component, which can be represented as R_{1212}. The other components are related to R_{1212} through equation (53) as follows:

$$R_{1212} = -R_{2112} = -R_{1221} = R_{2121} \quad \text{and} \quad R_{1111} = R_{1122} = R_{2222} = 0. \tag{68}$$

These formulae can be summarized smoothly as

$$R_{\delta\sigma\gamma\tau} = \Big(g_{\delta\gamma}g_{\sigma\tau} - g_{\delta\tau}g_{\sigma\gamma}\Big)\frac{R_{1212}}{g}, \tag{69}$$

where g is the determinant $g_{11}g_{22} - g_{12}^2$. If we contract δ with γ, it gives the Ricci tensor:

$$R_{\sigma\tau} = g_{\sigma\tau}\frac{R_{1212}}{g}, \tag{70}$$

and contracting σ and τ, it gives the scalar curvature:

$$R = 2\frac{R_{1212}}{g}. \tag{71}$$

Therefore, the curvature tensor is

$$R_{\delta\sigma\gamma\tau} = \frac{1}{2}R\Big(g_{\delta\gamma}g_{\sigma\tau} - g_{\delta\tau}g_{\sigma\gamma}\Big). \tag{72}$$

Therefore, the sectional curvature, discussed in Section 2, will be defined as:

$$k \equiv= -\frac{R}{2} = -\frac{R_{1212}}{g}. \tag{73}$$

In three dimensions, equation (66) implies that the curvature tensor $C_3 = 6$ independent components. Moreover, this is the same number of independent components as the Ricci tensor $R_{\sigma\tau}$ in three dimensions. This suggests the possibility that $R_{\delta\sigma\gamma\tau}$ might be expressed solely in terms of $R_{\sigma\tau}$.

Theorem 2 *By using the covariant, symmetry, and contraction properties of* $R_{\delta\sigma\gamma\tau}$*, we have:*

$$R_{\delta\sigma\gamma\tau} = g_{\delta\gamma}R_{\sigma\tau} - g_{\delta\tau}R_{\sigma\gamma} - g_{\sigma\gamma}R_{\delta\tau} + g_{\sigma\tau}R_{\delta\gamma} - \frac{1}{2}\Big(g_{\delta\gamma}g_{\sigma\tau} - g_{\delta\tau}g_{\sigma\gamma}\Big)R \tag{74}$$

Proof To prove (74), first we need to adopt the coordinate systems such that the $g_{\sigma\gamma}$ vanishes for $\sigma \neq \gamma$ at some point X. This can be managed by choosing $\frac{\partial x'^{\sigma}}{\partial x^{\delta}}$ at X as the orthogonal matrix that diagonalizes $g_{\sigma\gamma}$ at X. Therefore, in this system, at X we have:

$$R_{12} = g^{33}R_{1323}, \qquad R_{1323} = g_{33}R_{12}. \tag{75}$$

The equation (75) is in agreement with (74). Furthermore,

$$R_{11} = g^{22}R_{1212} + g^{33}R_{1313}, \qquad R_{22} = g^{33}R_{2323} + g^{11}R_{2121}, \tag{76}$$

so,

$$\begin{aligned} & g_{22}R_{11} + g_{11}R_{22} = 2R_{1212} + g^{33}\Big(g_{22}R_{1313} + g_{11}R_{2323}\Big) \\ & = R_{1212} + G_{11}g_{22}\Big(g^{11}g^{22}R_{1212} + g^{11}g^{33}R_{1313} + g^{22}g^{33}R_{2323}\Big) \\ & R_{1212} = g_{22}R_{11} + g_{11}R_{22} - \frac{1}{2}g_{11}g_{22}R, \end{aligned} \tag{77}$$

also the equation (77) is in agreement with (74). The other independent components of $R_{\delta\sigma\gamma\tau}$ are R_{1223}, R_{1213}, R_{2323}, and R_{3131}, which can be obtained from R_{1323} and R_{1212} by permuting the values $1, 2, 3$, so, (74) holds for these independent components as well. Since (74) holds in a coordinate system that is orthogonal at X and patently covariant, it generally holds. □

So, four or more dimensions of the Riemann-Christoffel tensor $R_{\delta\sigma\gamma\tau}$ are needed to describe the curvature in spacetime. For example, in four dimensions, (66) gives the curvature tensor $C_4 = 20$ independent components, whereas $R_{\sigma\tau}$ has only 10 independent components, so $R_{\delta\sigma\gamma\tau}$ has ten components beyond those which can be expressed in terms of $R_{\sigma\tau}$.

Therefore, the $\frac{1}{12}N^2(N^2-1)$ components of $R_{\delta\sigma\gamma\tau}$ describes the curvature of a general N-dim spacetime [33], but it does not do so in a consistent manner. Hence, these components' values depend on the intrinsic properties and the particular coordinate system chosen. When we talk about the invariant characterization of the curved space, it must be in terms of scalars which are constructed from $R_{\delta\sigma\gamma\tau}$ and $g_{\sigma\gamma}$. Also, let us see how to count the scalars out there. Let us see the N^2 quantities $\frac{\partial x'^{\sigma}}{\partial x^{\gamma}}$ for the general coordinate transformation as $x \to x'$ it is everything in a given point X. Hence, there are $\frac{1}{12}N^2(N^2-1)$ independent components of $R_{\delta\sigma\gamma\tau}$ and the $\frac{1}{2}N(N+1)$ independent components of $g_{\sigma\gamma}$, so at this point, general coordinate transformations are subjected to N^2 algebraic conditions. Therefore, the number of scalars that can be constructed from $R_{\delta\sigma\gamma\tau}$ and $g_{\sigma\gamma}$ are:

$$\frac{1}{12}N^2(N^2-1) + \frac{1}{2}N(N+1) - N^2 = \frac{1}{12}N(N-1)(N-2)(N+3). \tag{78}$$

The case $N = 2$ is an exception to this argument since 2 dimensional spacetime has one parameter subgroup of coordinate transformations that has no effect on $g_{\sigma\gamma}$ and $R_{\delta\sigma\gamma\tau}$, the correct number of invariants here is not zero but one, that is the scalar curvature R itself. However, this exception does not occur in higher dimension spacetime, so (78) holds for $N \geq 3$. Therefore, for $N = 3$, the equation (78) tells that there is 3 scalar curvature, which suitably is chosen as the three roots of the equation:

$$Det\Big(R_{\sigma\gamma} - \delta g_{\sigma\gamma}\Big) = 0, \tag{79}$$

which is equivalent to the three quantities as $R, R_{\sigma\gamma}R^{\sigma\gamma}$, and $\dfrac{DetR}{Detg}$. For $N=4$, the equation (78) tells that there are 14 independent components of scalar curvature; in order to enumerate them, we need to decompose $R_{\delta\sigma\gamma\tau}$ into terms that only depend on Ricci tensor $R_{\sigma\gamma}$ plus a term $W_{\delta\sigma\gamma\tau}$ which does not contain nontrivial contractions. In $N \geq 3$ dimensions, the decompositions are:

$$R_{\delta\sigma\gamma\tau} = \frac{1}{N-2}\Big(g_{\delta\gamma}R_{\sigma\tau} - g_{\delta\tau}R_{\sigma\gamma} - g_{\sigma\gamma}R_{\delta\tau} + g_{\sigma\tau}R_{\delta\gamma}\Big) - \frac{R}{(N-1)(N-2)}\Big(g_{\delta\gamma}g_{\sigma\tau} - g_{\delta\tau}g_{\sigma\gamma}\Big) + W_{\delta\sigma\gamma\tau}. \tag{80}$$

