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Abstract

Let G be a group in which every non-subnormal subgroup has finite rank. This paper considers the question as to which extra conditions on such a group G ensure that G has all subgroups subnormal. For example, if G is torsion-free and locally soluble-by-finite then either G has finite 0-rank or G is nilpotent. Several results are obtained on soluble (respectively, locally soluble-by-finite) groups satisfying the stated hypothesis on subgroups.

Key Words: Subnormal subgroups; locally soluble-by-finite groups; finite Mal'cev rank.

1. Introduction

Let G be a group in which every non-subnormal subgroup has finite rank. Throughout this paper the term "finite rank" means "finite Prüfer (or Mal'cev, or special) rank": a group X has finite rank r if every finitely generated subgroup of X is r-generated. It was shown in [5] that if G is soluble and of infinite rank then G is a Baer group, that is, every finitely generated subgroup of G is subnormal, and in [6] it was established that a locally soluble-by-finite group with this restriction on non-subnormal subgroups is soluble (and hence a Baer group). The aim of this article is to present some results on groups in which all non-subnormal subgroups have finiteness of rank of a different kind. We need the following definitions. Let G be a group. (a) G has finite torsion-free rank, or *finite* 0*-rank*, denoted $r_0(G)$, if G has a finite subnormal series of subgroups the factors of which are either infinite cyclic or periodic. (b) For a given prime p, G has *finite section p-rank* if every elementary abelian p-section of G is finite, and *finite section rank* if every abelian section has both finite p-rank for every prime p and finite 0-rank. (c) G has *finite section total rank* if, for each abelian section X of G, $r_0(X) + \Sigma r_p(X)$ is finite, where the sum runs over all primes p (see [9;6.2]). (d) G is *minimax* if it has a finite subnormal series the factors of which satisfy either max or min. Our main result is the following, which is the "0-rank version" of Theorem 3 of [6].

Theorem 1.1 *Let G be a torsion-free locally soluble-by-finite group in which every subgroup of infinite 0-rank is subnormal. If G has infinite 0-rank then G is nilpotent.*

There is a similar result for the p-rank case; however, in view of the fact that there are non-nilpotent p-groups with all subgroups subnormal [2] the conclusion is necessarily somewhat weaker.We also remark that the hypothesis on periodic subgroups that appears in the following theorem cannot be omitted: an example is provided in [6] of a (soluble) group G of infinite rank in which every non-subnormal subgroup has finite rank, the torsion subgroup of G has finite rank, but not every subgroup of G is subnormal.

Theorem 1.2 *Let p be a prime and let G be a locally soluble-by-finite group in which every non-subnormal subgroup has finite section p-rank. If G contains a periodic subgroup of infinite section p-rank, then G is soluble and every subgroup of G is subnormal.*

We have also obtained the following results.

Theorem 1.3 *Let G be a soluble group in which every non-subnormal subgroup has finite 0-rank. If G has infinite 0-rank but its maximal normal torsion subgroup $P(G)$ has finite section rank then G is a Baer group.*

Theorem 1.4 *Let p be a prime and let G be a soluble group in which every non-subnormal subgroup has finite section p-rank. Suppose that G has infinite section p-rank but all periodic subgroups have finite section p-rank. Then $G/O_{p'}(G)$ is nilpotent.*

Theorem 1.5 *Let G be a soluble group in which every non-subnormal subgroup has finite section rank. If G has infinite section rank then G is a Baer group.*

Theorem 1.6 *Let G be a locally soluble-by-finite group in which every non-subnormal subgroup has finite section total rank. If G contains a periodic subgroup of infinite section total rank then every subgroup of G is subnormal.*

Theorem 1.7 *Let G be a soluble group in which every non-subnormal subgroup has finite section total rank. If G has infinite section total rank but all periodic subgroups of G have finite section total rank then G is nilpotent.*

Theorem 1.8 *Let G be a soluble group and suppose that every non-minimax subgroup of G is subnormal. If G is not minimax then every subgroup of G is subnormal.*

2. The proof of Theorem 1.1

As might be expected, the proof here uses some ideas from [6], though there are a few significant differences. We shall frequently use the well-known theorem of Mal'cev [10; Theorem 6.36] that if G is a locally nilpotent group in which all abelian subgroups have finite 0-rank then G modulo its torsion subgroup is nilpotent and of finite rank - thus, for a torsion-free locally nilpotent group G, the properties *finite rank* and *finite* 0*-rank* are equivalent and imply nilpotency. Suppose next that G is a group with all non-R subgroups subnormal, where R is any subgroup-closed class of groups, and let H be a non-R subgroup of G. Every subgroup of G that contains H is subnormal in G, and so there is a finite subnormal series from H to G each factor of which has all subgroups subnormal. Each such factor is soluble, by the theorem of Möhres [8], and it follows that some term of the derived series of G lies in H. This observation will be used quite often and without further reference. We now present a result that will reduce the proof of Theorem 1 to the establishing of the solubility of our group G. The maximal normal torsion subgroup of G is here denoted $P(G)$.