The tensor $W_{\delta\sigma\gamma\tau}$ is called the conformal tensor or Weyl tensor [32, 34]. It is a necessary and sufficient condition for the existence of coordinate systems in which $g_{\sigma\gamma}$ is proportional to a constant matrix throughout the space and $W_{\delta\sigma\gamma\tau}$ vanishes everywhere [31, 32]. This Weyl tensor has the same algebraic properties as $R_{\delta\sigma\gamma\tau}$ and in addition characteristics satisfies the $\frac{1}{2}N(N+1)$ conditions, $W^{\delta}_{\sigma\gamma\tau} = 0$, so the number of its linearly independent components as shown below, also represented in figure 2 :

$$\frac{1}{12}N^2(N^2-1) - \frac{1}{2}N(N+1) = \frac{1}{12}N(N+1)(N+2)(N-3). \tag{81}$$

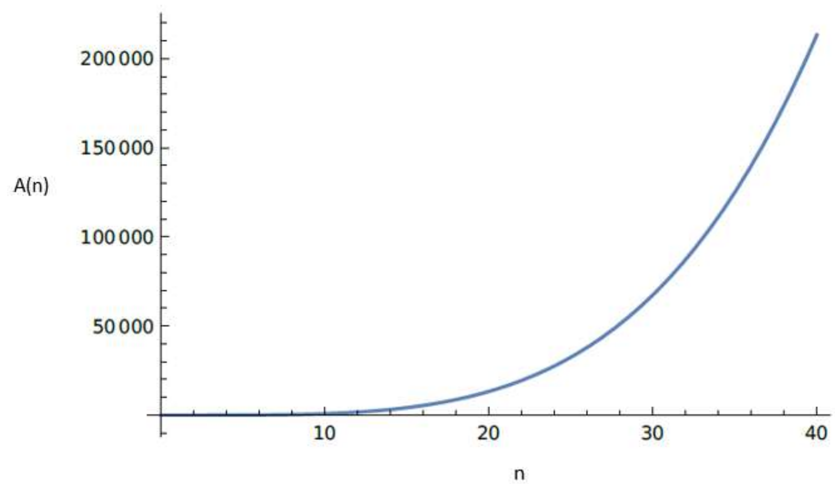


Figure 2. Weyl Tensor

Then from equation (74), $W_{\delta\sigma\gamma\tau} = 0$ for $N=3$. Without any misconception, the curvature invariants can be explained as comprising all the components of the Weyl tensor, except for a specific choice of coordinate axes that make $R_{\sigma\gamma}$ and $g_{\sigma\gamma}$ diagonal. In this case, the elements of $g_{\sigma\gamma}$ are $\pm 1's$ and $0's$, plus the N-eigenvalues of $R_{\sigma\gamma}$ [13, 35, 36]. Nevertheless, sometimes this enumerate breaks down when some eigenvalues of $R_{\sigma\gamma}$ are degenerate; also, we know that when the $R_{\sigma\gamma}=0$, it describes the gravitational fields in empty or vacuum space. Therefore, in this way, the curvature is invariant for $N=4$, have 10 vanishing components of $R_{\sigma\gamma}$ plus 4 quantities, $W^{\delta\sigma\gamma\tau}W_{\delta\sigma\gamma\tau}$, $\in^{\delta\sigma} W^{p\beta\gamma\tau}_{p\beta}W_{\delta\sigma\gamma\tau}$, $W_{\delta\sigma\gamma\tau}W^{\gamma\tau p\beta}W^{\delta\sigma}_{p\beta}$ and $W_{\delta\sigma\gamma\tau}\dfrac{W^{\gamma\tau p\beta} \in^{a\eta}_{p\beta} W^{\delta\sigma}_{a\eta}}{\sqrt{g}}$.

According to Petrov et al. [37], the Petrov type N has given an equivalent description of the 4 nonvanishing curvature invariants as roots of rooted equations and has classified various algebraic types of Weyl tensors according to the degeneracies of these roots. Furthermore, it should be highlighted that (78) gives the number of algebraically independent curvature invariants.
Also, we know that the Weyl tensor in pseudo-Riemannian metric g of the dimension n in terms of the Riemann curvature tensor, Ricci tensor, and scalar curvature is given as [16, 38, 39]

$$W_{\delta\sigma\gamma\tau} = R_{\delta\sigma\gamma\tau} - \frac{R\delta\gamma g_{\sigma\tau} - R_{\delta\tau} g_{\sigma\gamma} + R_{\sigma\tau} g_{\delta\gamma} - R_{\sigma\gamma} g_{\delta\tau}}{n-2} + \frac{R}{(n-1)(n-2)}(g_{\sigma\tau} g_{\delta\gamma} - g_{\sigma\gamma} g_{\delta\tau}) \tag{82}$$

which is the same as the equation given in (80).
Weisstein explained the independent component numbers in the Weyl tensor, which were discovered by Sloane in 1964. This information is listed in the On-Line Encyclopedia Integer Sequences (OEIS) and under the code A052472 [33], where n represents the integer starting from $N = 3, ..., 40$ which are the independent number for Weyl tensor in N-dimension as follows:

$$a(N) = \frac{N\,(N+1)\,(N+2)\,(N-3)}{12}, \tag{83}$$

and is illustrated in Figure 3.

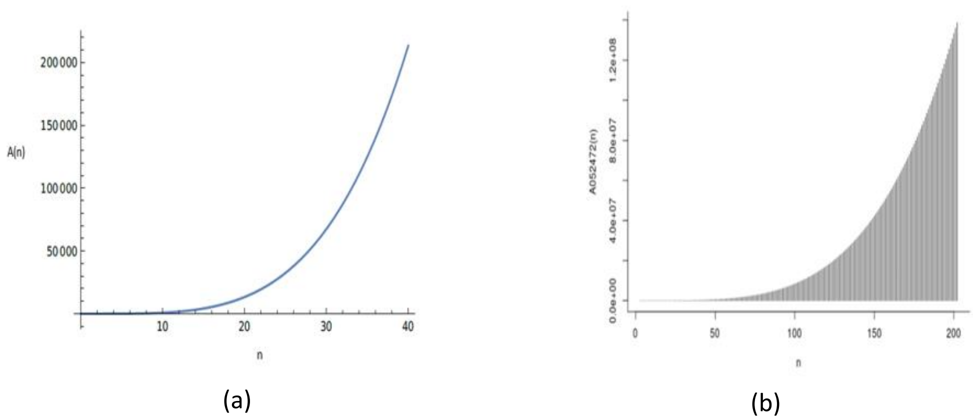


Figure 3. (a) and (b) show the decomposition of Weyl tensor in four components of Petrov type N in spacetime dimension, and it shows the gravitational fields locally obtained in a linear summation of those distinct components, and where the graphs are monotonically growing with respect to t.

The slope of the logarithmic plot also demonstrates that the Weyl scalar invariants transverse radiations in a long way as shown in Figure 4:

Additionally, the independent number of scalars that can be constructed from the Riemann tensor and the metric tensor in N dimensions is given by the OEIS code A050297 for $A050297$, $N = 3, ..., 40$ [33], as shown in the following equation:

$$b(N) = \frac{N^2\left(N^2-1\right)}{12}, \tag{84}$$

as illustrated in Figure 5.

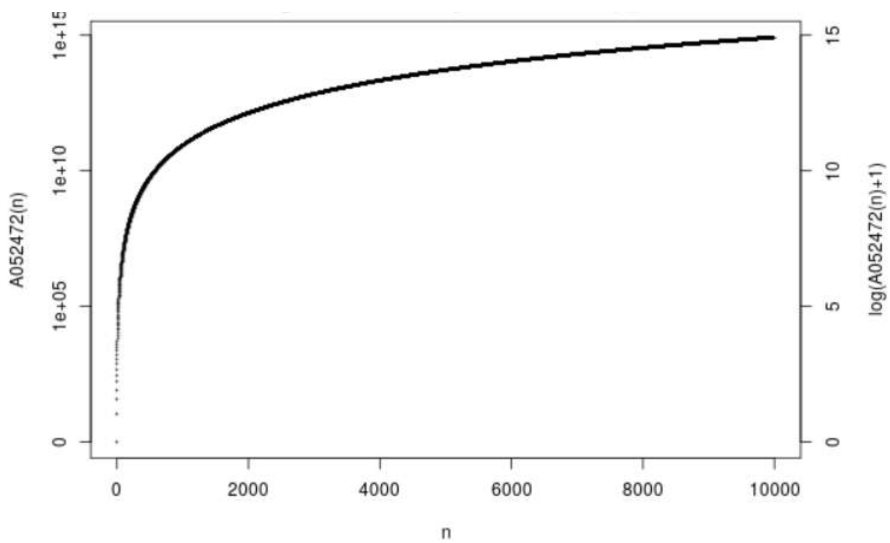
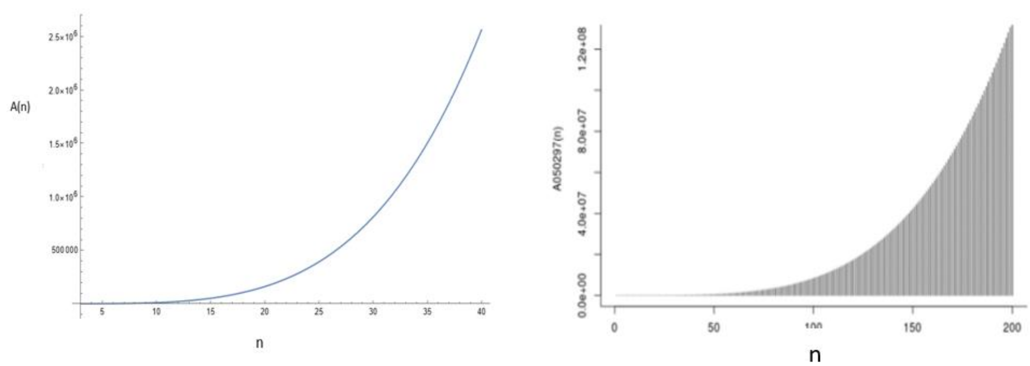


Figure 4. Weyl tensor: The transverse of radiations scalar invariants in the decomposition of Weyl tensor through linear summation of distinct components.



(a) (b)

Figure 5. Riemannian Curvature Tensor: The graphs shown in (a) and (b) exhibit monotonic growth with respect to t, similar to the behavior depicted in the graph of the Weyl tensor in Figure 5.

6. Weyl tensor in the Schwarzschild black hole

The contraction of the Weyl tensor, $W^{\delta\sigma\gamma\tau}W_{\delta\sigma\gamma\tau}$ is the scalar invariant principal for construction. The Weyl tensor vanishes everywhere and is conformally flat in the spacetime metric. However, Penrose's hypothesis states that some scalar invariants of the Weyl tensor, like $W^{\delta\sigma\gamma\tau}W_{\delta\sigma\gamma\tau}$ the functions, are monotone, which grows over time t. It can be identified as an arbitrary gravitational collapse of the object in the universe [40, 41]. Also, we understand the Penrose hypothesis better by conceptualizing the Friedman-Lemaitre-Robertson-Walker metric, which is based on the exact solution of the Einstein field equations of general relativity, and discovered that spacetime is conformally flat and the Weyl tensor vanishes, while the Ricci tensor persists [41]. Conversely, in the Schwarzschild black hole solution, the Ricci tensor vanishes while the Weyl tensor persists [40].

The Schwarzschild black hole metric is given by:

$$ds^2 = -\Big(1 + \frac{2GM}{c^2 r}\Big)c^2dt^2 + \Big(1 - \frac{2GM}{c^2 r}\Big)^{-1}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{85}$$

The sloping in logarithmic the Riemann curvature scalars as the gravitational radiations going far is represented in Figure 6: The Schwarzschild metric has the coordinates which correspond to the

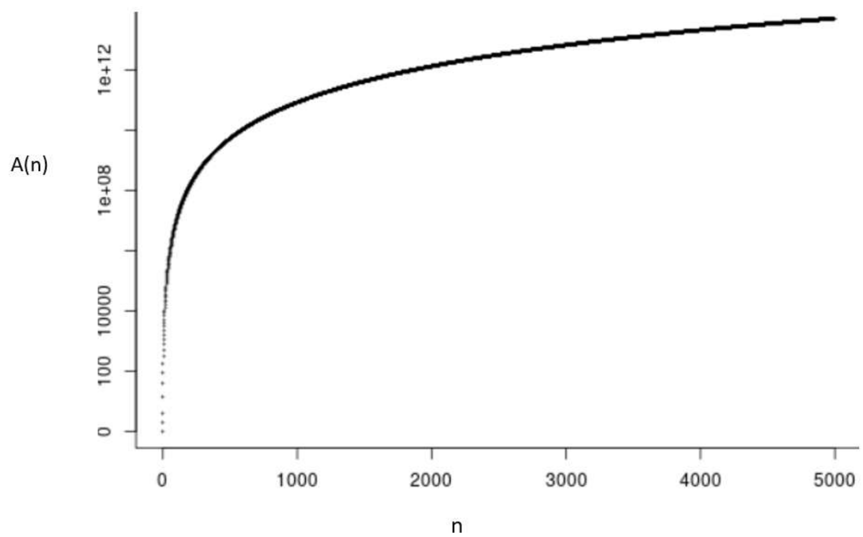


Figure 6. Transverse logarithmic graph of Riemann curvature scalar invariant.

Einstein metric $g_{\alpha\beta}(x)$ which is spherically symmetric; it has time-independent t, mass M, and Schwarzschild radius $r = \dfrac{2GM}{c^2}$. The displacement of Killing vector η associated with time coordinate t are spherically symmetric and $\xi = (t, r, \theta, \phi) = (1, 0, 0, 0)$ from equation (85) and the line element can be summarized as [42]:

$$d\Sigma^2 = r^2 d\theta^2 + r^2 sin^2\theta d\phi^2. \tag{86}$$

Regarding the changes of angles like θ and ϕ as in (85) and (86), the Schwarzschild geometry is spherical symmetric for ϕ-direction such that the metric is independent of ϕ-invariant under the rotations near the z-axis. From (86), the Killing vectors associated with the symmetry will be $\eta^\xi = (0, 0, 0, 1)$. The area of the Schwarzschild metric in two-dimensional spacetime at r and t is written as

$$A = 4\pi r^2, \tag{87}$$

which we get from (86) and also can be written on a flat surface as

$$dA = dl^2 dl^3 = \sqrt{g_{11} g_{22}} dx^1 dx^2. \tag{88}$$

The spatial metric of the homogeneous closed universe is written as

$$ds^2 = \frac{dr^2}{1 - (\dfrac{r}{a})^2} + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2, \tag{89}$$

then, the area of the two surfaces will be:

$$Area = \int dA = \int_0^{\pi} d\theta \int_0^{2\pi} d\phi r^2 sin\theta = 4\pi r^2. \tag{90}$$