Proposition 2.1 *Let G be a soluble group in which every non-subnormal subgroup has finite* 0*-rank, and suppose that G has infinite* 0*-rank. Then $G/P(G)$ is torsion-free nilpotent.*

The proof of this proposition requires the following result, which will be used again later on.

Lemma 2.2 *Let G be a hyperabelian group, T the maximal normal torsion subgroup of G, and suppose that every abelian subgroup of G/T has finite* 0*-rank. Then G has finite* 0*-rank, and G/T is soluble.*

Proof. Assume the result false, and suppose first that G/T is soluble. By considering an abelian normal series of G/T we see that there is a normal subgroup H/T of G/T such that H has finite 0-rank, while L/H is torsion-free abelian and of infinite 0-rank for some subgroup L of G. By [10;Lemma 9.34], H/T has a finite characteristic ascending series the factors of which are abelian and either finite or torsion-free (of finite rank). Let K/T denote the penultimate term of this series. If H/K is finite then it easy to see that there is a torsion-free abelian subgroup U/K of L/K that has infinite 0-rank. Now suppose that H/K is torsion-free of finite rank. If A/K is an abelian subgroup of L/K that has finite 0-rank then AH/H is of finite rank and so A/K has finite rank. But if every abelian subgroup of L/K has finite rank then L/K has finite rank [4], a contradiction. It follows (in either case) that L/K has an abelian subgroup of infinite 0-rank and hence a torsion-free such subgroup M/K, say. Repeating this argument as often as necessary we arrive at an abelian subgroup of G/T that has infinite 0-rank, a contradiction that establishes the result in the case where G/T is soluble. In the general case, let N/T denote the locally nilpotent radical of G/T, and note that N/T is torsion-free nilpotent of finite rank. If C denotes the centralizer in G of N/T then G/C is soluble [12]. But $C \leq N$ [10;Lemma 2.17], and we have the contradiction that G/T is soluble. □

Proof of Proposition 2.1 We may assume that $P(G) = 1$. Let B denote the Baer radical of G; it suffices to prove that $B = G$, since B is locally nilpotent and torsion-free and so Theorem 3 of [6] applies to give B nilpotent. By Lemma 2.2, G has an abelian subgroup of infinite 0-rank, and since this is subnormal we see that B has infinite 0-rank. Since B is nilpotent $B < g >$ is soluble for all $g \in G$, so that $B < g >$ has every subgroup of infinite rank subnormal and is therefore a Baer group, by the main result of [5]. But $B < g >$ is subnormal in G (since it has infinite 0-rank), and we deduce that $< g >$ is subnormal in G, giving $g \in B$ and hence $G = B$, as required. □

Another general structure result that we shall need is the following.

Lemma 2.3 *Let G be a locally (soluble-by-finite) group with finite* 0*-rank, and let T denote the torsion radical of G. Then G/T has a normal subgroup L/T of finite index* *such that L/T has a finite G-invariant series the factors of which are torsion-free abelian (and of finite rank).*

Proof. We may assume that G is not periodic and that $T = 1$, so that every normal subgroup of G has trivial torsion radical. Now G has a subnormal infinite cyclic subgroup $< x >$, and the normal closure K of $< x >$ in G is locally nilpotent and torsion-free of finite rank, so it is nilpotent of class c, say, and K clearly has a G-invariant series of the required kind. Let U/K denote the torsion radical of G/K; by induction on the 0-rank of G we may assume that M/U has a G-invariant series with torsion-free abelian factors, for some normal subgroup M of finite index in G. Let J denote an arbitrary upper central factor of K, and let C be the centralizer of J in U; then U/C embeds in $GL(r, Q)$ for some integer r and is therefore finite [13; Theorem 9.33], and we see that U has a G-invariant subgroup V of finite index such that $K \leq V$ and V centralizes every upper central factor of K. Clearly then $F/Z_c(F)$ is finite for every finitely generated subgroup F of V, so that $\gamma_{c+1}F$ is also finite for all such F [10; Corollary 2 to Theorem 4.21], and $\gamma_{c+1}V$ is locally finite and therefore trivial. It follows that V too has a G-invariant series of the required type, and we need only show that G/V has a normal subgroup L/V of finite index that has a G-invariant series with torsion-free abelian factors. Since M/U has such a series we may choose A/U normal in G/U with A/U torsion-free abelian (and non-trivial). If D/V is the centralizer of U/V in A/V then we have A/D finite, D/V normal in G/V and D/V nilpotent. It is easy to see that D^nV/V is torsion-free abelian for some positive integer n, and a further induction (on $r_0(G/V)$) completes the proof. □