The Schwarzschild black hole geometry is the unique spherically symmetric solution of the vacuum Einstein field equations. The exact form of the weak field in static metric (85) with the gravitational potential Φ is given by

$$\Phi = -\frac{GM}{r}, \tag{91}$$

where the weak metric with gravitational potential is given by [42]:

$$ds^2 = -\Big(1 + \frac{2\Phi(x^i)}{c^2}\Big)c^2dt^2 + \Big(1 - \frac{2\Phi(x^i)}{c^2}\Big)(dx^2 + dy^2 + dz^2). \tag{92}$$

The Newtonian theorem states that it is spherically symmetric outside of the falling object, which is governed by the Newtonian gravitational potential and is as given in (92), whether the objects changing the directions with time t or not. Therefore, outside of the falling objects, it could be there due to the conserved mass. Nevertheless, similarly, the general relativity theorem shows the distribution of mass is time-dependent and outside of the spherical symmetric geometry of the gravitational collapse is time-independent in Schwarzschild geometry.
The local coordinate of the inertial systems surrounding the geometry of a black hole at a certain point p in spacetime is given by

$$g_{\sigma\tau}(x) = \eta_{\sigma\tau}, \tag{93}$$

at given point of the curved spacetime, first derivative vanishes, and then, someone can say that the spacetime is flat as:

$$g^{'}_{\sigma\tau}(x^{'}_p) = \eta_{\sigma\tau}. \tag{94}$$

The Killing vector outside the event horizon surrounding Schwarzschild geometry generated through null vector fields are given by [42–44]:

$$\frac{\partial g^{'}_{\sigma\tau}}{\partial x^{'\gamma}}\bigg|_{x=x_p} = 0, \tag{95}$$

which can be also written as:

$$\frac{d^2x^{\gamma}}{dt^2}\bigg|_{p} = 0. \tag{96}$$

From equation (85), we see that the radius of Schwarzschild is $r = 2GM$, where G is the Newton gravitational force of the collapsing object and M is the total mass contained in the collapsing object. Therefore, the corresponding scalar invariant in the Schwarzschild metric (85) will be written as follows [40, 42, 43, 45];

$$R = 0, \quad R_{\sigma\tau}R^{\sigma\tau} = 0, \tag{97}$$

and the Riemann curvature and Weyl scalar invariants are written as:

$$R^{\delta\sigma\gamma\tau}R_{\delta\sigma\gamma\tau} = W^{\delta\sigma\gamma\tau}W_{\delta\sigma\gamma\tau} = \frac{48(GM)^2}{r^6}. \tag{98}$$

7. Weyl Tensor in the Reissner–Nordström (RN) Black Hole

The Reissner–Nordström black hole is a non-rotating black hole with a charged electromagnetic field, which is significant in the vacuum solutions of Einstein's field equations. It represents a charged black hole in an asymptotic Minkowski space with more than three dimensions [20, 21]. The Reissner–Nordström metric is the exact solution to the Einstein-Maxwell equations. These equations are spherically symmetric due to the absence of rotation but involve an electric charge.A distinctive feature of the Reissner–Nordström black hole, compared to the Schwarzschild black hole, is the presence of two horizons: the event horizon and the inner horizon. The inner horizon is believed to be unstable under small perturbations due to the mass inflation phenomenon. The Reissner–Nordström solution also has an ever-moving inward spacetime that, instead, ends on the Cauchy horizon, which can then travel into a region where singularity can be vividly seen but evaded [22, 23]. The correlations inside a black hole show perturbations and instability of the internal Cauchy horizon, which occurs, as the spacelike or null radiations where singularity emerges inside a charged Reissner–Nordström black hole [17, 46]. The RN solution is an exact solution to the Einstein and Maxwell field equations (Einstein–Maxwell) of general relativity in the presence of an electromagnetic field, exhibiting the same spacetime symmetries as the uncharged Schwarzschild solution [18, 19]. Recently, many researchers have focused on solutions for black holes constructed in the framework of power-law Maxwell theory [17–19, 46], where the power-law function can be written in the form of:

$$L = -\alpha(F_{\mu v}F^{\mu v})^k, \tag{99}$$

where α is a coupling constant and k is a power parameter. Therefore, the asymptotic behavior of the solutions depends on the power parameter k. However, we consider the solutions of black holes in the modified Maxwell field, including nonminimal coupling between the gravitational and electromagnetic fields [17], which shows that the coupling terms are used to modify the gravitation and electromagnetic structures of the charged black holes. A generalized electromagnetic theory with electrodynamics of Weyl correction, which involves coupling between Maxwell field and Weyl tensor, has been observed by [16, 17]. In this theory, the electromagnetic field of the Lagrangian density is modified to be:

$$L_{EM} = -\frac{1}{4}\Big(F_{\mu v}F^{\mu v} - 4\alpha W^{\mu v\rho\sigma}F_{\mu v}F_{\rho\sigma}\Big), \tag{100}$$

where $F_{\mu v}$ is the electromagnetic tensor, which is related to E_μ, an electromagnetic vector potential, in such a way that $F_{\mu v} = E_{\mu;v} - E_{v;\mu}$ and the coefficient α is a coupling constant with a squared length of dimensions. $W_{\mu v\rho\sigma}$ is the Weyl tensor, which is related to Riemann tensor $R_{\mu v\rho\sigma}$, the Ricci tensor $R_{\mu v}$, and the Ricci Scalar R, and can also be written as in equation (82):

$$W_{\mu v\rho\sigma} = R_{\mu v\rho\sigma} - \frac{2}{N-2}\Big[g_{\mu[\rho}R_{\sigma]v} - g_{v[\rho}R_{\sigma]\mu}\Big] + \frac{2}{(N-1)(N-2)}Rg_{\mu}[_{\rho}g_{\sigma}]_{v}, \tag{101}$$

where N refers to the dimensions, $g_{\mu v}$ is the metric, and the indices in brackets refer to the antisymmetric portion.

In order to study the RN black hole solutions, we use the Weyl tensor corrections for static and symmetrically charged black holes. The action of the gravity system, coupled with the Weyl tensor and electromagnetic field, is given by [16, 17, 47]:

$$S = \int d^4x\sqrt{-g}\Big[R - \frac{1}{4}F_{\mu v}F^{\mu v} + \alpha W^{\mu v\rho\sigma}F_{\mu v}F_{\rho\sigma}\Big]. \tag{102}$$