The final part of of the proof of Theorem 2 of [6] deals with the case where G is (countable and) locally polycyclic – it is shown that if G has infinite rank and every subgroup of infinite rank is subnormal then G is soluble, and the same argument deals with the locally polycylic case of our theorem, since what is used is the fact that the torsion-free ranks of finitely generated subgroups of G are unbounded. Thus our aim is to reduce to the locally polycyclic case. One important step in this reduction is provided by the following result.

Lemma 2.4 *Let G be an insoluble group with all non-subnormal subgroups of finite rank, and suppose that G is the ascending union of finitely generated soluble minimax subgroups $F_1 \leq F_2 \leq$ ...... Suppose also that every periodic subgroup of G has finite section rank, every proper image of G is soluble, periodic and locally nilpotent, and the intersection* *of all nontrivial normal subgroups of G is trivial. If G has infinite 0-rank, then F_n is nilpotent-by-finite for each positive integer n.*

Proof. Firstly we note that G is residually periodic and so every F_n is residually finite. In particular, F_n contains no nontrivial quasicyclic subgroups. Let L_n denote the Fitting radical of F_n for each n and let L be the subgroup generated by the L_n. Suppose that $r_0(L) \leq k$ for some integer k and that each L_n is torsion-free. By the well-known theorem of Zassenhaus [10; Theorem 2.25], soluble subgroups of $GL(r, Q)$ have derived length bounded in terms of r only, so some bounded term of the derived series of F_n centralizes every upper central factor of L_n and hence lies in L_n, giving the contradiction that the derived lengths of the F_n are bounded. Thus, still under the assumption that $r_0(L)$ is finite, we see that L must contain nontrivial elements of finite order and hence an element x of prime order p, say. Let $X =< x >^G$. If H is a nontrivial G-invariant subgroup of X then X/H is locally nilpotent and hence a p-group, so X is residually a p-group, and every periodic subgroup of X is therefore a p-group. Put $X_n = F_n \cap X, V_n = Fitt(X_n)$, for each n. Then V_n is normal in F_n and is therefore contained in L_n, while $L_n \cap X_n$ is a normal nilpotent subgroup of X_n, and so $L_n \cap X_n = V_n$ for each n. Set $V =< V_n | n \in N >$; then V is contained in L and so $r_0(V)$ is finite. If T_n denotes the torsion radical of V_n and T is the subgroup generated by all the T_n then T is a p-subgroup of finite section rank and is therefore Chernikov. Let P be the divisible radical of T and let P_1 be the subgroup of P consisting of all elements of order at most p. There is a non-trivial normal subgroup U of G such that $P_1 \cap U = 1$; then $X \cap U \cap P = 1$ and so $X \cap U \cap T$ is finite. Again, $X \cap U \cap T \cap W = 1$ for some nontrivial normal subgroup W of G, and $Y := X \cap U \cap W$ is nontrivial, while $Y \cap T = 1$. Let $Y_n = F_n \cap Y, R_n = Fitt(Y_n)$. Clearly $R_n = L_n \cap Y$, and so R_n is torsion-free (and of bounded 0-rank) for each n. Arguing as before, we have that Y is soluble; but G/Y is soluble, and we have a contradiction. Thus $r_0(L)$ is infinite, and it follows that some term K of the derived series of G is contained in L. Now G/K is periodic, and so for each $g \in G$ there is an integer $t = t(g)$ such that $g^t \in L$. If $g \in F_1$ then $g^t \in L \cap F_1$, that is, $g^t \in (L_1 L_2 ... L_k) \cap F_1$ for some positive integer k. But $(L_1 L_2 ... L_k) \cap F_1 = (L_1 L_2 ... L_k) \cap F_{k-1} \cap F_1 = (L_1 L_2 ... L_{k-1})(L_k \cap F_{k-1}) \cap F_1 = (L_1 L_2 ... L_{k-1}) \cap F_1 = ... = L_1$. Thus F_1/L_1 is periodic and hence finite. Now set $S =< L_n | n \geq 2 >$, a normal subgroup of L. Then $r_0(S)$ is infinite, and we can repeat the previous argument and obtain that F_2/L_2 is finite. Using induction on n we obtain that each F_n/L_n is finite, and the result follows. □

We are now ready to establish the solubility of G in the case where G is locally soluble.