Through the adoption of the Schwarzschild coordinates, the line elements of RN in static and spherically symmetric black hole spacetime can be written as:

$$ds^2 = f(r)dt^2 - (f(r))^{-1}dr^2 - H(r)(d\theta^2 + sin^2\theta d\phi^2), \tag{103}$$

where $f(r)$ and $H(r)$ are the metric coefficients which represent the functions of the polar coordinates r. However, assuming the electromagnetic fields inherit the static spherical symmetries, the potential of electric field for four-vectors can be expressed in the form of:

$$E_\mu = (\phi(r), 0, 0, 0). \tag{104}$$

Through inserting equations (101), (103), and (104) into equation (102) and verifying the actions with respect to $f(r)$, $H(r)$, and E_μ, we can have three coupled equations of motion as follows:

$$3\Big(H^{'}(r)^2 - 2H(r)H^{''}(r)\Big) + 4\alpha\frac{d}{dr}\Big[\phi^{'}(r)H(r)\Big(2H(r)\phi^{''}(r) + H^{'}(r)\phi^{'}(r)\Big)\Big] = 0, \tag{105}$$

$$\begin{aligned}&3\Big[\phi^{'}(r)^2H(r)^2 + f(r)\Big(H^{'}(r)^2 - 2H(r)H^{''}(r)\Big) - 2H(r)\Big(H(r)f^{''}(r) + H^{'}(r)f^{'}(r)\Big)\Big]\\ &- 4\alpha\Big[2H(r)\phi^{''}(r)^2\Big(\phi^{'}(r)f(r)\Big)^{'} + 2H(r)H^{'}(r)\Big(f(r)\phi^{'}(r)^2\Big)^{'} - \phi^{'}(r)^2\Big(H^{'}(r)^2f(r) + 2f^{''}(r)H(r)^2\Big)\\ &+ 2f(r)H(r)\phi^{'}(r)\Big(H(r)\phi^{'''}(r) + H^{''}(r)\phi^{'}(r)\Big)\Big] = 0,\end{aligned} \tag{106}$$

$$\frac{d}{dr}\Bigg\{\phi^{'}(r)H(r) + \frac{4\phi^{'}(r)\alpha}{3H(r)}\Big[f(r)\Big(H^{'}(r)^2 - H(r)H^{''}(r)\Big) + H(r)\Big(H^{'}(r)f^{''}(r) - f^{'}(r)H^{'}(r)\Big) - 2H(r)\Big]\Bigg\} = 0. \tag{107}$$

Therefore, we must solve these three coupling equations above to obtain the solutions of black holes attached with Weyl tensor correction. In equation (105), we can obtain the solutions of the Reissner–Nordström black hole by using $\alpha \to 0$. Nevertheless, it will be complicated for nonzero Weyl coupling constant α to obtain the analytical solutions of black holes. Otherwise, we can decide for ourselves in which case the deviation of the coupling parameter is minimal, starting from zero. For this justification, we can say it is a weak Weyl correction. The terms containing parameter α on the left-hand side of the three coupling equations are regarded as perturbation, and using the theory of perturbation, we get:

$$H(r) = H_0(r) + \alpha H_1(r) + \Phi(\alpha^2), \quad f(r) = f_0(r) + \alpha f_1(r) + \Phi(\alpha^2), \quad \phi(r) = \phi_0(r) + \alpha\phi_1(r) + \Phi(\alpha^2). \tag{108}$$

Substituting equation (108) into equations (105)–(107), we obtain a series of perturbation equations. Apparently, the normal Reissner–Nordström black hole is a solution of the zeroth-order equations, with

$$H_0(r) = r^2, \qquad f_0(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2}, \qquad \Phi_0(r) = \frac{q}{r}. \tag{109}$$

By solving the first order differential equations and using (109), we obtain

$$H(r) = r^2 + \frac{4\alpha q^2}{9r^2}, \quad f(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2} - \frac{4\alpha q^2}{3r^4}\Big[1 - \frac{10m}{3r} + \frac{26q^2}{15r^2}\Big], \quad \phi(r) = \frac{q}{r} + \frac{\alpha q}{r^3}\Big[\frac{m}{r} - \frac{37q^2}{45r^2}\Big]. \tag{110}$$

Clearly, the static electric potential and metric coefficients depend on the coupling parameter α, which means that the Weyl tensor corrections affect the behaviors of electric fields' and charged black holes' behaviors. Notably, we find static electric potential ϕ, which depends on the parameter m, sometimes different from regular RN black hole in which the static electric potential $\phi(r)$ depends only on the charged particle q because of the Weyl tensor coupling between the gravitational and electromagnetic fields. The radius of the black hole horizon is located when $f(r) = 0$ from the above equation (110) of the charged black hole with Weyl tensor correction. Though the equation $f(r) = 0$ has more than two real distinct roots, we focus only on the roots close to the radius of RN black hole horizon using the weak Weyl correction and leaving the rest.

8. Weyl tensor in the Kerr and Kerr–Newman (KN) black holes

Kerr black hole is the black hole solution that has no electric charge but only mass and angular momentum. Nevertheless, the Kerr–Newman black hole is a rotating and charged black hole. However, Kerr and Kerr–Newman black holes are solutions to Einstein's field equation of general relativity. They describe the rotating black holes, but there is a difference between the two: the Kerr–Newman black hole has an electric charge, while the Kerr black hole does not have an electric charge. In addition to mass and charge, the Kerr–Newman black hole also possesses angular momentum, which arises from its rotation. We will use the rotating black hole with a charged electromagnetic field for the investigation to get the solutions of Kerr and Kerr–Newman black holes through coupling between the Maxwell field and Weyl tensor. The investigation will be of a rotating charged black hole with minimal Weyl corrections that have also been done by [16, 17]. The electromagnetic field coupling with the Weyl tensor was introduced in [17] as stated in equation (102), where $F_{\mu v} = E_{\mu;v} - E_{v;\mu}$, which means $F_{\mu v} = \partial_v E_\mu - \partial_\mu E_v$ is the electromagnetic tensor and E_μ is the vector potential written in the form of (104). The significance of charged rotating spherically symmetric black hole spacetime with minimal Weyl tensor corrections was obtained in [17]. The chosen convention signature coordinates is $(-,+,+,+)$ with geometry units $(G = c = 1)$. Considering the spacetime formula written in equation (102) is an electrovacuum solution, then we can apply the method of complex transformation coordinates of the Newman-Janis Algorithm in Boyer–Lindquist coordinate (t, r, θ, ϕ) to the spacetime metric of the charged rotating black hole spacetime with minimal Weyl correction [17]. We know that the coupling between the Maxwell field and Weyl tensor causes difficulties in getting solutions for field equations for a rotating black hole. From previous static black hole solutions, we need to construct a rotating black hole with minimal Weyl corrections (110) to study the properties of spacetime black holes. Even if the previous method of perturbation is different, the solution is true since equation (102) is the only nongravitational mass-energy present with an electromagnetic field and the spacetime described in equation (102) is electrovacuum and the approach of Newman-Janis is substantiated for electrovacuum [3, 39, 48]. However, we will neglect calculations of order terms $\Phi(\alpha^2)$ from the above section and keep consistency with the terms of higher order in consideration of the minimal Weyl corrections from the above section. We can introduce new variable ω defined by [17]:

$$\omega = t - \int \frac{dr}{f(r)}, \tag{111}$$

so that we can rewrite the metric (110) as:

$$ds^2 = f(r)d\omega^2 + 2d\omega dr - H(r)^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{112}$$

The inverse metric using null tetrad vectors will be written as:

$$g^{v\omega} = -l^v n^\omega - l^\omega n^v + b^v \tilde{b}^\omega + b^\omega \tilde{b}^v, \tag{113}$$

such that using the differential equations obtained from (110), the null tetrad vectors will be:

$$\begin{aligned} l^v &= \delta_1^v, \quad n^v = \delta_0^v - \frac{1}{2}\Big[1 - \frac{2m}{r} + \frac{q^2}{r^2} - \frac{4\alpha q^2}{3r^4}\Big(1 - \frac{10m}{3r} + \frac{26q^2}{15r^2}\Big)\Big]\delta_1^v, \\ b^v &= \frac{1}{\sqrt{2H(r)}}\Big(\delta_2^v + \frac{i}{\sin\theta}\delta_3^v\Big), \quad \tilde{b}^v = \frac{1}{\sqrt{2H(r)}}\Big(\delta_2^v - \frac{i}{\sin\theta}\delta_3^v\Big). \end{aligned} \tag{114}$$