Proposition 2.5 *Let G be a torsion-free locally soluble group in which every subgroup of infinite 0-rank is subnormal, and suppose that G has infinite 0-rank. Then G is soluble.*

Proof. Let us assume for a contradiction that G is not soluble. Since G contains finitely generated subgroups of arbitrarily high derived length and 0-rank we may assume that G is countable. Let H be a subgroup of G that is an ascending union of G-invariant subgroups with successive factors abelian and which is such that G/H has no nontrivial normal abelian normal subgroups – note that such an H exists. If $r_0(H)$ is infinite then, by Lemma 2.2, H contains an abelian subgroup U of infinite 0-rank, and this implies that G is soluble, a contradiction that shows that $r_0(H)$ is finite. Again by Lemma 2.2, H is soluble, so that $Q := G/H$ is insoluble and has infinite 0-rank. Let P/H be an arbitrary periodic subgroup of G/H; then $r_0(P)$ is finite, and Lemma 2.3 implies that P has finite rank. Thus every periodic subgroup of Q has finite rank. Also by Lemma 2.3, every normal subgroup of Q that has finite 0-rank is soluble and therefore trivial. Now Q is locally soluble and is therefore not simple, while for every nontrivial normal subgroup B of Q, Q/B is soluble and locally nilpotent. The intersection of all such subgroups B must be trivial. By our earlier remarks, Q can have no soluble subgroups of infinite 0-rank; in particular every finitely generated subgroup of G is of finite 0-rank. Let L denote the intersection of all nontrivial normal subgroups N of Q such that Q/N is torsion-free. Each factor Q/N is locally nilpotent and hence nilpotent: by the remarks at the beginning of this section if Q/N has finite rank, or by Theorem 3 of [6] if Q/N has infinite rank, so if $L = 1$ then Q is residually torsion-free nilpotent and locally of finite 0-rank, hence locally nilpotent, as in the proof of Lemma 2 of [6]. By this contradiction, L is nontrivial. Suppose now that L has a nontrivial normal subgroup S such that L/S is not periodic. If S has finite 0-rank and K is the pre-image of S in G (recall that $S \leq Q = G/H$), then K is soluble, by Lemma 2.3, and K^G is hyperabelian and hence, by Lemma 2.2, soluble. It follows from the definition of H that S is trivial, a contradiction. Thus S has infinite 0-rank, L/S is soluble and locally nilpotent and thus has a nontrivial torsion-free image L/U (where $S \leq U$). Now some term R of the derived series of Q lies in U, and it follows that Q/R is locally nilpotent (since every nontrivial normal subgroup of L has infinite 0-rank, as was the case for S). But this easily leads to a contradiction to the definition of L, and we conclude that every proper image of L is periodic, also soluble and locally nilpotent. The Fitting subgroup of L has finite 0-rank and is therefore trivial. Since L is countable it is an ascending union of finitely generated subgroups F_n where, for each n, F_n is soluble and, by Lemma 2.2, of finite rank (using the fact that every periodic subgroup of F_n has finite rank). Thus F_n is minimax [10;Theorem 10.38], and Lemma 2.4 now implies that F_n is nilpotent by-finite, so that L is locally polycyclic. By the remarks preceding the statement of Lemma 2.4, L is therefore soluble, and we have our final contradiction. □

Proof of Theorem 1.1 With G as stated, every locally soluble subgroup of G that has finite 0-rank has finite rank, by Lemma 2.3, so if every locally soluble subgroup has finite 0-rank then G has finite rank, by [1], a contradiction. Thus G contains a locally soluble subgroup L of infinite 0-rank, and L is soluble, by Proposition 2.5. Finally, L contains some term of the derived series of G and the result follows. □