We treat radius r as the complex variable, then;

$$\begin{aligned} l^v &= \delta_1^v, \quad n^v = \delta_0^v - \frac{1}{2}\Big[1 - m(\frac{1}{r} + \frac{1}{\tilde{r}}) + \frac{q^2}{r\tilde{r}} - \frac{4\alpha q^2}{3r^2\tilde{r}^2}\Big(1 - \frac{5m}{3}(\frac{1}{r} + \frac{1}{\tilde{r}}) + \frac{26q^2}{15r\tilde{r}}\Big)\Big]\delta_1^v, \\ b^v &= \frac{1}{\sqrt{2H(r\tilde{r})}}\Big(\delta_2^v + \frac{i}{\sin\theta}\delta_3^v\Big), \quad \tilde{b}^v = \frac{1}{\sqrt{2H(r\tilde{r})}}\Big(\delta_2^v - \frac{i}{\sin\theta}\delta_3^v\Big), \end{aligned} \tag{115}$$

where $\tilde{r}$ is the complex conjugates of r. Now we need to perform complex transformation coordinates [3, 17, 39, 48]:

$$v' = v - ia\cos\theta, \quad r' = r + ia\cos\theta, \quad \theta' = \theta, \quad \phi' = \phi, \tag{116}$$

then, we find transformed tetrad as follows:

$$\begin{aligned} l'^v &= \delta_1^v, \\ n'^v &= \delta_0^v - \frac{1}{2}\Big[1 - \frac{2mr' - q^2}{r'^2 + a^2\cos^2\theta} - \frac{4\alpha q^2}{3(r'^2 + a^2\cos^2\theta)^2}\Big(1 - \frac{50mr' - 26q^2}{25(r'^2 + a^2\cos^2\theta)}\Big)\Big]\delta_1^v, \\ b'^v &= \frac{1}{\sqrt{2H(r')}}\Big[ia\sin\theta(\delta_0^v - \delta_1^v) + \delta_2^v + \frac{i}{\sin\theta}\delta_3^v\Big], \\ \tilde{b}'^v &= \frac{1}{\sqrt{2H(r')}}\Big[-ia\sin\theta(\delta_0^v - \delta_1^v) + \delta_2^v - \frac{i}{\sin\theta}\delta_3^v\Big]. \end{aligned} \tag{117}$$

We can check the metric of rotating charged black hole using Weyl corrections with the help of new formed tetrad by:

$$g'^{v\omega} = -l'^v n'^\omega - l'^\omega n'^v + b'^v \tilde{b}'^\omega + b'^\omega \tilde{b}'^v, \tag{118}$$

and the transformed coordinates (v', r', θ', ϕ') the covariant derivatives of the metric components in equation (118) can be written as:

$$g'_{00} = \frac{f(r',\theta')}{\Delta(r',\theta')}, \quad g'_{01} = 1, \quad g'_{13} = -a\sin^2\theta',$$
$$g'_{22} = -\Delta_1(r',\theta'), \quad g'_{03} = \Big(1 - \frac{f(r',\theta')}{\Delta(r',\theta')}\Big)a\sin^2\theta', \tag{119}$$
$$g'_{33} = -\frac{\sin^2\theta'}{\Delta(r',\theta')}\Big[\Delta(r',\theta')\Delta_1(r',\theta') + a^2\sin^2\theta'(2\Delta(r',\theta') - f(r',\theta'))\Big],$$

where

$$\Delta(r',\theta') = r'^2 + a^2\cos^2\theta', \quad \Delta_1(r',\theta') = r'^2 + a^2\cos^2\theta' + \frac{4\alpha q^2}{9(r'^2 + a^2\cos^2\theta')},$$
$$f(r',\theta') = r'^2 + a^2\cos^2\theta' - 2mr' + q^2 - \frac{4\alpha q^2}{3(r'^2 + a^2\cos^2\theta')}\Big[1 - \frac{50mr' - 26q^2}{15(r'^2 + a^2\cos^2\theta')}\Big]. \tag{120}$$

Now we need to use transformations of coordinates (v', r', θ', ϕ') to remove the elements g'_{01} and g'_{13} [17]:

$$dv' = dt - \Omega(r',\theta')dr, \quad r' = r, \quad \theta' = \theta, \quad d\phi' = d\phi - G(r',\theta)dr, \tag{121}$$

where

$$\Omega(r',\theta') = \frac{g'_{33}g'_{01} - g'_{13}g'_{03}}{g'_{33}g'_{00} - g'^2_{03}} = \frac{\Delta(r',\theta')\Delta_1(r',\theta') + a^2\sin^2\theta'}{f(r',\theta')\Delta_1(r',\theta') + \Delta(r',\theta')a^2\sin^2\theta'},$$
$$G(r',\theta) = \frac{g'_{13}g'_{00} - g'_{03}g'_{01}}{g'_{33}g'_{00} - g'^2_{03}} = \frac{a\Delta(r',\theta')}{f(r',\theta')\Delta_1(r',\theta') + \Delta(r',\theta')a^2\sin^2\theta'}. \tag{122}$$

Therefore, the charged rotating black hole metric with Weyl correction tensor is:

$$ds^2 = \frac{f(r,\theta)}{\Delta(r,\theta)}dt^2 + 2\Big(1 - \frac{f(r,\theta)}{\Delta(r,\theta)}\Big)a\sin^2\theta dt d\phi - \frac{\Delta(r,\theta)\Delta_1(r,\theta)dr^2}{f(r,\theta)\Delta_1(r,\theta) + a^2\sin^2\theta\Delta(r,\theta)}$$
$$-\Delta_1(r,\theta)d\theta^2 - \frac{\sin^2\theta}{\Delta(r,\theta)}\Big[\Delta(r,\theta)\Delta_1(r,\theta) + a^2\sin^2\theta(2\Delta(r,\theta) - f(r,\theta))\Big]d\phi^2. \tag{123}$$

where

$$\Delta(r,\theta) = r^2 + a^2\cos^2\theta, \quad \Delta_1(r,\theta) = r^2 + a^2\cos^2\theta + \frac{4\alpha q^2}{9(r^2 + a^2\cos^2\theta)},$$
$$f(r,\theta) = r^2 + a^2\cos^2\theta - 2mr + q^2 - \frac{4\alpha q^2}{3(r^2 + a^2\cos^2\theta)}\Big[1 - \frac{50mr - 26q^2}{15(r^2 + a^2\cos^2\theta)}\Big]. \tag{124}$$

Notice that, in the whole process, m represents the mass of black holes, $a = \frac{J}{m}$ is the rotational parameter, J is the angular momentum of the black hole, q is the electric charge, and α is the coupling constant parameter which has the squared-length of dimension. Therefore, the above metric

(123) is reduced to Kerr–Newman black hole metric in Boyer–Lindquist coordinates as $\alpha = 0$ and the same equation (123) to be reduced to Kerr Black hole solutions, we should have the electric charge $q = 0$ and the parameter $\alpha = 0$. When a parameter of the charged rotating black hole a vanishes, we can have solutions of the static and spherically symmetric black hole with Weyl correction as in (110). When the terms $a = q = 0$, the metric (123) reduces to Schwarzschild black hole metric as in (85).

9. Implications of study findings for current astrophysical observations

The initial theme developed for advanced astrophysics connects astronomical observers with the underlying physical phenomena that initiate our cosmos. Effective planning is essential for the tools and mechanisms used to observe astrophysical phenomena. Laboratory astrophysics includes both experimental and theoretical studies to produce tangible observed processes in astrophysical research. Six areas of physics are relevant to astrophysics and astronomy, with observational science focusing on detecting protons generated by atomic, molecular, and condensed matter physics [49–51]. Additionally, chemistry plays a role in condensed matter and molecular physics. The growth of space-based in situ observations of the solar system bodies necessitates advancements in astrophysical activities. These advancements should be conducted through sophisticated laboratory experiments to detect gravitational waves in cosmic objects [49].