3. Proofs

Proof of Theorem 1.2 Let G be as stated and let R be a periodic subgroup of G that has infinite section p-rank. Then there exists a countably infinite elementary abelian p-section V/U of R, and by Lemma 1.D.4 of [3] there is a p-subgroup Y of R such that $V = UY$. Since Y has infinite rank it contains an elementary abelian subgroup A of infinite rank, e.g. by Theorem 3.32 of [10]. Then some term of the derived series of G is contained in A and G is soluble. Let $g \in G$ and let $K = < A, g >, W = A^K$. Since A is subnormal in K we see that W is a p-group, and it follows that every subgroup of K that has finite section p-rank has finite rank, so that every non-subnormal subgroup of K has finite rank. By Theorem 2 of [5] K is therefore a Baer group. In particular we have $< g >$ subnormal in K, which in turn is subnormal in G. It follows that G is a Baer group. Let P denote the p-component of the torsion subgroup T of G, and note that P has infinite section p-rank. It suffices to prove that every subgroup of G that has finite section p-rank is subnormal in G. If H denotes such a subgroup then certainly PH is subnormal, so we may as well assume that $G = PH$. Furthermore, if Q is the p'-radical of T then $Q \cap H$ is normal in PH, so we may factor and hence assume that $Q \cap H$ is trivial. But now G/P is torsion-free, locally nilpotent and of finite section p-rank, so every abelian subgroup of G/P has finite 0-rank. It follows that G/P is (nilpotent and) of finite rank, so every subgroup of infinite rank is subnormal in G. Since the torsion subgroup P of G has infinite rank, we may apply Theorem 5 of [6] to conclude that every subgroup of G is subnormal. The result follows. □

For the proof of Theorem 1.3 we need the following lemma.

Lemma 3.1 *Let G be a group, g an element of G, and let A, B be $< g >$-invariant subgroups of G satisfying the following: $A \leq Z(B), A$ has finite 0-rank, $[B, g] \leq A$ and B/A is abelian and of infinite 0-rank. Then $C_G(g)$ contains an abelian subgroup of infinite 0-rank.*

Proof. The mapping $b \rightarrow [b, g]$ for all b in B is a homomorphism whose kernel is $C_B(g)$ and whose image has finite 0-rank. Thus $C_B(g)$ has infinite 0-rank and, since it is nilpotent, it has an abelian subgroup of infinite 0-rank. □

Proof of Theorem 1.3 Let T be the torsion radical of G. By Proposition 2.1, G/T is nilpotent. Let $g \in G$ - it suffices to prove that $< g >$ is subnormal in G. Let K/T be a maximal normal abelian subgroup of G/T; then K/T is self-centralizing and so G/K embeds in $Aut(K/T)$, and it follows that K/T has infinite 0-rank. Applying Lemma 3.1 we obtain a subgroup C/T of K/T that has infinite 0-rank and is such that $[C, g] \leq T$. Since $< g > C$ is subnormal in G we may as well assume that G/T is free abelian and of countably infinite rank, say with free generators $g_1, g_2, \ldots$ modulo T. Suppose first that T is abelian, and let F be an arbitrary finitely generated free abelian subgroup of G, g an element of G. Then $[F, < g >]$ is finitely generated as an $< F, g >$-group and therefore finite, as the Sylow p-subgroups of T are Chernikov. So $[F, < g >]$ is centralized by some nontrivial element x of $< g >$, and $< F, x >$ is nilpotent, and some nontrivial element y of $< x >$ (and hence of $< g >$) therefore centralizes F. Beginning with $F =< g_1 >$ and iterating the above construction (with $g = g_{i+1}$ at the *ith* step), we obtain a free abelian subgroup A of G such that G/TA is periodic. Since A is of infinite 0-rank it is subnormal in G, and it follows that the product TA is nilpotent and hence, by Lemma 3.1, contains an abelian subgroup C that has infinite 0-rank and centralizes g. In the general case, we may use the fact that T is soluble and repeat this argument sufficiently often to obtain an abelian subgroup C of infinite 0-rank that centralizes g. Then $C < g >$ is subnormal in G and $< g >$ is normal in $C < g >$, and the result follows. □