Advanced laboratory work requires both experimental and theoretical contributions to understand our universe comprehensively. Studying and conducting experiments in atomic, molecular, condensed matter, plasma, nuclear, and particle physics are key components. These aspects are crucial to observe during experiments and theoretical investigations in advanced laboratories. Understanding the effects of strong magnetic fields on atomic structure necessitates analyzing spectra from the vicinity of magnetized compact objects like neutron stars and black holes. Agencies concerned with astrophysics should prioritize laboratory astrophysics, recognizing its potential contributions to national research in plasma, magnetized objects, and high-energy plasma. The experiments and theoretical insights gained from laboratory astrophysics will serve as a foundation for generations to come. It extends beyond mere detector and instrumental development, playing a pivotal role in maximizing the scientific returns from astronomical observations.

10. Conclusion

The Einstein field equations (1) in the distributions of matter and energy with the radial null vector field, μ^σ in the manifold of a collapsing object, can be written in the form of [39, 52]:

$$R_{\sigma\tau}\mu^\sigma\mu^\tau \geq 0, \tag{125}$$

where $R_{\sigma\tau}$ is the Ricci tensor. By applying the field equations (1), equation (125) implies the radial null energy conditions for stress-energy tensor of matter, denoted as $T_{\sigma\tau}$, such that:

$$T_{\sigma\tau}\mu^\sigma\mu^\tau \geq 0. \tag{126}$$

From equation (97), we observe that the Ricci scalar equals zero in the Schwarzschild metric, while the Riemann scalar curvature and Weyl scalar invariants are as expressed in equation (98). This implies the presence of the Weyl tensor in the Schwarzschild black hole. Suppose $T_{\sigma\tau} = 0$ due to the zero Ricci scalar. However, this does not imply the absence of energy distribution; it could be less, and this condition holds in vacuum situations, akin to vacuum Maxwell field equations [53]. In a vacuum, $T_{\sigma\tau}$ vanishes from the equation:

$$R_{\sigma\tau} = -8\pi G\Big(T_{\sigma\tau} - \frac{1}{2}g_{\sigma\tau}T^{\lambda}_{\lambda}\Big). \tag{127}$$

Therefore, when $T_{\sigma\tau} = 0$, from (127), we conclude that the Einstein field equations are empty space as $R_{\sigma\tau} = 0$ and $T_{\sigma\tau}$ vanish. Additionally, the equations (23, 24, 25, 26, 27) and (52, 53, 54, 55) are the identities that show the stress- energy-momentum tensor equal to zero due to the source of Ricci scalar-tensor in the matter distribution. Furthermore, the divergence occurs near the event horizon in a stagnant Schwarzschild geometry with a proper acceleration [41, 53]. Indeed, in scenario, spacetime resembles the flat and static Schwarzschild black hole [7, 16, 52, 54, 55]. Therefore, in the stated conditions, the spacetime black hole might be flat, stagnant, and uncharged. Nevertheless, when we refer to Figure 7, comparisons of the Weyl scalar invariant and the Riemann curvature tensor show that the functions grow monotonically in the positive direction with respect to time t. Based

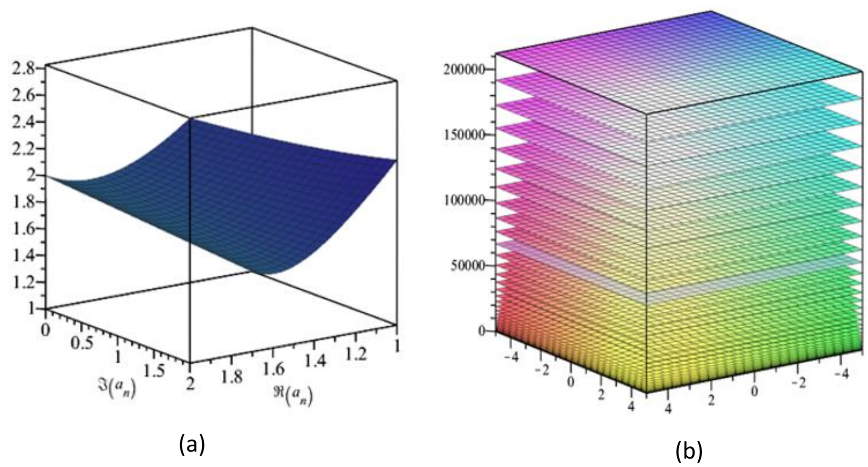


Figure 7. (a) and (b) show the comparison of the behaviors of the Riemann curvature and Weyl tensors, which almost behave the same.

on these interpretations, we can infer that the collapsing model could represent an uncharged and nonrotating body in spacetime, assuming it is far from external disturbances. This collapse may lead to the formation of a static Schwarzschild black hole, accompanied by the emergence of the Weyl tensor. Furthermore, we observe the presence of the Weyl tensor in the formation of RN, Kerr, and KN black holes, indicating a coupling between the Einstein–Maxwell equations and the Weyl tensor through minimal Weyl corrections

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References

[1] C. Ranzan, "The Nature of Gravitational Collapse," *American Journal of Astronomy and Astrophysics* **4**(2016) 15–33.

[2] C. Barcelo, R. Carballo-Rubio, L. J. Garay, "Where does the physics of extreme gravitational collapse reside?," *Universe* **2** (2016) 7 .

[3] S. Hawking, "Gravitationally collapsed objects of very low mass," *Monthly Notices of the Royal Astronomical Society* **152** (1971) 75–78.

[4] D. Malafarina, "Classical collapse to black holes and quantum bounces: A review," *Universe* **3** (2017) 48.

[5] J. Ovalle, R. Casadio, E. Contreras, A. Sotomayor, "Hairy black holes by gravitational decoupling," *Physics of the Dark Universe* **31** (2021) 100744.

[6] W. G. Boskoff, S. Capozziello, "General Relativity and Relativistic Cosmology. In A Mathematical Journey to Relativity: Deriving Special and General Relativity with Basic Mathematics," *Cham: Springer International Publishing* (2020) 257–366.

[7] B. Carter, "Republication of: Black hole equilibrium states part ii. general theory of stationary black hole states," *General Relativity and Gravitation* **42** (2010) 653–744.

[8] M. D. Mkenyeleye, R. Goswami, S. D. Maharaj, "Gravitational collapse of generalized Vaidya spacetime," *Physical Review D* **90** (2014) 064034.

[9] N. J. Gogoi, G. K. Mahanta, P. Phukon, "Geodesics in geometrothermodynamics (GTD) type II geometry of 4D asymptotically anti-de-Sitter black holes," *The European Physical Journal Plus* **138** (2023) 345.

[10] M. Arminjon, F. Reifler, "Dirac equation: Representation independence and tensor transformation," *Brazilian Journal of Physics* **38** (2008) 248-258.

[11] G. E. Volovik, "Black hole and Hawking radiation by type-II Weyl fermions," *JETP letters* **104** (2016) 645-648.

[12] H. Weyl, "Gravitation and the electron," *roceedings of the National Academy of Sciences* **15** (1929) 323-334.

[13] T. Friedrich, K. D. Kirchberg, "Eigenvalue estimates for the Dirac operator depending on the Weyl tensor," *Journal of Geometry and Physics* **41** (2002) 196-207.

[14] L. G. Gomez, "Constraining light fermionic dark matter with binary pulsars," *Physics of the Dark Universe* **26** (2019) 100343.

[15] S. A. Hayward, " Quasilocal gravitational energy," *Physical Review D* **49** (1994) 831.

[16] G. Abbas, A. Mahmood, M. Zubair, "Strong deflection gravitational lensing for photon coupled to Weyl tensor in a charged Kiselev black hole," *Physics of the Dark Universe* **31** (2021) 100750.

[17] S. Chen, J. Jing. "Rotating charged black hole with Weyl corrections," *Physical Review D* **89** (2014) 104014.

[18] G. Dotti, J. M. F. Tío. "Black hole nonmodal linear stability: Even perturbations in the Reissner-Nordström case," *Physical Review D* **101** (2020) 024034.

[19] S. G. Ghosh. "A nonsingular rotating black hole," *The European Physical Journal C* **75** (2015) 532.

[20] X. Wang, R. Li, J. Wang. "Islands and Page curves of Reissner-Nordström black holes," *Journal of High Energy Physics* **4** (2021) 1–20.

[21] P. A. Cano, S. Chimento, R. Linares, T. Ortin, P. F. Ramirez. "α' corrections of Reissner-Nordstrom black holes," *Journal of High Energy Physics* **2** (2020) 1–31.

[22] C. Corda, F. Feleppa, F. Tamburini. "On the quantization of the extremal Reissner-Nordstrom black hole," *Europhysics Letters* **132** (2020) 30001.

[23] J. P. Lemos, V. T. Zanchin. "Regular black holes: Electrically charged solutions, Reissner-Nordström outside a de Sitter core," *Physical Review D* **83** (2011) 124005.

[24] T. Rapcsák, "Sectional curvatures in nonlinear optimization," *Journal of Global Optimization* **40** (2008) 375–388.

[25] E. Fairchild, F. Owen, B. B. Healy, "Sectional curvature in Riemannian manifolds," *The Mathematica Journal* **22** (2020).

[26] Z. Hu, H. Li, U. Simon, "Schouten curvature functions on locally conformally flat Riemannian manifolds," *Journal of Geometry* **88** (2008) 75–100.

[27] V. R. Krym, "The Schouten Curvature Tensor and the Jacobi Equation in Sub-Riemannian Geometry," *Journal of Mathematical Sciences* **255** (2021) 184–192.

[28] M. Pieterzack, R. Pina, "Prescribed diagonal Schouten tensor in locally conformally flat manifolds," *Journal of Geometry* **104** (2013) 341–355.

[29] M. Glogowska, "Semi-Riemannian manifolds whose Weyl tensor is a Kulkarni-Nomizu square," *Publications de l'Institut Mathematique* **72** (2002) 95–106.

[30] H. Baltazar, "Besse conjecture for compact manifolds with pinched curvature," *Archiv der Mathematik* **115** (2020) 229–239.

[31] G. Catino, "Generalized quasi-Einstein manifolds with harmonic Weyl tensor," *Mathematische Zeitschrift* **271** (2012) 751–756.

[32] C. A. Mantica, Y. J. Suh, "Pseudo-Z symmetric space-times with divergence-free Weyl tensor and pp-waves," *International Journal of Geometric Methods in Modern Physics* **13** (2016) 1650015.

[33] A. Coley, D. McNutt, "Identification of black hole horizons using scalar curvature invariants," *Classical and Quantum Gravity* **35** (2017) 025013.

[34] S. M. Kuzenko, M. Ponds, "Conformal geometry and (super) conformal higher-spin gauge theories," *Journal of High Energy Physics* **5** (2019) 1–59.

[35] R. J. Slagter, S. Pan, "A new fate of a warped 5D FLRW model with a U(1) scalar gauge field," Foundations of Physics **46** (2016) 1075–1089.

[36] J. Z. Yang, S. Shahidi, T. Harko, "Black hole solutions in the quadratic Weyl conformal geometric theory of gravity," *The European Physical Journal C* **82** (2022) 1171.

[37] R. R. Lompay, A. N. Petrov, "Covariant differential identities and conservation laws in metric-torsion theories of gravitation. I. General consideration," *Journal of Mathematical Physics* **54** (2013) 062504.

[38] J. T. Wheeler, "Weyl geometry," *General Relativity and Gravitation* **50** (2018) 1–36.

[39] P. S. Negi, "Exact solutions of Einstein's field equations," International Journal of Theoretical Physics **45** (2006) 1684–1702.

[40] N. Li, X. L. Li, S. P. Song, "An exploration of the black hole entropy via the Weyl tensor," *The European Physical Journal C* **76** (2016) 1-6.

[41] B. Holdom, J. Ren, "Not quite a black hole," *Physical Review D* **95** (2017) 084034.

[42] J. B. Hartle, "Gravity. An Introduction To Einstein's General relativity *Pearson Education Ltd., Einbrugh Gate* **71** (2003).

[43] J. Podolsky, R. Svarc, V. Pravda, A. Pravdova, "Black holes and other exact spherical solutions in Quadratic Gravity," *Physical Review D* **101** (2020) 024027.

[44] R. Kumar, S. K. Srivastava, "Expansion-free self-gravitating dust dissipative fluids," *General Relativity and Gravitation* **50** (2018) 1–16.

[45] W. C. dos Santos, "Notes on the Weyl tensor, decomposition of Riemann tensor, Ruse-Lanczos identity and duality of the curvature tensor," Zenodo (2020).

[46] E. Poisson, I. Vlasov. "Geometry and dynamics of a tidally deformed black hole," Physical Review D **81** (2010) 024029.

[47] S. Chen, J. Jing. "Strong gravitational lensing for the photons coupled to Weyl tensor in a Schwarzschild black hole spacetime," *Journal of Cosmology and Astroparticle Physics* **10** (2015) 002.

[48] T. E. Kiess. "Exact solutions to Einstein's field equations for perfect spherically symmetric static fluids," *Classical and Quantum Gravity* **26** (2009) 025011.

[49] D. W. Savin, N. S. Brickhouse, J. J. Cowan, R. P. Drake, S. R. Federman et al. "The impact of recent advances in laboratory astrophysics on our understanding of the cosmos," *Reports on Progress in Physics* **75** (2012) 036901.

[50] R. Abbott, T. D. Abbott, S. Abraham, F. Acernese, K. Ackley et al. "Observation of gravitational waves from two neutron star–black hole coalescences," The Astrophysical journal letters **915** (2021) L5.

[51] M. Fasano, K. W. Wong, A. Maselli, E. Berti, V. Ferrari, B. S. Sathyaprakash. "Distinguishing double neutron star from neutron star-black hole binary populations with gravitational wave observations," *Physical Review D* **102**, (2020) 023025.

[52] H. Kawai, Y. Yokokura, "Phenomenological description of the interior of the Schwarzschild black hole," *International Journal of Modern Physics A* **30** (2015) 1550091.

[53] P. Huang, F. F. Yuan, "Uniformly expanding vacuum: A possible interpretation of the dark energy," *Physics of the Dark Universe* **12** (2016) 45–49.

[54] D. L. Meier, "The Non-Thermal Exhaust System I. Rotating Magnetospheres that Drive the Turbo Exhaust. In Black Hole Astrophysics: The Engine Paradigm," *Berlin, Heidelberg: Springer Berlin Heidelberg* (2012) 591–654.

[55] C. J. Gao, "Arbitrary Dimensional Schwarzschild-FRW Black Holes," *Classical and Quantum Gravity* **21** (2004) 4805.