Proof of Theorem 1.4 We may assume that $O_{p'}(G) = 1$. Every Sylow p-subgroup of G is Chernikov and so the maximal normal torsion subgroup T of G is Chernikov, by a result of Kargapolov [3;Theorem 3.17]. If $r_0(G)$ is finite then G/T has finite rank [7;Theorem 3] and so G has finite section p-rank, a contradiction; hence $r_0(G)$ is infinite. If H is a subgroup of G that has infinite 0-rank then H has a free abelian section with infinite 0-rank and hence an abelian section with infinite p-rank. Thus every non-subnormal subgroup of G has finite 0-rank, and so G/T is nilpotent, by Theorem 1.1. Furthermore G is a Baer group, by Theorem 1.3. If D is the divisible component of T then T/D is finite and so G/D is nilpotent, while if If D has rank r then it lies in $Z_r(G)$ - here we may consider an arbitrary subgroup of the form DF, where F is finitely generated, and use the fact that G is Baer. Thus G is nilpotent, and the result follows. □

Proof of Theorem 1.5 If G contains a periodic subgroup of infinite p-rank for some prime p then Theorem 1.2 applies. Otherwise, letting T denote the maximal normal torsion subgroup of G, we see that every p-subgroup of T is Chernikov and so, as in the proof of Theorem 1.4, G/T has infinite 0-rank and every non-subnormal subgroup of G has finite 0-rank. Theorem 1.3 gives the result. □

Proof of Theorem 1.6 Let R be a periodic subgroup of infinite section total rank. Since R is not Chernikov it contains a non-Chernikov abelian subgroup [3; Theorem 5.8], and so (as before) G is soluble. Thus every non-subnormal subgroup of G has finite rank, and by the main result of [5] G is a Baer group. By Theorem 1.2 we may assume that all periodic subgroups have finite section p-rank for all primes p, so that every Sylow p-subgroup of the torsion subgroup T of G is Chernikov. Suppose for a contradiction that there is a non-subnormal subgroup H of G. Then HT is subnormal in G and we may as well assume that $G = HT$. Now $H \cap T$ is Chernikov and therefore contained in a G-invariant subgroup S of T such that $T = S \times U \times V$ for some G-invariant subgroups U, V that have infinite section total rank. But HU and HV are subnormal in G, and hence $H = HU \cap HV$ is also subnormal, a contradiction that concludes the proof. □

Proof of Theorem 1.7 Let T be the torsion radical of G. Then T is Chernikov and, as in the proof of Theorem 1.4, $r_0(G)$ is infinite. Since every non-subnormal subgroup has finite 0-rank we have G/T nilpotent, by Theorem 1.1, and G is a Baer group, by Theorem 1.3. Again as in the proof of Theorem 1.4, G is nilpotent. □

Proof of Theorem 1.8 If G has infinite (section) total rank then we may apply Theorems 1.6 and 1.7, since every minimax subgroup of G has finite total rank. Suppose then that G has finite total rank. We claim that G is nilpotent, and in order to establish this it suffices to show that G is Baer, for a (soluble) Baer group with finite total rank is easily shown to be nilpotent (see p.38 of Volume II of [10]). Since G is not minimax it has an abelian subgroup H that is not minimax, by a result of Baer and Zaičev [11; 15.2.8]. Since H is contained in the Baer radical of G its normal closure $A = H^G$ is nilpotent and not minimax. Then A/A' is non-minimax [10; Theorem 2.26], while if G/A' is nilpotent then so is G [10; Theorem 2.27]. Factoring, we may assume that A is abelian. Let $g \in G$. It suffices to prove that $< g >$ is subnormal in G, and since $A < g >$ is subnormal we may assume that $G = A < g >$. There is a finitely generated subgroup F of A such that A/F is periodic; write $D = F^{<g>}$, a normal subgroup of G. Since $< F, g >$ has finite rank it is minimax [10; Theorem 10.38], and $< F, g >$ is residually finite, by a result of P. Hall [10; Theorem 9.51]. The torsion subgroup of D is therefore finite, and D has a G-invariant torsion-free subgroup B of finite index, which in turn contains a finitely generated subgroup C such that B/C is the direct product of finitely many quasicyclic groups. The set of primes occurring here is the *spectrum* $Sp(B)$ of B, and if p is any prime not contained in $Sp(B)$ then B/B^p is nontrivial; indeed, the intersection of all B^p is trivial. It is easy to see that, for each such prime $p, A/B^p$ has $< g >$-invariant non-minimax subgroups $U/B^p, V/B^p$ such that $U \cap V \leq B^p$ and $U < g > \cap V < g >= B^p < g >$. Each of $U < g >, V < g >$ is subnormal in G, as therefore is $B^p < g >$. If r is the rank of B then we have $[B,_r < g >] \leq B^p$ for all such p and so $[B,_r < g >] = 1$. Since $B < g >$ is subnormal in G we deduce that $< g >$ is subnormal in G, as required. □

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